УДК 512.542

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On Modular Property for Fitting Classes

1. Introduction. In the theory of classes of groups, the lattice properties have been extensively studied by a number of authors. Sel'kin [1] has proved that the lattice of all Schunck classes of finite groups is distributive. Al-Dababseh [2] proved that the lattice of all Schunck classes of finite n-groups is distributive. Skiba [3] has shown that in the lattice of all formations of finite groups, the modular law holds. We also note that Guo and Shum [4] proved that the set of all Schunck classes of Mal'cev algebras satisfying the maximun condition for subalgebras is a distributive lattice and its sublattice consisting of all Schunck classes of finite algebras is algebraic. However, up to now, we do not know whether the modular law holds for the lattice of all Fitting classes of finite groups (in particular, for the lattice of all soluble Fitting classes of finite groups). This is just the open problem 14.47 in [5]. In this direction, N.T. Vorob'ev [6] obtained only some results about sublattice of the lattice of Fitting classes.

In this paper, we give a condition under which the modular law holds for the Fitting classes of finite groups.

2. Preliminaries. Throughout this paper, all groups considered are finite groups. Recall that a class of groups $\mathfrak{F}$ is called a Fitting class if the following conditions hold:

i) if $G \in \mathfrak{F}$ and $N \lhd G$, then $N \in \mathfrak{F}$;

ii) if $G = MN$, $M \lhd G$, $N \lhd G$ and $M \in \mathfrak{F}$, $N \in \mathfrak{F}$, then $G \in \mathfrak{F}$.

It is easy to see from the condition ii) that if $\mathfrak{F}$ is a nonempty Fitting class, then every group G has a unique $\mathfrak{F}$-maximal normal subgroup which is called the $\mathfrak{F}$-radical of G and denote by $G_{\mathfrak{F}}$.

Let $\mathfrak{F}$ and $\mathfrak{H}$ be two Fitting classes. Then the class of groups $\mathfrak{F}\mathfrak{H} = (G \mid G/G_{\mathfrak{F}} \in \mathfrak{H})$ is called the product of Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$.

It is well known that the product of Fitting classes is also a Fitting class and the associative law of the multiplication holds.

We denote by $\mathfrak{F} \vee \mathfrak{H}$ the Fitting class generated by $\mathfrak{F}$ and $\mathfrak{H}$, i.e., the intersection of all those Fitting class which contains the union of $\mathfrak{F}$ and $\mathfrak{H}$.

A Fitting class $\mathfrak{F}$ is said to be a radical homomorph [7] if it is closed under homomorphic images.

Let $\mathfrak{L} = \{\mathfrak{X} \mid \mathfrak{X}$ is a Fitting class and $\mathfrak{X} \subseteq \mathfrak{G}\}$, where $\mathfrak{G}$ is the class of all finite groups. Obviously, $\mathfrak{L}$ by inclusion relation $\subseteq$ is a partially ordered set. The partially ordered set is a lattice. In fact, for any two elements $\mathfrak{F}$, $\mathfrak{H}$ in $\mathfrak{L}$ have a least upper bound $\mathfrak{F} \vee \mathfrak{H}$ and a greatest lower bound $\mathfrak{F} \cap \mathfrak{H}$.

We say that the modular law holds in $\mathfrak{L}$ provided the following condition is satisfied:

if $\mathfrak{F}$, $\mathfrak{H}$ and $\mathfrak{X}$ are Fitting classes in $\mathfrak{L}$ and $\mathfrak{F} \subseteq \mathfrak{X}$, then

$$(\mathfrak{F} \vee \mathfrak{H}) \cap \mathfrak{X} = \mathfrak{F} \vee (\mathfrak{H} \cap \mathfrak{X}).$$

For other notation and terminology not mentioned in this paper are standard, the reader may refer to ref. [8, 9].

3. Main result. To proving our theorem, we need using the Cusack's result in [10] which was proved throughout the class $\mathfrak{S}$ of all finite soluble groups. But, by using the same method it can be proved that the result holds throughout the class $\mathfrak{E}$ of all finite groups. We now cite it as the following lemma.

Lemma 3.1 [10, Theorem 2.9]. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Fitting classes such that* $\mathfrak{X} \vee \mathfrak{Y} = S_n(G|G = G_{\mathfrak{X}}G_{\mathfrak{Y}})$. *If $\mathfrak{F}$ is a Fitting class with $\mathfrak{X} \subseteq \mathfrak{F}$, then*

$$(\mathfrak{X} \vee \mathfrak{Y}) \cap \mathfrak{F} = \mathfrak{X} \vee (\mathfrak{Y} \cap \mathfrak{F}).$$

Theorem 3.2. *Let $\mathfrak{X}$, $\mathfrak{Y}$ be Fitting classes and $\mathfrak{F}$, $\mathfrak{H}$ be radical homomorphs such that $\mathfrak{F} \cap \mathfrak{H} = (1)$, $\mathfrak{X} \subseteq \mathfrak{Y}\mathfrak{F}$ and $\mathfrak{Y} \subseteq \mathfrak{X}\mathfrak{H}$. If $\mathfrak{X} \subseteq \mathfrak{F}$, then*

$$(\mathfrak{X} \vee \mathfrak{Y}) \cap \mathfrak{F} = \mathfrak{X} \vee (\mathfrak{Y} \cap \mathfrak{F}).$$

Proof. Let $\mathfrak{M} = (G \mid G = G_{\mathfrak{X}}G_{\mathfrak{Y}})$. We first prove that $\mathfrak{X} \vee \mathfrak{Y} = \mathfrak{M}$.

If $G \in \mathfrak{M}$, then $G = G_{\mathfrak{X}}G_{\mathfrak{Y}}$. By the definition of $\mathfrak{X}$-radical, we have that $G_{\mathfrak{X}} \in \mathfrak{X} \subseteq \mathfrak{X} \vee \mathfrak{Y}$. Analogously, $G_{\mathfrak{Y}} \in \mathfrak{X} \vee \mathfrak{Y}$. However, since $\mathfrak{X} \vee \mathfrak{Y}$ is a Fitting class, we see that $G = G_{\mathfrak{X}}G_{\mathfrak{Y}} \in \mathfrak{X} \vee \mathfrak{Y}$. This shows that

$$\mathfrak{M} \subseteq \mathfrak{X} \vee \mathfrak{Y} \qquad (1)$$

On the other hand, since $\mathfrak{X} \subseteq \mathfrak{X}\mathfrak{H} \cap \mathfrak{Y}\mathfrak{F}$ and $\mathfrak{Y} \subseteq \mathfrak{X}\mathfrak{H} \cap \mathfrak{Y}\mathfrak{F}$, we have that

$$\mathfrak{X} \vee \mathfrak{Y} \subseteq \mathfrak{X}\mathfrak{H} \cap \mathfrak{Y}\mathfrak{F} \qquad (2)$$

Let $\mathfrak{X}\mathfrak{H} \cap \mathfrak{Y}\mathfrak{F} = \mathfrak{D}$ and $G \in \mathfrak{D}$. Then, $G/G_{\mathfrak{X}} \in \mathfrak{H}$, and $G/G_{\mathfrak{Y}} \in \mathfrak{F}$. But, by the condition of the theorem, $\mathfrak{F}$ and $\mathfrak{H}$ are radical homomorphs, so we have that $(G/G_{\mathfrak{X}})/(G_{\mathfrak{X}}G_{\mathfrak{Y}} / G_{\mathfrak{X}}) \in \mathfrak{H}$ and $(G/G_{\mathfrak{Y}})/(G_{\mathfrak{X}}G_{\mathfrak{Y}} / G_{\mathfrak{Y}}) \in \mathfrak{F}$.

Since $(G/G_{\mathfrak{X}}) / (G_{\mathfrak{X}}G_{\mathfrak{Y}} / G_{\mathfrak{X}}) \simeq G / G_{\mathfrak{X}}G_{\mathfrak{Y}}$ and $(G/G_{\mathfrak{Y}}) / (G_{\mathfrak{X}}G_{\mathfrak{Y}} / G_{\mathfrak{Y}}) \simeq G/ G_{\mathfrak{X}}G_{\mathfrak{Y}}$, we obtain that $G / G_{\mathfrak{X}}G_{\mathfrak{Y}} \in \mathfrak{F} \cap \mathfrak{H} = (1)$. This shows that $G = G_{\mathfrak{X}}G_{\mathfrak{Y}}$ and hence $\mathfrak{D} \subseteq \mathfrak{M}$. Then, by the inclusion (2), we obtain

$$\mathfrak{X} \vee \mathfrak{Y} \subseteq \mathfrak{M}. \qquad (3)$$

From the inclusions (1) and (3) follows that

$$\mathfrak{M} = \mathfrak{X} \vee \mathfrak{Y}.$$

Now, by using Lemma 3.1, we obtain that $(\mathfrak{X} \vee \mathfrak{Y}) \cap \mathfrak{F} = \mathfrak{X} \vee (\mathfrak{Y} \cap \mathfrak{F})$. The theorem is thus proved.

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SUMMARY

In this paper, we give a condition under which the modular law holds for the Fitting classes of finite groups.

Поступила в редакцию 24.02.2004