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MATHEMATICAL LOGIC

Study guide

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The study guide is prepared in accordance with the curriculum of the course "Discrete Mathematics & Mathematical Logic" for the students of the Faculty of Mathematics. It covers basic topics of the course such like "Mathematical Logi" and "Sets." Each section contains theoretical material and practical assignments. The teaching aid designed for students of specialty "Applied Informatics" (English-Medium).

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Preface (in Russian)

Дисциплина «Дискретная математика и математическая логика» направлена на формирование математической культуры студента, повышение уровня фундаментальной подготовки математика-программиста, овладение современным математическим аппаратом для дальнейшего использования в приложениях.

Настоящее учебное издание охватывает основополагающие разделы дисциплины: «Математическая логика» и «Множества». Каждый раздел содержит теоретический материал (часть A) и практические задания (часть B). По каждой теме приведены необходимые определения и теоретические сведения, а также разобраны примеры решения задач. Издание содержит список литературы, необходимой для более глубокого и разностороннего изучения материала, а также для закрепления практических навыков самостоятельного решения конкретных задач, в качестве приложения приведен перечень определений и обозначений основных понятий.

Методические рекомендации предназначены для студентов специальности «Прикладная информатика» (English-Medium) с обучением на английском языке, а также для организации самостоятельной работы студентов математического факультета специальностей «Прикладная информатика», «Прикладная математика», «Математика и информатика» дневной и заочной форм обучения.

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Introduction

Mathematical logic is a subfield of mathematics exploring the applications of formal logic to mathematics. Topically, mathematical logic bears close connections to metamathematics, the foundations of mathematics, and theoretical computer science. The unifying themes in mathematical logic include the study of the expressive power of formal systems and the deductive power of formal proof systems.

It is important to be aware of the reasons that we study logic. There are three very significant reasons. First, the truth tables we studied tell us the exact meanings of the words such as “*and*,” “*or*,” “*not*” and so on. For instance, whenever we use or read the “*If ..., then*” construction in a mathematical context, logic tells us exactly what is meant. Second, the rules of inference provide a system in which we can produce new information (statements) from known information. Finally, logical rules such as DeMorgan’s laws help us correctly change certain statements into (potentially more useful) statements with the same meaning. Thus logic helps us understand the meanings of statements and it also produces new meaningful statements.

Logic is the glue that holds strings of statements together and pins down the exact meaning of certain key phrases such as the “*if ..., then*” or “*for all*” constructions. Logic is the common language that all mathematicians use, so we must have a firm grip on it in order to write and understand mathematics. But despite its fundamental role, logic’s place is in the background of what we do, not the forefront. From here on, the beautiful symbols $\sim$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$ and $\exists$ are rarely written. But we are aware of their meanings constantly. When reading or writing a sentence involving mathematics we parse it with these symbols, either mentally or on scratch paper, so as to understand the true and unambiguous meaning.

We shall study a formal language called *propositional logic*. This language abstracts from ordinary language the properties of the propositional connectives. These are “*not*”, “*and*”, “*or*”, “*if*”, and “*if and only if*”. Formal languages differ from natural languages such as English or Russian in that the syntax of a formal language is precisely given. This is not the case with English: authorities often disagree as to whether a given English sentence is grammatically correct. *Mathematical logic may be defined as that branch of mathematics which studies formal languages.*

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I Logic

A1 Logical Connectives

Mathematics works according to the laws of logic, which specify how to make valid deductions. In order to apply the laws of logic to mathematical statements, you need to understand their logical forms. Proofs are composed of statements. A statement is a declarative sentence that can be either true or false.

Remark 1. Many real proofs contain things which aren't really statements — questions, descriptions, and so on. They're there to help to explain things for the reader. When I say "*Proofs are composed of statements*", I'm referring to the actual mathematical content with the explanatory material removed.

Example 1. "*Bob is a math major*" is a statement. You'd need to know more about Bob and math majors to know whether the statement is true or false.
"0 = 1" is a statement which is false (assuming that "0" and "1" refer to the real numbers 0 and 1).
"*Do you have a pork barbecue sandwich?*" is not a statement — it's a question. Likewise, "*Eat your vegetables!*" is not a statement — it's an imperative sentence, i.e. an order to do something.

"1+1 = 2" is a statement. An easy way to tell is to read it and see if it's a complete declarative sentence which is either true or false. This statement would read (in words): "*One plus one equals two*." You can see that it's a complete declarative sentence (and it happens to be a true statement about real numbers).
On the other hand, "1+1" is not a statement. It would be read "*One plus one*", which is not a sentence since it doesn't have a verb. (Things like "1 + 1" are referred to as ***terms*** or ***expressions***.)

Since proofs are composed of ***statements***, you should never have isolated phrases (like 1+1 or "a+3b") in your proofs. Be sure that every line of a proof is a statement. Read each line to yourself to be sure.
In terms of logical form, statements are built from simpler statements using ***logical connectives***. The basic connectives of sentential logic are:

1	**Negation**	(*"not"*)	denoted $\sim$
2	**Conjunction**	(*"and"*)	denoted $\wedge$
3	**Disjunction**	(*"or"*)	denoted $\vee$
4	**Conditional**	(*"if-then"* or *"implication"*)	denoted $\rightarrow$
5	**Biconditional**	(*"if and only if"* or *"double implication"*)	denoted $\leftrightarrow$

Later I'll discuss the quantifiers "*for all*" (denoted $\forall$) and "*there exists*" (denoted $\exists$).

Remark 2. You may see different symbols used by other people. For example, some people use ¬ for negation. And => is sometimes used for the conditional, in which case <=> is used for the biconditional.

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Example 2. Represent the following statements using logical connectives.

	Statement	***Logical translation***
1	P or not Q.	$P \vee \sim Q$
2	If P and R, then Q.	$(P \wedge R) \rightarrow Q$
3	P if and only if (Q and R).	$P \leftrightarrow (Q \wedge R)$
4	Not P and not Q.	$\sim P \wedge \sim Q$
5	It is not the case that if P, then Q.	$\sim (P \rightarrow Q)$
6	If P and Q, then R or S.	$(P \wedge Q) \rightarrow (R \vee S)$

Other words or phrases may occur in statements. Here's a list of some of them and how they are translated.

	Phrase	***Logical translation***
1	P, but Q	$P \wedge Q$
2	Either P or Q	$P \vee Q$
3	P or Q, but not both	$(P \vee Q) \wedge \sim (P \wedge Q)$
4	P if Q	$Q \rightarrow P$
5	P is necessary for Q	$Q \rightarrow P$
6	P is sufficient for Q	$P \rightarrow Q$
7	P only if Q	$P \rightarrow Q$
8	P is equivalent to Q	$P \leftrightarrow Q$
9	P whenever Q	$Q \rightarrow P$

Consider the word "*but*", for example. If I say "*Bob is here, but Sam is there*", I mean that Bob is here and Sam is there. My intention is that both of the statements should be true. That is the same as what I mean when I say "*Bob is here and Sam is there*".

In practice, mathematicians tend to a small set of phrases over and over. It doesn't make for exciting reading, but it allows the reader to concentrate on the meaning of what is written. For example, a mathematician will usually say "*if Q, then P*", rather than the logically equivalent "*P whenever Q*". The second statement is less familiar, and therefore more apt to be misunderstood.

This is a good time to discuss the way the word "or" is used in mathematics. When you say "*I'll have dinner at MacDonald's or at Pizza Hut*", you probably mean "*or*" in its exclusive sense: You'll have dinner at MacDonald's or you'll have dinner at Pizza Hut, but not both. The "*but not both*" is what makes this an ***exclusive or***.

Mathematicians use "*or*" in the inclusive sense. When "*or*" is used in this way, "*I'll have dinner at MacDonald's or at Pizza Hut*" means you'll have dinner at MacDonald's or you'll have dinner at Pizza Hut, *or possibly both*. Obviously, I'm not guaranteeing that both will occur; I'm just not ruling it out.

Example 3. Translate the following statements into logical notation, using the following symbols:
S = "*The stromboli is hot*."
L = "*The lasagne is cold*."
P = "*The pizza will be delivered*."

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	Phrase	Logical translation
1	*The stromboli is hot and the pizza will not be delivered*	$S \wedge \sim P$
2	*If the lasagne is cold, then the pizza will be delivered*	$L \rightarrow P$
3	*Either the lasagne is cold or the pizza won't be delivered*	$L \vee \sim P$
4	*If the pizza won't be delivered, then both the stromboli is hot and the lasagne is cold*	$\sim P \rightarrow (S \wedge L)$
5	*The lasagne isn't cold if and only if the stromboli isn't hot*	$\sim L \leftrightarrow \sim S$
6	*The pizza will be delivered only if the lasagne is cold*	$P \rightarrow L$
7	*The stromboli is hot and the lasagne isn't cold, but the pizza will be delivered*	$S \wedge \sim L \wedge P$

The order of precedence of the logical connectives is:

1. Negation
2. Conjunction
3. Disjunction
4. Implication
5. Double implication

As usual, parentheses override the other precedence rules. *In most cases, it's best for the sake of clarity to use parentheses even if they aren't required by the precedence rules.* For example, it's better to write

$$(P \wedge Q) \vee R \quad \text{rather than} \quad P \wedge Q \vee R.$$

Precedence would group P and Q anyway, but the first expression is clearer.
It's not common practice to use parentheses for grouping in ordinary sentences. Therefore, when you're translating logical statements into words, you may need to use certain expressions to indicate grouping.

1) The combination "*Either . . . or . . .* " is used to indicate that everything between the "*either*" and the "*or*" is the first part of the "or" statement.
2) The combination "*Both . . . and . . .* " is used to indicate that everything between the "*both*" and the "*and*" is the first part of the "and" statement.

In some cases, the only way to say something unambiguously is to be a bit wordy. Fortunately, mathematicians find ways to express themselves which are clear, yet avoid excessive linguistic complexity.

Example 4. Suppose that

C = "*The cheesesteak is good*."
F = "The french fries are greasy."
W = "The wings are spicy."

Translate the following logical statements into words (with no logical symbols):

	Statement	Translation
1	$(\sim C \wedge F) \rightarrow W$	"***If the cheesesteak isn't good and the french fries are greasy, then the wings are spicy***."

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2	$\sim (C \vee W)$	If I say "*It's not the case that the cheesesteak is good or the wings are spicy*", it might not be clear whether the negation applies only to "*the cheesesteak is good*" or to the disjunction "*the cheesesteak is good or the wings are spicy*". So it's better to say "***It's not the case that either the cheesesteak is good or the wings are spicy***", since the "*either*" implies that "*the cheesesteak is good*" or "*the wings are spicy*" are <u>grouped together</u> in the or-statement. In this case, the "*either*" blocks the negation from applying to "*the cheesesteak is good*", so the negation has to apply to the whole "*or*" statement.
3	$\sim (\sim W \wedge C)$	"***It's not the case that <u>both</u> the wings aren't spicy and the cheesesteak is good***." As with the use of the word "either" in (2), I've added the word "*both*" to indicate that the initial negation applies to the conjunction "*the wings aren't spicy and the cheesesteak is good*". In this case, the "*both*" blocks the negation from applying to "*the wings aren't spicy*", so the negation has to apply to the whole "*and*" statement.
4	$\sim (\sim F)$	The literal translation is "*It's not the case that the french fries aren't greasy*". Or (more awkwardly) you could say "*It's not the case that it's not the case that the french fries are greasy*". Of course, this is logically the same as saying "***The french fries are gresy***". But the question did not ask you to simplify the original statement — only to translate it, which you should do verbatim.

Example 5. Here are some examples of actual mathematical text.

(a) **Theorem.** *In the semi-simple ring R, let L = Re be a left ideal with generating idempotent e. Then L is a minimal left ideal if and only if eRe is a skew field.*

You could express this using logical connectives in the following way. Let

A = "R is a semi-simple ring".
B = "$L = Re$ is a left ideal with generating idempotent e".
C = "L is a minimal left ideal".
D = "eRe is a skew field".

The statement can be translated as $(A \wedge B) \rightarrow (C \leftrightarrow D)$.

Notice that to determine the *logical form*, you don't have to know what the words mean! Mathematicians use the word "let" to introduce *hypotheses* in the statement of a theorem. From the point of view of logical form, the statements that accompany "let" form the ***antecedent*** — the first part — of a conditional, as statements A and B do here.

(b) **Proposition**. *Let X and Y be CW-complexes. Then* $X \times Y$ (*with the compactly generated topology*) *is a CW complex, and* $X \vee Y$ *is a subcomplex.*

Let

P = "X and Y are CW-complexes".
Q = "$X \times Y$ (with the compactly generated topology) is a CW complex".
R = "$X \vee Y$ is a subcomplex".

The proposition can then be written in logical notation as $P \rightarrow (Q \wedge R)$.

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Notice that you can often translate a statement in different ways. For example, I could have let

A = "X is a CW-complex".

B = "Y is a CW-complex".

C = "$X \times Y$ (with the compactly generated topology) is a CW complex".

D = "$X \vee Y$ is a subcomplex".

Now the proposition becomes $(A \wedge B) \to (C \wedge D)$. There is no difference in mathematical content, and no difference in terms of how you would prove it.

As the last example shows, logical implications often arise in mathematical statements. Here's some terminology. If $P \to Q$ is an implication, then:

(a) P is the ***antecedent*** or ***hypothesis*** and Q is the ***consequent*** or ***conclusion***.
(b) The ***converse*** is the conditional $Q \to P$.
(c) The ***inverse*** is the conditional $\sim P \to \sim Q$.
(d) The ***contrapositive*** is the conditional $\sim Q \to \sim P$.

Example 6. Find the antecedent and the consequent of the following conditional statement: "If $x > y$, then $y >$ Bob."
Construct the converse, the inverse, and the contrapositive.
The antecedent is "$x > y$" and the consequent is "$y >$ Bob".
The converse is "If $y >$ Bob, then $x > y$".
The inverse is "If $x \leq y$, then $y \leq$ Bob."
The contrapositive is "If $y \leq$ Bob, then $x \leq y$".
Later on, I'll show that a conditional statement and its contrapositive are ***logically equivalent***.

Example 7. Find the antecedent and the consequent of the following conditional statement: "*If Bob gets a hot dog, then Bob doesn't get a soda*."
Construct the converse, the inverse, and the contrapositive.
The antecedent is "*Bob gets a hot dog*" and the consequent is "*Bob doesn't get a soda*".
The converse is "*If Bob doesn't get a soda, then Bob gets a hot dog*".
The inverse is "*If Bob doesn't get a hot dog, then Bob gets a soda*". (Note that the literal negation of the consequent is "*It is not the case that Bob doesn't get a soda*". But the two negations cancel out — this is called ***double negation*** — so I get "*Bob gets a soda*".)
The contrapositive is "*If Bob gets a soda, then Bob doesn't get a hot dog*".

B1 Problem Set

Problem 1. Write the negation (in words) of each statement.

(a) "82 *is an odd integer*."
(b) "$\sqrt{5}$ *is a rational number*."
(c) "*x is a positive real number*."
(d) "3.1 *is an integer*."
(e) "π *is a real number*."
(f) "23 *is a prime number*."
(g) "0 *is a natural number*."
(h) "0.5 *is an irrational number*".
(i) "15 *is an even integer*."
(j) "–2 *is a rational number*."

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Problem 2. Suppose

P is the statement "100 *is an even integer*".

Q is the statement "52 *is a prime number*".

(a) Write the statement "52 is a prime number or 100 is not an even integer" in symbols.

(b) Write the statement "100 is not an even integer and 52 is not a prime number" in symbols.

(c) Write the statement "52 is not a prime number but 100 is an even integer" in symbols.

(d) Write the statement "52 is a prime number if and only if 100 is not an even integer" in symbols.

(e) Write the statement "If 100 is an even integer, then 52 is a prime number" in symbols.

(f) Write the statement $P \vee Q$ in words, and determine whether the statement is true or false.

(g) Write the statement $P \wedge Q$ in words, and determine whether the statement is true or false.

(h) Write the statement $Q \to \sim P$ in words, and determine whether the statement is true or false.

(i) Write the statement $\sim (P \leftrightarrow Q)$ in words, and determine whether the statement is true or false.

(j) Write the statement $\sim (Q \to P)$ in words, and determine whether the statement is true or false.

Problem 3. Suppose that

C = *"Bob likes cheddar."*

S = *"Bob likes swiss."*

M = *"Bob likes mozzarella."*

F = *"You have the flu."*

E = *"You miss the final examination."*

P = *"You pass the course."*

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(i) Translate the following logical statements into words:

(a) $C \to \sim S$.

(b) $(C \vee S) \leftrightarrow \sim M$.

(c) $\sim (\sim M \wedge \sim C)$.

(d) $M \wedge (C \leftrightarrow \sim S)$.

(e) $\sim (M \vee C \vee S)$.

(f) $\sim F \to (E \vee P)$.

(g) $\sim (E \to F)$.

(h) $(P \vee E\ ) \leftrightarrow \sim F$.

(i) $\sim F \to E \to P$.

(j) $P \leftrightarrow (\sim F \wedge E)$.

(ii) Translate the following statements into logical symbols:

(a) *"Bob likes cheddar if he likes mozzarella."*

(b) *"Bob doesn't like either cheddar or mozzarella."*

(c) *"Bob doesn't like cheddar if and only if either he doesn't like swiss or he doesn't like mozzarella."*

(d) *"Bob likes mozzarella whenever he doesn't like cheddar."*

(e) *"Bob doesn't like mozzarella but if he likes cheddar then he likes swiss."*

(f) *"If you have the flu, you miss the final examination."*

(g) *"You pass the course if and only if you take the final exam."*

(h) *"If you miss the final examination, you will not pass the course."*

(i) *"You either have the flu, miss the final examination, or pass the course."*

(j) *"Either you have the flu and miss the final exam, or you don't miss the final exam and pass the course."*

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A2 Truth Tables

Mathematics normally works with a ***two-valued logic***: Every statement is either ***True*** or ***False***. You can use ***truth tables*** to determine the truth or falsity of a complicated statement based on the truth or falsity of its simple components.

A statement in sentential logic is built from simple statements using the logical connectives ~, ∧, ∨, →, and ↔. I'll construct tables which show how the truth or falsity of a statement built with these connective depends on the truth or falsity of its components.

Here's the table for negation:

P	$\sim P$
T	F
F	T

This table is easy to understand. If P is *true*, its negation $\sim P$ is *false*. If P is *false*, then $\sim P$ is *true*.

$P \wedge Q$ should be *true* when both P and Q are *true*, and *false* otherwise:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

$P \vee Q$ is *true* if either P is *true* or Q is *true* (or both). It's only *false* if both P and Q are *false*.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Here's the table for logical implication:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

To understand why this table is the way it is, consider the following example: "*If you get an* A*, then I'll give you a dollar*." The statement will be true if I keep my promise and false if I don't.

Suppose it's true that you get an A and it's true that I give you a dollar. Since I kept my promise, the implication is true. This corresponds to the first line in the table.

Suppose it's true that you get an A but it's false that I give you a dollar. Since I didn't keep my promise, the implication is false. This corresponds to the second line in the table.

What if it's false that you get an A? Whether or not I give you a dollar, I haven't broken my promise. Thus, the implication can't be false, so (since this is a two-valued logic) it must be true. This explains the last two lines of the table.

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$P \leftrightarrow Q$ means that P and Q are ***equivalent***. So the double implication is *true* if P and Q are both *true* or if P and Q are both *false*; otherwise, the double implication is *false*

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

You should remember — or be able to construct — the truth tables for the logical connectives. You'll use these tables to construct tables for more complicated sentences. It's easier to demonstrate what to do than to describe it in words, so you'll see the procedure worked out in the examples.

Remark 1. When you're constructing a truth table, you have to consider all possible assignments of True (T) and False (F) to the component statements. For example, suppose the component statements are P, Q, and R. Each of these statements can be either true or false, so there are $2^3 = 8$ possibilities. When you're listing the possibilities, you should assign truth values to the component statements in a systematic way to avoid duplication or omission. The easiest approach is to use *lexicographic ordering*. Thus, for a compound statement with three components P, Q, and R, I would list the possibilities this way:

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Remark 2. There are different ways of setting up truth tables. You can, for instance, write the truth values "under" the logical connectives of the compound statement, gradually building up to the column for the "primary" connective. I'll write things out the long way, by constructing columns for each "piece" of the compound statement and gradually building up to the compound statement.

Example. Construct a truth table for the formula $\sim P \wedge (P \rightarrow Q)$.

P	Q	$\sim P$	$P \rightarrow Q$	$\sim P \wedge (P \rightarrow Q)$
T	T	F	T	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

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Summary

A statement is a sentence that is either true or false. A *closed sentence* is an objective statement which is either true or false. An *open sentence* is a statement which contains a variable and becomes either true or false depending on the value that replaces the variable.
The <u>negation</u> of statement P is "not P", symbolized by $\sim P$. A statement and its negation have opposite truth values.
A <u>conjunction</u> is a compound statement formed by joining two statements with the connector "and". The conjunction "P and Q" is symbolized by $P \wedge Q$. A conjunction is true when both of its combined parts are true; otherwise it is false.
A <u>disjunction</u> is a compound statement formed by joining two statements with the connector "or". The disjunction "P or Q" is symbolized by $P \vee Q$. A disjunction is false if and only if both statements are false; otherwise it is true.
A <u>conditional</u> statement, symbolized by $P \to Q$, is an if-then statement in which P is a *hypothesis* and Q is a *conclusion*. The conditional is defined to be true unless a true hypothesis leads to a false conclusion.
A <u>biconditional</u> statement is defined to be true whenever both parts have the same truth value. The biconditional operator is denoted by a double-headed arrow $\leftrightarrow$. The biconditional $P \leftrightarrow Q$ represents "P if and only if Q" where P is a *hypothesis* and Q is a *conclusion*.

B2 Problem Set

Problem 1. Given

(i) R = "I am an honor student", S = "I play football",
(ii) R = "I eat cold cereal for breakfast", S = "I play football",
(iii) R = "I love mathematics", S = "I can dance the samba",
(iv) R = "I like winter", S = "I live in Vitebsk",
(v) R = "I am a famous singer", S = "I work in a hospital".

Write the statement represented by each of the following

(a) $\sim R$ ____
(b) $R \wedge S$ ____
(c) $R \wedge \sim S$ ____
(d) $\sim R \wedge S$ ____
(e) $R \vee S$ ____
(f) $R \vee \sim S$ ____

Problem 2. (i) Given

$$A = \text{"}y < 6\text{"},$$
$$B = \text{"}y > 1\text{"}.$$

Let the domain $U = \{1, 2, 3, 4, 5, 6, 7\}$.

(a) For what values y is $A \wedge B$ true? ______________________________
(b) For what values y is $A \vee B$ true? ______________________________

(ii) Given

$$C = \text{"}z \le 8\text{"},$$
$$D = \text{"}z > 4\text{"}.$$

Let the domain $U = \{4, 5, 6, 7, 8, 9\}$.

(a) For what values z is $C \wedge D$ true? ______________________________
(b) For what values z is $C \vee D$ true? ______________________________

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(iii) Given

$$M = \text{“}t < 4\text{”},$$
$$N = \text{“}t \geq 0\text{”}.$$

Let the domain $U = \{-1, 0, 1, 2, 3, 4\}$.

(a) For what values t is $M \wedge N$ true? ________________________________

(b) For what values t is $M \vee N$ true? ________________________________

(iv) Given

$$P = \text{“}x < 4\text{”},$$
$$Q = \text{“}x > 1\text{”}.$$

Let the domain $U = \{1, 2, 3, 4, 5\}$.

(a) For what values x is $P \wedge Q$ true? ________________________________

(b) For what values x is $P \vee Q$ true? ________________________________

(v) Given

$$R = \text{“}v > 0\text{”},$$
$$S = \text{“}v < -1\text{”}.$$

Let the domain $U = \{-4, -3, -2, -1, 0, 1\}$.

(a) For what values v is $P \wedge Q$ true? ________________________________

(b) For what values v is $P \vee Q$ true? ________________________________

Problem 3. Create the following truth table for the statements given in problem 1.

R	S	$\sim R$	$R \wedge S$	$\sim S$	$R \wedge \sim S$	$\sim R \wedge S$	$R \vee S$	$R \vee \sim S$
T	T							
T	F							
F	T							
F	F							

Problem 4. Complete the truth table below using your knowledge of symbolic logic.

P	Q	$\sim Q$	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$Q \rightarrow P$	$P \leftrightarrow Q$	$P \rightarrow \sim Q$	$P \vee Q \rightarrow P$	$P \wedge Q \rightarrow Q$
T	T									
T	F									
F	T									
F	F									

Problem 5. Construct a truth table for

(a) $(\sim P \wedge Q) \rightarrow P$,

P	Q	$\sim P$	$\sim P \wedge Q$	$(\sim P \wedge Q) \rightarrow P$
T	T			
T	F			
F	T			
F	F			

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(b) $(\sim P \leftrightarrow Q) \rightarrow P$,

P	Q	$\sim P$	$\sim P \leftrightarrow Q$	$(\sim P \leftrightarrow Q) \rightarrow P$
T	T			
T	F			
F	T			
F	F			

(c) $\sim Q \vee \sim Q \wedge P$,

P	Q	$\sim Q$	$\sim Q \wedge P$	$\sim Q \vee \sim Q \wedge P$
T	T			
T	F			
F	T			
F	F			

(d) $\sim (\sim P \rightarrow Q)$,

P	Q	$\sim P$	$\sim P \rightarrow Q$	$\sim(\sim P \rightarrow Q)$
T	T			
T	F			
F	T			
F	F			

(e) $P \leftrightarrow (Q \leftrightarrow \sim P)$,

P	Q	$\sim P$	$Q \leftrightarrow \sim P$	$P \leftrightarrow (Q \leftrightarrow \sim P)$
T	T			
T	F			
F	T			
F	F			

(f) $P \vee (Q \vee \sim P)$,

P	Q	$\sim P$	$Q \vee \sim P$	$P \vee (Q \vee \sim P)$
T	T			
T	F			
F	T			
F	F			

(g) $P \wedge \sim Q \vee P \wedge Q$,

P	Q	$\sim Q$	$P \wedge \sim Q$	$P \wedge Q$	$P \wedge \sim Q \vee P \wedge Q$
T	T				
T	F				
F	T				
F	F				

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(h) $P \leftrightarrow \sim Q \wedge \sim P$,

P	Q	$\sim Q$	$\sim P$	$\sim Q \wedge \sim P$	$P \leftrightarrow \sim Q \wedge \sim P$
T	T				
T	F				
F	T				
F	F				

(i) $(P \vee \sim Q) \to (P \wedge Q)$,

P	Q	$\sim Q$	$P \vee \sim Q$	$P \wedge Q$	$(P \vee \sim Q) \to (P \wedge Q)$
T	T				
T	F				
F	T				
F	F				

(j) $P \wedge Q \vee P \wedge \sim Q \vee \sim P \vee \sim Q$.

P	Q	$P \wedge Q$	$\sim Q$	$P \wedge \sim Q$	$\sim P$	$\sim P \vee \sim Q$	$P \wedge Q \vee P \wedge \sim Q$	$P \wedge Q \vee P \wedge \sim Q \vee \sim P \vee \sim Q$
T	T							
T	F							
F	T							
F	F							

Problem 6. Construct a truth table for
(a) $P \vee Q \to P \vee R$,

P	Q	R	$P \vee Q$	$P \vee R$	$P \vee Q \to P \vee R$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

(b) $P \vee \sim Q \leftrightarrow R$,

P	Q	R	$\sim Q$	$P \vee \sim Q$	$P \vee \sim Q \leftrightarrow R$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

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(c) $P \leftrightarrow ((Q \to P) \wedge R)$,

P	Q	R	$Q \to P$	$(Q \to P) \wedge R$	$P \leftrightarrow ((Q \to P) \wedge R)$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

(d) $P \vee Q \vee R \to \sim Q$,

P	Q	R	$P \vee Q$	$P \vee Q \vee R$	$\sim Q$	$P \vee Q \vee R \to \sim Q$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

(e) $P \leftrightarrow Q \wedge \sim R \leftrightarrow Q$,

P	Q	R	$\sim R$	$Q \wedge \sim R$	$P \leftrightarrow Q \wedge \sim R$	$P \leftrightarrow Q \wedge \sim R \leftrightarrow Q$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

(f) $\sim P \to (P \vee Q \vee \sim R)$,

P	Q	R	$\sim R$	$P \vee Q$	$P \vee Q \vee \sim R$	$\sim P$	$\sim P \to (P \vee Q \vee \sim R)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

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(g) $P \wedge Q \wedge \sim R \vee R$,

P	Q	R	$\sim R$	$P \wedge Q$	$P \wedge Q \wedge \sim R$	$P \wedge Q \wedge \sim R \vee R$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

(h) $(P \vee \sim Q) \rightarrow (Q \wedge \sim R)$,

P	Q	R	$\sim Q$	$P \vee \sim Q$	$\sim R$	$Q \wedge \sim R$	$(P \vee \sim Q) \rightarrow (Q \wedge \sim R)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

(i) $\sim (P \rightarrow \sim Q) \rightarrow (P \rightarrow \sim R)$,

P	Q	R	$\sim Q$	$P \rightarrow \sim Q$	$\sim (P \rightarrow \sim Q)$	$\sim R$	$P \rightarrow \sim R$	$\sim (P \rightarrow \sim Q) \rightarrow (P \rightarrow \sim R)$
T	T	T						
T	T	F						
T	F	T						
T	F	F						
F	T	T						
F	T	F						
F	F	T						
F	F	F						

(j) $(P \leftrightarrow Q \vee \sim R) \vee (\sim Q \rightarrow R)$.

P	Q	R	$\sim R$	$Q \vee \sim R$	$P \leftrightarrow Q \vee \sim R$	$\sim Q$	$\sim Q \rightarrow R$	$(P \leftrightarrow Q \vee \sim R) \vee (\sim Q \rightarrow R)$
T	T	T						
T	T	F						
T	F	T						
T	F	F						
F	T	T						
F	T	F						
F	F	T						
F	F	F						

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A3 Logical Equivalence

A ***tautology*** is a formula which is "always true" — that is, it is true for every assignment of truth values to its simple components. You can think of a tautology as a *rule of logic*. The opposite of a tautology is a ***contradiction***, a formula which is "always false". In other words, a contradiction is false for every assignment of truth values to its simple components.

Example 1. Show that $(P \to Q) \vee (Q \to P)$ is a tautology.

I construct the truth table for $(P \to Q) \vee (Q \to P)$ and show that the formula is always true.

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \vee (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

The last column contains only T's. Therefore, the formula is a tautology.

Two statements X and Y are ***logically equivalent*** if $X \leftrightarrow Y$ is a tautology. Another way to say this is: For each assignment of truth values to the *simple statements* which make up X and Y, the statements X and Y have identical truth values.

From a practical point of view, you can replace a statement in a proof by any logically equivalent statement.

To test whether X and Y are logically equivalent, you could set up a truth table to test whether $X \leftrightarrow Y$ is a tautology — that is, whether $X \leftrightarrow Y$ "has all T's in its column". However, it's easier to set up a table containing X and Y and then check whether the columns for X and for Y are the same.

Example 2. Show that $P \to Q$ and $\sim P \vee Q$ are logically equivalent.

P	Q	$P \to Q$	$\sim P$	$\sim P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the columns for $P \to Q$ and $\sim P \vee Q$ are identical, the two statements are logically equivalent. This tautology is called ***Conditional Disjunction***. You can use this equivalence to replace a conditional by a disjunction.

There are an infinite number of tautologies and logical equivalences; I've listed a few below.

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List of Tautologies

	Symbolic form	Name
1	$P \lor \sim P$	*Law of the excluded middle*
2	$\sim (P \land \sim P)$	*Contradiction*
3	$[(P \to Q) \land P] \to Q$	*Modus ponens*
4	$[(P \to Q) \land \sim Q] \to \sim P$	*Modus tollens*
5	$\sim \sim P \leftrightarrow P$	*Double negation*
6	$[(P \to Q) \land (Q \to R)] \to (P \to R)$	*Law of the syllogism*
7	$(P \land Q) \to P$	*Decomposing a conjunction*
	$(P \land Q) \to Q$	*Decomposing a conjunction*
8	$P \to (P \lor Q)$	*Constructing a disjunction*
	$Q \to (P \lor Q)$	*Constructing a disjunction*
9	$(P \leftrightarrow Q) \leftrightarrow [(P \to Q) \land (Q \to P)]$	*Definition of the biconditional*
10	$(P \land Q) \leftrightarrow (Q \land P)$	*Commutative law for* $\land$
11	$(P \lor Q) \leftrightarrow (Q \lor P)$	*Commutative law for* $\lor$
12	$[(P \land Q) \land R] \leftrightarrow [P \land (Q \land R)]$	*Associative law for* $\land$
13	$[(P \lor Q) \lor R] \leftrightarrow [P \lor (Q \lor R)]$	*Associative law for* $\lor$
14	$\sim (P \lor Q) \leftrightarrow (\sim P \land \sim Q)$	*DeMorgan's law*
15	$\sim (P \land Q) \leftrightarrow (\sim P \lor \sim Q)$	*DeMorgan's law*
16	$[P \land (Q \lor R)] \leftrightarrow [(P \land Q) \lor (P \land R)]$	*Distributivity*
17	$[P \lor (Q \land R)] \leftrightarrow [(P \lor Q) \land (P \lor R)]$	*Distributivity*
18	$(P \to Q) \leftrightarrow (\sim Q \to \sim P)$	*Contrapositive*
19	$(P \to Q) \leftrightarrow (\sim P \lor Q)$	*Conditional disjunction*
20	$[(P \lor Q) \land \sim P] \to Q$	*Disjunctive syllogism*
21	$(P \lor P) \leftrightarrow P$	*Simplification*
22	$(P \land P) \leftrightarrow P$	*Simplification*

When a tautology has the form of a biconditional, the two statements which make up the biconditional are logically equivalent. Hence, you can replace one side with the other without changing the logical meaning.

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Example 3. Write down the negation of the following statements, simplifying so that only simple statements are negated.

(a) $(P \vee \sim Q)$	$\sim(P \vee \sim Q) \leftrightarrow \sim P \wedge \sim\sim Q$	**DeMorgan's law**
	$\leftrightarrow \sim P \wedge Q$	**Double negation**
(b) $(P \wedge Q) \rightarrow R$	$\sim[(P \wedge Q) \rightarrow R] \leftrightarrow \sim\ [\sim(P \wedge Q) \vee R]$	**Conditional Disjunction**
	$\leftrightarrow \sim\sim(P \wedge Q) \wedge \sim R$	**DeMorgan's law**
	$\leftrightarrow (P \wedge Q) \wedge \sim R$	**Double negation**

I showed that $(A \rightarrow B)$ and $(\sim A \vee B)$ are logically equivalent in an earlier example.

Example 4. Use DeMorgan's Law to write the *negation* of the following statement, simplifying so that only simple statements are negated: "*If Phoebe buys a pizza, then Bob buys popcorn.*"

Let P be the statement "*Phoebe buys a pizza*" and let C be the statement "*Bob buys popcorn*". The given statement is $P \rightarrow C$. To simplify the negation, I'll use the ***Conditional Disjunction*** tautology which says

$$(P \rightarrow Q) \leftrightarrow (\sim P \vee Q).$$

That is, I can replace $P \rightarrow Q$ with $\sim P \vee Q$ (or vice versa). Here, then, is the negation and simplification:

$\sim(P \rightarrow C) \leftrightarrow \sim(\sim P \vee C)$	Conditional Disjunction
$\leftrightarrow \sim\sim P \wedge \sim C$	DeMorgan's Law
$\leftrightarrow P \wedge \sim C$	Double negation

The result is "*Phoebe buys the pizza and Bob doesn't buy popcorn*".

Example 5. Show that the inverse and the converse of a conditional are logically equivalent.
Let $P \rightarrow Q$ be the conditional. The inverse is $\sim P \rightarrow \sim Q$. The converse is $Q \rightarrow P$.
I could show that the inverse and converse are equivalent by constructing a truth table for

$$(\sim P \rightarrow \sim Q) \leftrightarrow (Q \rightarrow P).$$

I'll use some known tautologies instead.
Start with $\sim P \rightarrow \sim Q$:

$\sim P \rightarrow \sim Q \leftrightarrow \sim\sim Q \rightarrow \sim\sim P$	**Contrapositive**
$\leftrightarrow Q \rightarrow P$	**Double negation**

Remember that I can replace a statement with one that is logically equivalent. For example, in the last step I replaced $\sim\sim Q$ with Q, because the two statements are equivalent by Double negation.

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Remark. Suppose we know that a statement of the form $P \to Q$ is true. This tells us that whenever P is true, Q will also be true. By itself, $P \to Q$ being true does not tell us that either P or Q is true (they could both be false, or P could be false and Q true). However if in addition we happen to know that P is true then it must be that Q is true. This is called a ***logical inference***: Given two true statements we can infer that a third statement is true. In this instance true statements $P \to Q$ and P are "*added together*" to get Q. This is described below with $P \to Q$ and P stacked one atop the other with a line separating them from Q. The intended meaning is that $P \to Q$ combined with P produces Q.

B3 Problem Set

Problem 1. Answer questions 1 through 7 based on the truth table above (see B2.Problem 4).

1. Which statement is negation?____
2. Which statements are conditionals?__________________________________
3. Which statement is a disjunction?___________________________________
4. Which statement is a biconditional?_________________________________
5. Which statement is a conjunction?__________________________________
6. Which statements are logically equivalent?____________________________
7. Which statement is a tautology?_____________________________________

Problem 2. In each case, determine whether the statement is true or false. Explain your answers.

(a) "If $\sqrt{2} = 1.414$, then John likes bagels with lox and cream cheese."
(b) "If $\sqrt{4} = 2$, then $\sqrt{9} = -3$."
(c) "If $11^2 = 121$, then $11 \cdot 12 > 121$."
(d) "If 3 is a divisor of 12, then $12 \geq 3$."
(e) "If $2 - 5 = -3$, then C++ is a computer game."
(f) "If $5 > 4$, then $5 > 2$."
(g) "If $1 + 1 = 3$, then dogs can fly."
(h) "If Juan has a smartphone, then $2 + 3 = 5$."
(i) "If $5 > -1$, then Wolfgang Mozart is a dancer."
(j) "If π is a real number, then Abuja is the capital of Nigeria."
Note: You should not need a truth table to do this problem.

Problem 3. Use DeMorgan's laws to negate each statement, then write the negation in words.

(a) "Bob buys the stromboli or Sam does not eat the hamburger."
(b) "π is not rational and $(\sin x)^2 + (\cos x)^2 = 53$."
(c) "If Sam has chicken pox, then the class will be dismissed."
(d) "Tomas learns discrete mathematics and Tomas will find a good job."
(e) "2 is a divisor of 15 or 7 is a divisor of 15."
(f) "If π is a rational number, then π is a real number."
(g) "Heather will go to the concert and Steve will go to the concert."
(h) "If Jan is rich, then Jan is happy."
(i) "$2.5 + 1 = 3.5$ and $2.5 + 0.01 = 2.51$."
(j) "$-5 > -10$ or an apple is blue."

In (c), (f) and (h) use conditional disjunction to write the "if-then" statement as an "or" statement first.

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Problem 4. Use DeMorgan's laws to simplify the statement, so that only simple statements (if any) are negated.

(a) $\sim (P \vee \sim (Q \wedge R))$,
(b) $\sim (A \wedge B \wedge C)$,
(c) $\sim (R \vee S) \wedge \sim (R \vee \sim T)$,
(d) $\sim ((P \wedge Q) \vee (Q \wedge R))$,
(e) $\sim (R \wedge S \wedge T \vee \sim R \wedge \sim T)$,
(f) $\sim (P \rightarrow Q) \vee (Q \rightarrow \sim R)$,
(g) $\sim (P \rightarrow Q \rightarrow \sim R)$,
(h) $\sim (P \wedge \sim Q \vee Q \vee R \wedge \sim Q)$,
(i) $(A \wedge \sim B \rightarrow C) \rightarrow \sim A$,
(j) $\sim (P \wedge Q \rightarrow R) \rightarrow (P \rightarrow Q \rightarrow R)$.

In (f), (g), (i) and (j) use conditional disjunction to write the "if-then" statement as an "or" statement first.

Problem 5. Use a truth table to prove:

(a) $P \vee Q$ and $\sim(\sim P \wedge \sim Q)$ are logically equivalent (this is a version of DeMorgan's law),
(b) $P \wedge Q \vee P$ and P are logically equivalent,
(c) $P \rightarrow Q$ and $\sim P \wedge Q$ are not logically equivalent,
(d) $\sim (P \rightarrow Q)$ and $\sim P \vee Q$ are not logically equivalent,
(e) $P \vee Q$ and $(P \wedge \sim Q) \vee (\sim P \wedge Q) \vee (P \wedge Q)$ are logically equivalent,
(f) $P \leftrightarrow Q$ and $(\sim P \vee Q) \wedge (\sim Q \vee P)$ are logically equivalent,
(g) $(P \rightarrow Q) \rightarrow R$ and $P \rightarrow (Q \rightarrow R)$ are not logically equivalent,
(h) $(P \rightarrow Q) \wedge (Q \rightarrow R)$ and $P \rightarrow R$ are not logically equivalent,
(i) $(P \vee Q) \wedge \sim Q \wedge R$ and $P \wedge \sim Q \wedge R$ are logically equivalent,
(j) $(P \vee \sim Q) \wedge (\sim P \vee R)$ and $P \wedge R \vee \sim Q \wedge \sim P \vee \sim Q \wedge R$ are logically equivalent.

Problem 6. Use a (single) truth table to show:

(a) $R \vee \sim R$ is a tautology,
(b) $R \rightarrow \sim R$ is a not tautology,
(c) $(R \vee S) \rightarrow (R \wedge S)$ is a not tautology,
(d) $(R \wedge S) \rightarrow (R \vee S)$ is a tautology,
(e) $\sim (R \vee S) \leftrightarrow (\sim R \wedge \sim S)$ is a tautology,
(f) $R \rightarrow (S \rightarrow R \wedge S)$ is a tautology,
(g) $(\sim R \rightarrow \sim S) \leftrightarrow (S \rightarrow R)$ is a tautology,
(h) $(R \rightarrow S) \rightarrow ((S \rightarrow T) \rightarrow (R \rightarrow T))$ is a tautology,
(i) $\sim (P \wedge Q \wedge R) \rightarrow \sim P \wedge \sim Q \wedge \sim R$ is a not tautology,
(j) $(S \rightarrow T) \rightarrow ((R \vee S) \rightarrow (R \vee T))$ is a tautology.

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A4 Quantifiers

Here is a (true) statement about real numbers:

Every real number is either rational or irrational.

I could try to translate the statement as follows: Let

P = "*x is a real number*"

Q = "*x is rational*"

R = "*x is irrational*"

The statement can be expressed as the implication $P \to (Q \vee R)$.

The simple statements contain a variable x, and you might find it difficult to translate these statements without using a *variable* (or, what is the same thing, a *pronoun*). The reason is that the original statement is meant to apply to every element of a set — in this case, *every* element of the set of real numbers.

You can see that I'm cheating in making my translation: "x is a real number" is not a single statement about a uniquely specified object "x". It is a different *kind* of statement than "π is a real number", which talks about a specific real number π.

I can use quantifiers to translate statements like these so as to capture this meaning. Mathematicians use two ***quantifiers***:

$\forall$, the ***universal quantifier***, which is read "for all", "for every", or "for each".

$\exists$, the ***existential quantifier***, which is read "there is" or "there exists".

Here are some examples which show how they're used.

Example 1. Let $P(x)$ mean "*x likes pizza*". Then:

$\forall x(P(x))$ means "*Everyone likes pizza*".

$\exists x(P(x))$ means "*Someone likes pizza*".

Note that if "*Someone likes pizza*" is true, it may be true that "*Everyone likes pizza*". On the other hand, if "*Everyone likes pizza*" — and assuming that the set of people is nonempty — it must be true that "*Someone likes pizza*".

$\sim \forall x(P(x))$ means "*Not everyone likes pizza*".

$\sim \exists x(P(x))$ means "*No one likes pizza*".

Again, if "*Not everyone likes pizza*", it may be true that "*No one likes pizza*". On the other hand, if "*No one likes pizza*", it must be true that "*Not everyone likes pizza*." Note also that if "*Not everyone likes pizza*," it may be true that "*Someone likes pizza*."

Example 2. Translate the statement "*Every real number is either rational or irrational*" using universal or existential quantifiers. Then determine whether the statement is true or false.

Let $R(x)$ = "*x is rational*" and $I(x)$ = "*x is irrational*". The statement may be translated as

$$\forall x(R(x) \vee I(x)).$$

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(Some people prefer to write the initial x as a subscript: $\forall_x(R(x) \vee I(x))$. Use whichever form you prefer.) Here are the details.

First, when you use a quantifier you do so within the context of some universe of applicable objects. For this statement, the universe would be the set of real numbers. (It would be of little use to let x be a car, or an orange, or the right to freedom of speech.)

How do you know what the universe is for a given quantified statement? Sometimes, it is apparent from the context: In a mathematical discussion, it would probably be clear that the statement above was intended to apply to real numbers. In any case where confusion might arise, you should name the universe for quantification explicitly. In this case, the first part of the statement ("*Every real number*") makes it clear that the universe is the set of real numbers.

Notice that there is no conditional ("→") in the quantified translation.

The statements $R(x)$ and $I(x)$ depend on the variable x. That is, $R(3)$ would mean "3 *is rational*", $R(\sqrt{5})$ would mean "$\sqrt{5}$ *is rational*" and so on. $R(x)$ and $I(x)$ are called ***one-place predicates*** or ***single variable predicates***. Finally, note that while this statement happens to be true, truth value is distinct from how you translate the statement.

If you know a quantified statement is true, you can draw certain conclusions.

<u>Universal Quantifiers.</u> If you know $\forall x P(x)$, then for any element c in the universe, $P(c)$ is true. Thus, if a and b are elements of the universe, $P(a)$ is true and $P(b)$ is also true.

<u>Existential Quantifiers.</u> If you know $\exists x P(x)$, then you can say there is an element c such that $P(c)$. In a proof, you will usually say something like: "Let c satisfy $P(c)$", or "Let c be such that $P(c)$". When you say "Let c ... ", you create an element named c — in this case, satisfying $P(c)$. From then on, c acts like a constant. In particular, you can't assign a value to it arbitrarily.

In addition, the existence statement only guarantees the existence of *at least one thing* satisfying $P(x)$. So having said "*Let c satisfy* $P(c)$", you can't say in addition "*And let d satisfy* $P(d)$", since this creates *another thing d* which satisfies $P(x)$. On the other hand, you can't assume that c is *the only thing* which satisfies $P(x)$. Thus, *there might* be an element d such that $P(d)$ — but you're not justified in saying there is.

Example 3. Let $P(x)$ mean "x likes pepperoni", and let $O(x)$ mean "x likes onions". Translate each statement into logical symbols.

(a) "*Some people don't like pepperoni*." The statement is that there are people who don't like pepperoni. In logical symbols, this is $\exists x\,(\sim P(x))$.

(b) "*Everyone who likes pepperoni likes onions*." The statement means that if someone likes pepperoni, then the person likes onions. In logical symbols, this is

$$\forall x(P(x) \to O(x)).$$

(c) "*Everyone likes pepperoni and onions*." The statement means that everybody likes both pepperoni and onions. In logical symbols, this is $\forall x(P(x) \wedge O(x))$.

Do you understand the difference between the statements in (b) and (c)? In (b), you know that *if there is* a person who likes pepperoni, *then* the person likes onions. But there might not be anyone who likes pepperoni! In (c), everyone likes both pepperoni and onions, so in particular, there are certainly many people who like pepperoni.

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Remark. What is the negation of "*Everyone likes pizza*"?
Let $P(x)$ mean "*x likes pizza*". The statement may be written in quantified form as $\forall x(P(x))$. The negation is (literally) $\sim \forall x(P(x))$: "*It is not the case that everyone likes pizza*." What does this *mean*?
A common mistake is to think that this means "*No one likes pizza*". However, ask yourself what it would take to show that the original statement was false. If I knew, for instance, that Bob doesn't like pizza, that's enough to prove that "*Everyone likes pizza*" is false.
In other words, for "*Everyone likes pizza*" to be false — or equivalently, for "*It is not the case that everyone likes pizza*" to be true — it's enough if I find someone who doesn't like pizza. So the negation actually means "*There exists someone who doesn't like pizza*" — in symbols, $\exists x(\sim P(x))$.

Let $Q(x)$ mean "*x likes lasagne*". What is the negation of "*Someone likes lasagne*"?
In symbols, "*Someone likes lasagne*" becomes $\exists x(Q(x))$. The negation is $\sim \exists x(Q(x))$. In words, this is: "*It is not the case that someone likes lasagne*". This is the same as "*No one likes lasagne*", which is $\forall x(\sim Q(x))$.
To summarize:

$$[\sim \forall x(P(x))] \leftrightarrow [\exists x(\sim P(x))]$$
$$[\sim \exists x(P(x))] \leftrightarrow [\forall x(\sim P(x))]$$

In other words, to negate a quantified statement, change the quantifier to the "*other*" quantifier — $\forall$ to $\exists$ and $\exists$ to $\forall$ — and negate the "*stuff inside*".

B4 Problem Set

Problem 1. (a) Show by specific counterexample that the following universal statement about real numbers is false:

(i) $\forall\, x \in \mathbf{R}\ [(x-3)^2 > 0]$,
(ii) $\forall\, x \in \mathbf{R}\ (4x > 4)$,
(iii) $\forall\, x \in \mathbf{R}\ (\sin x < 1)$,
(iv) $\forall\, n \in \mathbf{N}\ [n(n+1) = n^2 + 1]$,
(v) $\forall\, k \in \mathbf{Z}\ (2.5k = 25)$.

(b) Show by specific example that the following existence statement about real numbers is true:

(i) $\exists\, x \in \mathbf{R}\ (x^2 = -x^2)$,
(ii) $\exists\, x \in \mathbf{R}\ (x^2 + 1 > 26)$,
(iii) $\exists\, x \in \mathbf{R}\ (0.1x > 2)$,
(iv) $\exists\, m \in \mathbf{N}\ (m < 3)$,
(v) $\exists\, m \in \mathbf{N}$ ($m + 1$ is a prime number).

Problem 2. Suppose

$C(x)$ = "x likes chicken",
$H(x)$ = "x likes ham",
$D(x, y)$ means "x has dinner with y",
$I(x)$ = "x has an Internet connection",
$T(x,y)$ = "x and y have chatted over the Internet".

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(a) Translate the following logical statements into words:

(i) $\forall x\, (H(x) \wedge C(x))$,

(ii) $\forall x\, (H(x) \rightarrow C(x))$,

(iii) $\exists x\, (T(x, \text{Daria}))$,

(iv) $\exists x\, (T(x, \text{Nikita}) \leftrightarrow T(x, \text{Anna}))$,

(v) $\forall x\, (\exists y\, (T(x,y) \rightarrow I(x)))$.

(b) Translate the following statements into logical symbols:

(i) "Some people don't like either chicken or ham."

(ii) "Someone has dinner with everyone who likes chicken."

(iii) "Not everyone has an Internet connection."

(iv) "Someone does not have an Internet connection."

(v) "Someone has an Internet connection but has not chatted with anyone."

Problem 3. Negate each quantified statement, simplifying so that only the simple statements are negated. Show each step of your work.

(a) $\forall x\, (\sim P(x) \wedge Q(x))$,

(b) $\exists x\, (Q(x) \rightarrow \sim P(x))$,

(c) $\exists x\, (\sim Q(x) \vee P(x))$,

(d) $\exists x\, (\sim Q(x) \rightarrow \sim P(x))$,

(e) $\forall x\, (\sim (Q(x) \rightarrow P(x)))$,

(f) $\exists x\, (P(x) \wedge \sim Q(x))$,

(g) $\forall x\, (\sim Q(x) \vee P(x))$,

(h) $\forall x\, (\sim P(x) \rightarrow Q(x))$,

(i) $\exists x\, (\sim (P(x) \vee \sim Q(x)))$,

(j) $\forall x\, (\sim P(x) \wedge \sim Q(x))$.

Problem 4. In each case, negate the given statement, simplify so that only simple statements are negated, and write the answer in words. Show your work!

(a) "Everyone likes hot dogs and onion rings."

(b) "There's someone who likes hot dogs but doesn't like onion rings."

(c) "There is someone who can speak Russian and who knows C++."

(d) "There is someone who can speak Russian but who doesn't know C++."

(e) "Everyone either can speak Russian or knows C++."

(f) "No one can speak Russian or knows C++."

(g) "For all integers n, $n > 0$ and $5n < 1$."

(h) "There exists an integer k such that $k > 3$ and $k^2 < 2$."

(i) "For each natural number k, if $k < 3$ then k is a prime number".

(j) "For all natural numbers n, if n is divisible by 2 then n^2 is divisible by 4."

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II Sets

A5 Introduction to Sets

All of mathematics can be described with sets. This becomes more and more apparent the deeper into mathematics you go. It will be apparent in most of your upper level courses, and certainly in this course. The theory of sets is a language that is perfectly suited to describing and explaining all types of mathematical structures.

A ***set*** is a collection of things. The things in the collection are called ***elements*** of the set. We are mainly concerned with sets whose elements are mathematical entities, such as numbers, points, functions, etc. A set is often expressed by listing its elements between commas, enclosed by braces.

Example 1. The collection $\{2, 4, 6, 8\}$ is a set which has four elements, the numbers 2, 4, 6 and 8. Some sets have infinitely many elements. For example, consider the collection of all integers, $\{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$.

Here the dots indicate a pattern of numbers that continues forever in both the positive and negative directions. A set is called an ***infinite*** set if it has infinitely many elements; otherwise it is called a ***finite*** set.

Two sets are ***equal*** if they contain exactly the same elements.

Example 2. $\{2, 4, 6, 8\} = \{4, 2, 8, 6\}$ because even though they are listed in a different order, the elements are identical; but $\{2, 4, 6, 8\} \neq \{2, 4, 6, 7\}$. Also

$$\{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\} = \{0, -1, 1, -2, 2, -3, 3, -4, 4, ...\}.$$

We often let uppercase letters stand for sets. In discussing the set $\{2, 4, 6, 8\}$ we might declare $A = \{2, 4, 6, 8\}$ and then use A to stand for $\{2, 4, 6, 8\}$. To express that 2 is an element of the set A, we write $2 \in A$, and read this as "2 *is an element of A*," or "2 *is in A*," or just "2 *in A*." We also have $4 \in A$, $6 \in A$ and $8 \in A$, but $5 \notin A$. We read this last expression as "5 *is not an element of A*," or "5 *not in A*." Expressions like $6, 2 \in A$ or $2, 4, 8 \in A$ are used to indicate that several things are in a set.

Here is some notation for some sets that occur often in mathematics.

N is the set of *natural numbers.*

Z is the set of *integers*.

Q is the set of *rational numbers*.

R is the set of *real numbers*.

C is the set of *complex numbers*.

Sets need not have just numbers as elements. The set $B = \{\mathrm{T}, \mathrm{F}\}$ consists of two letters, perhaps representing the values "*true*" and "*false*." The set $C = \{a, e, i, o, u\}$ consists of the lowercase vowels in the English alphabet.

If X is a finite set, its ***cardinality*** or ***size*** is the number of elements it has, and this number is denoted as $|X|$. Thus for the sets above, $|A| = 4$, $|B| = 2$ and $|C| = 5$.

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There is a special set that, although small, plays a big role. The ***empty set*** is the set {} that has no elements. We denote it as $\varnothing$ so $\varnothing = \{\}$. Whenever you see the symbol $\varnothing$ it stands for {}. Observe that $|\varnothing| = 0$. The empty set is the only set whose cardinality is zero.

A special notation called ***set-builder notation*** is used to describe sets that are too big or complex to list between braces.

Example 3. Consider the infinite set of even integers $E = \{..., -6, -4, -2, 0, 2, 4, 6, ...\}$.

In set-builder notation this set is written as $E = \{2n : n \in \mathbf{Z}\}$. We read the first brace as "*the set of all things of form*," and the colon as "*such that*." So the expression $E = \{2n : n \in \mathbf{Z}\}$ is read as "*E equals the set of all things of form* 2*n*, *such that n is an element of* **Z**." The idea is that E consists of all possible values of $2n$, where n takes on all values in **Z**.

In general, a set X written with set-builder notation has the syntax $X = \{\text{expression} : \text{rule}\}$, where the elements of X are understood to be all values of "*expression*" that are specified by "*rule*."

Remark 1. You have to be a little bit careful in constructing sets in order to avoid set-theoretic paradoxes. The philosopher and mathematician Bertrand Russell (1872–1970) did groundbreaking work on the theory of sets and the foundations of mathematics. He was probably among the first to understand how the misuse of sets can lead to bizarre and paradoxical situations. He is famous for an idea that has come to be known as *Russell's paradox*.

Consider "*the set S of all sets which are not members of themselves*". Is S an element of S? If S is an element of S, then by definition S is not a member of itself — contradicting my assumption that S is an element of S. If S is not an element of S, then S is an element of S, since S consists of sets which are not members of themselves. This is also a contradiction. By the Law of the Excluded Middle, either S is an element of S, or it is not — but both alternatives lead to contradictions. Paradoxes of this kind are avoided by setting up axioms for set theory which do not allow the construction of sets such as this one.

Given two sets A and B, it is possible to "*multiply*" them to produce a new set denoted as $A \times B$. This operation is called the *Cartesian product*. To understand it, we must first understand the idea of an ordered pair. An ***ordered pair*** is a list (x, y) of two things x and y, enclosed in parentheses and separated by a comma.

Example 4. For example, $(2, 4)$ is an ordered pair, as is $(4, 2)$. These ordered pairs are different because even though they have the same things in them, the order is different. We write $(2, 4) \neq (4, 2)$.

The ***Cartesian product*** of two sets A and B is another set, denoted as $A \times B$ and defined as $A \times B = \{(a, b) : a \in A, b \in B\}$. Thus $A \times B$ is a set of ordered pairs of elements from A and B.

Example 5. If $A = \{k, l, m\}$ and $B = \{q, r\}$, then

$$A \times B = \{(k, q), (k, r), (l, q), (l, r), (m, q), (m, r)\}.$$

For another example, $\{0, 1\}\times\{2, 1\} = \{(0, 2), (0, 1), (1, 2), (1, 1)\}$.

Remark 2. If A and B are finite sets, then $|A \times B| = |A|\cdot|B|$.

We can also define Cartesian products of three or more sets by moving beyond ordered pairs. An ordered triple is a list (x, y, z). The Cartesian product of the three sets **R**, **N** and **Z** is

$$\mathbf{R} \times \mathbf{N} \times \mathbf{Z} = \{(x, y, z) : x \in \mathbf{R}, y \in \mathbf{N}, z \in \mathbf{Z}\}.$$

Of course there is no reason to stop with ordered triples. In general,

$$A_1 \times A_2 \times \ldots \times A_n = \{(x_1, x_2, ..., x_n) : x_i \in A_i \text{ for each } i = 1, 2, ..., n\}.$$

We can also take ***Cartesian powers*** of sets. For any set A and positive integer n, the power A^n is the Cartesian product of A with itself n times:

$$A^n = A \times A \times \ldots \times A = \{(x_1, x_2, ..., x_n) : x_1, x_2, ..., x_n \in A\}.$$

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B5 Problem Set

Problem 1. Write each of the following sets by listing their elements between braces. What is the cardinality of each of these sets?

(a) $\{x \in \mathbf{N} : -2 \leq x < 7\}$,
(b) $\{x \in \mathbf{Z} : -2 \leq x < 7\}$,
(c) $\{x \in \mathbf{R} : x^2 = 4\}$,
(d) $\{x \in \mathbf{R} : (x-3)(2x-5)(x+11) = 0\}$,
(e) $\{x : x$ is the square of an integer and $x < 100\}$,
(f) $\{x : x$ is a positive integer less than $12\}$,
(g) $\{5x - 1 : x \in \mathbf{N}, x < 5\}$,
(h) $\{x^2 : x \in \mathbf{Z}, |x| \leq 6\}$,
(i) $\{x : x \in \mathbf{Z}, x^2 = 2\}$,
(j) $\{x : x \in \mathbf{R}, x^2 + 5 = 0\}$.

Problem 2. Write each of the following sets in set-builder notation:

(a) $\{2, 4, 8, 16, 32, 64, \ldots\}$,
(b) $\{0.1, 0.01, 0.001, 0.0001, \ldots\}$,
(c) $\{\ldots, -6, -3, 0, 3, 6, 9, 12, \ldots\}$,
(d) $\{0, 1, 4, 9, 16, 25, 36, \ldots\}$,
(e) $\{1, 3, 5, 7, 9, \ldots\}$,
(f) $\{0, 4, 16, 36, 64, 100, \ldots\}$,
(g) $\{8, 9, 10, 11, \ldots, 35, 36\}$,
(h) $\{3, 4, 5, 6, 7, 8\}$,
(i) $\{-4, -3, -2, -1, 0, 1, 2\}$,
(j) $\{\ldots, 0.125, 0.25, 0.5, 1, 2, 4, 8, 16, \ldots\}$.

Problem 3. Write out the indicated sets by listing their elements between braces.

(a) $\{1, 2, 3, 4\} \times \{a, c\}$,
(b) $\{a, c\} \times \{1, 2, 3, 4\}$,
(c) $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$,
(d) $\{1, 2, 3, 4\} \times \varnothing$,
(e) $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\} \times \{a, c\}$,
(f) $\{a, c\}^3$,
(g) $\{x \in \mathbf{R} : x^2 = 4\} \times \{1, 5\}$,
(h) $\{x \in \mathbf{Q} : x^2 = 4\} \times \{1, 5\}$,
(i) $\{x \in \mathbf{Z} : -3 < x < 3\} \times \{x^2 : x \in \mathbf{Z}, |x| = 6\}$,
(j) $\{x \in \mathbf{Z} : -2 \leq x \leq 2\} \times \{x^2 : x \in \mathbf{Z}, |x| \leq 2\}$.

Problem 4. Sketch these Cartesian products on the x-y plane $\mathbf{R}^2$ (or $\mathbf{R}^3$ for the last two).

(a) $\{1, 2, 3\} \times \{-1, 0, 1\}$,
(b) $\{-1, 0, 1\} \times \{1, 2, 3\}$,
(c) $[0, 1] \times [0, 1]$,
(d) $[-1, 1] \times [1, 2]$,
(e) $\{1, 1.5, 2\} \times [1, 2]$,
(f) $[1, 2] \times \{1, 1.5, 2\}$,
(g) $\{1\} \times [-3, 3]$,
(h) $\mathbf{Z} \times \{2\}$,
(i) $[0, 1] \times [0, 1] \times [0, 1]$,
(j) $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\} \times [0, 1]$.

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A6 Subsets

It can happen that every element of some set A is also an element of another set B. For example, each element of $A = \{0, 2, 4\}$ is also an element of $B = \{0, 1, 2, 3, 4\}$. When A and B are related this way we say that A is a *subset of* B.

Definition 1. Suppose A and B are sets. If every element of A is also an element of B, then we say A is a ***subset*** of B, and we denote this as $A \subseteq B$. We write $A \not\subset B$ if A is *not* a subset of B, that is, if it is not true that every element of A is also an element of B. Thus $A \not\subset B$ means that there is at least one element of A that is *not* an element of B.

Remark 1. Some people use "$T \subset S$" to mean that T is a subset of S, but is not equal to S. A subset of S other than S itself is called a ***proper subset*** of S. With this convention, you write $T \subseteq S$ mean that T is a subset of S, possibly S itself.

Example 1. Be sure you understand why each of the following is true.

1. $\{2, 3, 7\} \subseteq \{2, 3, 4, 5, 6, 7\}$.
2. $\{2, 3, 7\} \not\subset \{2, 4, 5, 6, 7\}$.
3. $\{2, 3, 7\} \subseteq \{2, 3, 7\}$.
4. $\{2n : n \in \mathbf{Z}\} \subseteq \mathbf{Z}$.

This brings us to a significant fact: If B is any set whatsoever, then $\varnothing \subseteq B$. To see why this is true, look at the last sentence of Definition 1. It says that $\varnothing \not\subset B$ would mean that there is at least one element of $\varnothing$ that is not an element of B. But this cannot be so because $\varnothing$ contains no elements! Thus it is not the case that $\varnothing \not\subset B$, so it must be that $\varnothing \subseteq B$.

Remark 2. *The empty set is a subset of every set, that is, $\varnothing \subseteq B$ for any set B.* Here is another way to look at it. Imagine a subset of B as a thing you make by starting with braces $\{\}$, then filling them with selections from B. For example, to make one particular subset of $B = \{a, b, c\}$, start with $\{\}$, select b and c from B and insert them into $\{\}$ to form the subset $\{b, c\}$. Alternatively, you could have chosen just a to make $\{a\}$, and so on. But one option is to simply select nothing from B. This leaves you with the subset $\{\}$. Thus $\{\} \subseteq B$. More often we write it as $\varnothing \subseteq B$.

Remark 3. If a finite set has n elements, then it has 2^n subsets.

Given a set, you can form a new set with the *power set* operation, defined as follows.
Definition 2. If A is a set, the ***power set*** of A is another set, denoted as $\mathcal{P}(A)$ and defined to be the set of all subsets of A. In symbols, $\mathcal{P}(A) =\{X : X \subseteq A\}$.

Example 2. Suppose A = $\{1, 2, 3\}$. The power set of A is the set of all subsets of A. We learned how to find these subsets in the previous section, and they are $\{\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$. Therefore the power set of A is

$$\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

As we saw the previous remark, if a finite set A has n elements, then it has 2^n subsets, and thus its power set has 2^n elements.

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Remark 4. I If A is a finite set, then $|\mathcal{P}(A)| = 2^{|A|}$.

Example 3. You should examine the following statements and make sure you understand how the answers were obtained. In particular, notice that in each instance the equation $|\mathcal{P}(A)| = 2^{|A|}$ is true.

1. $\mathcal{P}(\{0, 1, 3\}) = \{\varnothing, \{0\}, \{1\}, \{3\}, \{0, 1\}, \{0, 3\}, \{1, 3\}, \{0, 1, 3\}\}$.
2. $\mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.
3. $\mathcal{P}(\{1\}) = \{\varnothing, \{1\}\}$.
4. $\mathcal{P}(\varnothing) = \{\varnothing\}$.

If A is finite, it is possible (though maybe not practical) to list out $\mathcal{P}(A)$ between braces as was done in the above example. That is not possible if A is infinite.

Example 4. For example, consider $\mathcal{P}(\mathbf{N})$. If you start listing its elements you quickly discover that $\mathbf{N}$ has infinitely many subsets, and it's not clear how (or if) they could be arranged as a list with a definite pattern:

$$\mathcal{P}(\mathbf{N}) = \{\varnothing, \{1\}, \{2\}, \ldots, \{1, 2\}, \{1, 3\}, \ldots., \{39, 47\}, \ldots, \{3, 87, 131\}, \ldots, ?, \ldots\}.$$

B6 Problem Set

Problem 1. Decide if the following statements are true or false. Explain.

(a) $\{2, 3, 5\} \subseteq \{2, 3, 5, 8, 10\}$,
(b) $\{-5, 0, 10, 25, 35\} \subseteq \mathbf{Z}$,
(c) $\{2, 4, 8, 16\} \subseteq \{2^n : n \in \mathbf{N}\}$,
(d) $\mathbf{Z} \subseteq \mathbf{N}$,
(e) $\mathbf{Q} \subseteq \mathbf{R}$,
(f) $\{x \in \mathbf{R} : 2x^2 + 1 = 0\} \subseteq \{1, 2, 5\}$,
(g) $\{-5, 5\} \subseteq \{x \in \mathbf{R} : x^2 = 25\}$,
(h) $\{(x, y) : x^2 - 1 = 0\} \subseteq \{(x, y) : x^2 - x = 0\}$,
(i) $[0, 1] \times [0, 1] \times [1, 2] \subseteq \mathbf{R}^3$,
(j) $\mathbf{R}^2 \subseteq \mathbf{R}^3$.

Problem 2. Find the indicated sets.

(a) $\mathcal{P}(\{1, 5\})$,
(b) $\mathcal{P}(\{x, y, z\})$,
(c) $\mathcal{P}(\{3, 9, 27, 81\})$,
(d) $\mathcal{P}(\{5m - 8 : m \in \mathbf{N}, 3 < m < 7\})$,
(e) $\mathcal{P}(\{4^x : x \in \mathbf{R}, (2x - 1)(x - 2)(x + 2) = 0\})$,
(f) $\mathcal{P}(\{x \in \mathbf{R} : x^2 = 5\})$,
(g) $\mathcal{P}(\{x \in \mathbf{N} : x^2 = 5\})$,
(h) $\mathcal{P}(\{a, b\} \times \{1, 2\})$,
(i) $\mathcal{P}(\{1, 2\} \times \{3\})$,
(j) $\mathcal{P}(\{1, 5, 9\} \times \varnothing)$.

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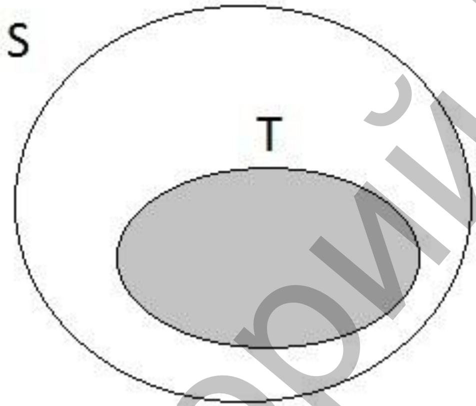
A7 Venn Diagrams

In thinking about sets, it is sometimes helpful to draw informal, schematic diagrams of them. In doing this we often represent a set with a circle (or oval), which we regard as enclosing all the elements of the set. Such diagrams can illustrate how sets combine using various operations. Such graphical representations of sets are called ***Venn diagrams***, after their inventor, British logician John Venn, 1834–1923. Though you are unlikely to draw Venn diagrams as a part of a proof of any theorem, you will probably find them to be useful "scratch work" devices that help you to understand how sets combine, and to develop strategies for proving certain theorems or solving certain problems.
This picture illustrates $T \subset S$:



When one is discussing sets, there is usually a "big set" which contains all the sets under discussion. This "big set" is usually called the ***universe***; usually, it will be clear from the context what it is. For example, if I'm discussing sets of integers, the universe is the set of integers **Z**. If I'm discussing sets of real numbers, the universe is the set of real numbers **R**.

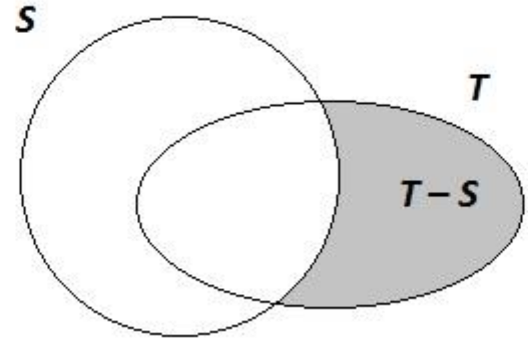
Definition 1. Let S and T be sets in the universe X. The ***complement*** of S is the set of elements of the universe which are not contained in S; it is denoted $\overline{S}$ or $X - S$. Thus,

$$\overline{S} = \{x \in X : x \notin S\}.$$

The complement of S in T is the set of elements of T which are not elements of S; it is denoted $T - S$. Thus,

$$T - S = \{x \in T : x \notin S\}.$$

Here's a picture; the shaded area is $T - S$.



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Set complements function likes logical negations; the analogy even extends to DeMorgan's Laws, as I'll show below.
Example 1. If X is the universe, $\overline{X} = \varnothing$ and $\overline{\varnothing} = X$.

Example 2. Let the universe be {1, 2, 3, 4, apple}. Then

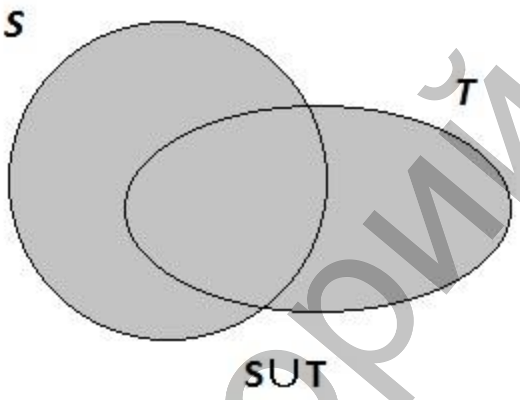
$$\overline{\{1,4\}} = \{2,3,\text{apple}\} \quad \text{and} \quad \overline{\{2,3,\text{apple}\}} = \{1,4\}.$$

In fact, for any set S, $\overline{\overline{S}} = S$. In this situation, $\overline{\{1,4\}} = \{2,3,\text{apple}\}$, but $\overline{\{1,4\}} \not\subset \{2,3,\text{apple}\}$.

Definition 2. Let S and T be sets. The ***union*** of S and T is the set whose elements are elements of S or elements of T; it is denoted $S \bigcup T$. That is,

$$S \bigcup T = \{x : x \in S \text{ or } x \in T\}.$$

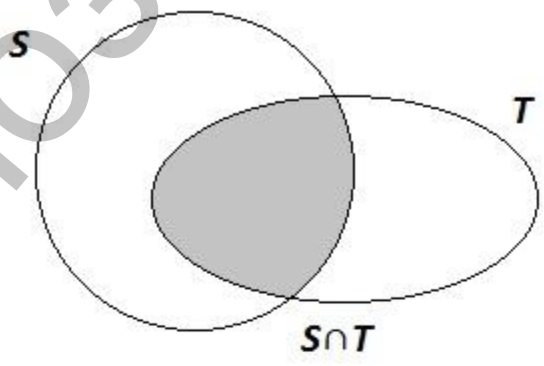
Picture:



Definition 3. Let S and T be sets. The ***intersection*** of S and T is the set whose elements are elements of *both* S and T; it is denoted $S \bigcap T$. That is,

$$S \bigcap T = \{x : x \in S \text{ and } x \in T\}.$$

Picture:



Note that the "or" and "and" in the last two definitions are the logical "or" and "and".

Example 3. Let S = {1, 2, 3, 4, 5}, let T be the set of even integers, and suppose the universe is **Z**. Then

$$S \bigcup T = \{\ldots,-8,-6,-4,-2,\ 0, 1, 2, 3, 4, 5, 6, 8, 10, \ldots\}. \qquad S \bigcap T = \{2, 4\}.$$

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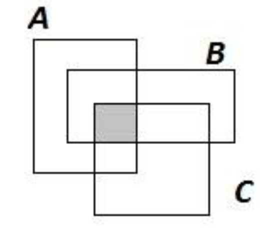
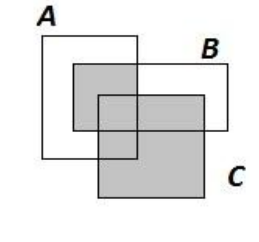
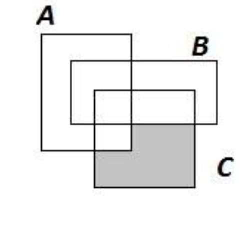
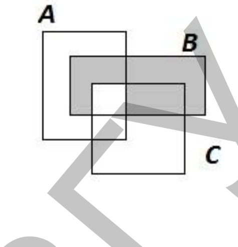
$S - T = \{1, 3, 5\}$. $\quad T - S = \{\text{all even integers except 2 and 4}\}$. $\quad \bar{T} = \{\text{all odd integers}\}$.

Remark 1. Here are the Venn diagrams for some set constructed using complements, unions, and intersection:

			
$A \cap B \cap C$	$(A \cap B) \cup C$	$C - (A \cup B)$	$B - (A \cap C)$

Remark 2. Important points:

1) If an expression involving sets uses only $\cup$, then parentheses are optional.
2) If an expression involving sets uses only $\cap$, then parentheses are optional.
3) If an expression uses both $\cup$ and $\cap$, then parentheses are ***essential***.

Students in their first advanced mathematics classes are often surprised by the extensive role that sets play and by the fact that most of the proofs they encounter are proofs about sets. Perhaps you've already seen such proofs in your linear algebra course, where a vector space was defined to be a set of objects (called vectors) that obey certain properties. Your text proved many things about vector spaces, such as the fact that the intersection of two vector spaces is also a vector space, and the proofs used ideas from set theory. As you go deeper into mathematics, you will encounter more and more ideas, theorems and proofs that involve sets.

There are a lot of rules involving sets; you'll probably become familiar with the most important ones simply by using them a lot. Usually you can check informally (for instance, by using a Venn diagram) whether a rule is correct; if necessary, you should be able to write a proof. In most cases, you can give a proof by going back to the definitions of set constructions in terms of elements.

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B7 Problem Set

Problem 1. Decide if the following statements are true or false. Explain.

(a) $5 \in \{1, 5, 10\} \cap \{2, 4, 11\}$,
(b) $5 \in \{1, 5, 10\} \cup \{2, 4, 11\}$,
(c) $4 \in \{2, 4, 6, 8\} - \{1, 2, 8\}$,
(d) $\sqrt{2} \in \mathbf{R} - \mathbf{Q}$,
(e) $3 \notin \mathbf{Z} - \mathbf{N}$,
(f) $2.5 \in \mathbf{R} - \mathbf{Q}$,
(g) $5 \in \{x \in \mathbf{R} : x > 4\} \cap \{x \in \mathbf{R} : x + 2 = 10\}$,
(h) $-2 \in \{x \in \mathbf{R} : x > 4\} \cup \{x \in \mathbf{R} : x + 2 \geq 0\}$,
(i) $(1, 1) \in (\{1, 5\} \cup \{2\}) \times \mathbf{N}$,
(j) $\mathbf{N} = \mathbf{Z} \cap \mathbf{N}$.

Problem 2. Suppose
(i) $A = \{4, 3, 6, 7, 1, 9\}$, $B = \{5, 6, 8, 4\}$, $C = \{5, 8, 4\}$;
(ii) $A = \{0, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7\}$, $C = \{2, 8, 4\}$.

Find:
(a) $A \cup B$, (b) $A \cap B$, (c) $A - B$, (d) $A - C$, (e) $B - A$,
(f) $A \cap C$, (g) $B \cap C$, (h) $B \cup C$, (i) $C - A$, (j) $C - B$.

Problem 3. Suppose
(i) $A = \{4, 3, 6, 7, 1, 9\}$, $B = \{5, 6, 8, 4\}$, the universe $U = \{0, 1, 2, \dots, 10\}$;
(ii) $A = \{0, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7\}$, the universe $U = \{0, 1, 2, \dots, 8\}$.

Find:
(a) $\overline{A}$, (b) $\overline{B}$, (c) $A \cap \overline{A}$,
(d) $A \cup \overline{A}$, (e) $A - \overline{A}$, (f) $A - \overline{B}$,
(g) $\overline{A} - \overline{B}$, (h) $\overline{A \cap B}$, (i) $\overline{A} \cup \overline{B}$,
(j) $\overline{A} \times B$.

Problem 4. Draw a Venn diagram for:
(a) $\overline{A}$, (b) $B - A$, (c) $\overline{\overline{A} \cap \overline{B}}$,
(d) $(A - B) \cap C$, (e) $(A \cup B) - C$, (f) $\overline{A} \cap B - C$,
(g) $\overline{A - B - C}$, (h) $\overline{A - B} - C$, (i) $(A \cup B) \cap (A \cup C)$,
(j) $A \cup (B \cap C)$.

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Glossary

biconditional statement — A biconditional statement is defined to be true whenever both parts have the same truth value. The biconditional operator is denoted by a double-headed arrow $\leftrightarrow$.

cardinality of a set — The cardinality of a set is the number of elements in the set. The cardinality of S is denoted by $|S|$

complement of a set — Given set A, the complement of A is the set of all elements in the universal set that are not in A, denoted by $\overline{A}$.

conditional statement — A conditional statement, symbolized by $p \rightarrow q$, is an if-then statement in which p is a hypothesis and q is a conclusion.

disjunction — A disjunction is a compound statement formed by joining two statements with the connector "or". The disjunction of p and q is denoted by $p \vee q$.

empty set — The empty set (or null set) is the set which contains no objects and is denoted by $\varnothing$.

finite set — A finite set has a finite (bounded) number of elements.

infinite set — An infinite set is a set with an infinite (countless) number of elements.

intersection — The intersection of two sets is the set of all the elements they have in common. The intersection of S and T is denoted by $S \cap T$.

negation — The negation of statement p is "not p", denoted by $\sim p$.

proper subset — If the set A is a subset of B, but is not equal to B, then A is said to be a proper subset of B and it is denoted by $A \subset B$.

set — A set is a collection of objects that have something in common or follow a rule.

set equality — For any two sets, if A is a subset of B and B is a subset of A, then $A = B$. Thus A and B are equivalent.

set-bulder notation — Set-builder notation is a shorthand used to write sets, often sets with an infinite number of elements.

statement — A statement is a sentence that is either true or false, but not both simultaneously.

subset — The set A is a subset of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

union — The union of two sets S and T, denoted by $S \cup T$, is the set of elements which are in S or in T or in both.

universe — A universe is the set of all elements under consideration, denoted by capital U.

Venn diagram — In a Venn diagram sets are represented by shapes; usually circles or ovals. The elements of a set are labelled within the circle. Venn diagrams are especially useful for showing relationships between sets.

truth table — A truth table helps us find all possible truth values of a statement. Each statement is either true (T) or false (F), but not both.

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