Инъекторы локальных классов Фиттинга

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Подгруппу V группы G называют F-инъектором группы G, если V \cap N является F-максимальной подгруппой в N для всякой субнормальной подгруппы N из G. В настоящей работе найдена формула F-инъектора π -разрешимой группы для произвольного локального класса Фиттинга F. В частности, получен метод построения инъектора разрешимой группы для локального класса Фиттинга.

Ключевые слова: инъекторы групп, локальный класс Фиттинга, Н-функция.

Injectors of local Fitting classes

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Summary. A «V » subgroup of a «G » group is called an F -injector of «G » if $V \cap N$ is an F -maximal subgroup of «N »« for every subnormal «N » subgroup of «G ». In this paper a formula of F -injector of a π -soluble group is found for voluntary local Fitting « F » class. This yields a new method of definition of soluble groups for finite injectors for local Fitting classes.

1. Introduction. In the theory of classes of finite soluble groups the basic result which generalizes fundamental theorems of Sylow and Hall is the theorem of Gaschütz [1] on existence and conjugacy of F -projectors in groups *G* for every soluble local formation F. Recall that a subgroup *F* is called [2] an F -projector of *G* if FN/N is an F -maximal subgroup in G/N for all normal subgroups *N* of *G*.

In the sequel many researchers were interested in finding a formula for an F-projector. This task was solved by Doerk [3], D'Arcy [4] and N.T. Vorob'ev [5].

In 1967 an importat bright result in the theory of finite soluble groups was established by Gaschütz, Fischer and Hartley in their paper [6], where another generalization of the theorems of Sylow and Hall was presented. It was proved [6] that in every group $G \in S$ a unique conjugacy class F-injectors exist for all soluble Fitting classes. Recall that a subgroup V of a group G is called [2] an F-injector of G if $V \cap N$ is an F-maximal subgroup of N for every subnormal subgroup N of G.

In 1975 L.A. Shemetkov [7] extended the Gaschütz-Fischer-Hartley's result on the case of π -soluble groups for all Fitting classes F where $\pi = \pi(F) = \bigcup \{\pi(G) \mid G \in F\}$. In this connection in the

theory of Fitting classes the task of finding an formula for an F-injector of a finite π -soluble group for an arbitrary local Fitting class F is not completed. In the class S it was solved by Hartley [8], N.T. Vorob'ev and I.V. Dudkin [9] only for some special cases of a local class F.

The main result of this paper is a full solution of the task specified above in the class of $\pi(F)$ -soluble groups for a local Fitting class F. We refer to [2] for the notation and basic results on classes of groups.

2. Preliminaries. All groups considered in this paper are finite.

For the first time a local method of studying of finite soluble groups by means of radicals and Fitting classes was introduced by Hartley [8]. Every map $f: P \rightarrow \{ Fitting \ classes \}$ is called a Hartley function or an *H*-function [10] where *P* designate the set of primes. Let $LR(f) = \mathbb{E}_{\pi} \cap (\bigcap_{p \in \pi} f(p) \mathbb{N}_p \mathbb{E}_{p'})$, where $\pi = \pi(F)$. A Fitting class F is called local [10], if F = LR(f) for some *H*-function *f*.

A Hartley function f of a Fitting class F is called full integrated, if $f(p)N_p = f(p)$ and $f(p) \subseteq F$ for all prime numbers p. Such a Hartley function exist for every local Fitting class [11]. Let π be a set of primes. A finite group *G* is said to be π -soluble if every chief factor of *G* is either an abelian π -group or a π '-group.

A set Σ of pairwise permutable Sylow *p*-subgroups of *G*, exactly one for each $p \in \pi(G)$, together with the identity subgroup, is called a Sylow basis of *G*.

If Σ is a set of subgroups of a group *G* and if $R \le G$, we denote by $\Sigma \cap R$ the following set of subgroups of G $\Sigma \cap R = \{S \cap R \mid S \in \Sigma\}$.

In general, $\Sigma \cap R$ is not a Sylow basis of R. However, when it happens that $\Sigma \cap R$ is a Sylow basis of R, we say that Σ reduces into R and write $\Sigma \ W R$. If Σ is a Sylow basis of G, then Σ reduces into R^g for some $g \in G$. In particular, if $R \triangleleft G$, then $\Sigma \ W R$. If Σ is a Sylow basis of a group G, the subgroup $N_G(\Sigma) = \{g \in G | H = H^g \text{ for all } H \in \Sigma\}$ is called the normalizer of Σ . A system normalizer of G is a subgroup of the form $N_G(\Sigma)$ for some Sylow basis of a group G.

Recall that if H/K is a section of a group G, then the group $N_G(H/K) = \{g \in G | (hK)^g \in H/K \text{ for all } h \in H\}$ is called the normalizer of a section H/K in a group G. Let A be a subgroup of a normalizer $N_G(H/K)$ of a section H/K of a group G. Then the group $C_A(H/K)$ is defined

as $C_A(H/K) = \{a \in A \mid h^a K = hK \text{ for all } h \in H\}$. If $K \triangleleft H$ it is easy to see that $A \cap K \leq C_A(H/K)$.

Lemma 2.1 (see Theorem 2.2 in [7]). *If* F *is a Fitting class and G is a* π *-soluble group* ($\pi = \pi(F)$) *then G possesses exactly one conjugacy class of* F*-injectors.*

Lemma 2.2 (see Theorem 2.3 in [7]). Let F be a Fitting class and V an F -injector of a π -soluble group G, where $\pi = \pi(\mathsf{F})$. If $V \le A \le G$, then V is an F -injector of A.

Lemma 2.3. Let F be a Fitting class and $\mathsf{F} \subseteq \mathsf{E}_{\pi}$. If V is an F -injector of a Hall π -subgroup G_{π} of a π -soluble group G, then V is also an F -injector of G. If V is an F -injector of a π -soluble group G, then V is an F -injector of some Hall π -subgroup G_{π} of G.

Proof. The result follows from lemma 2.1 and lemma 2.2.

Lemma 2.4 (see Theorem 7 in [12]). Let F be a full integrated H-function of a local Fitting class F and V an F-subgroup of a finite group G. If Sis a p-subgroup of G for some $p \in \pi(F)$, then $V \cdot C_S(V/V_{F(p)}) \in \mathsf{F}$. In the case $V \triangleleft G$, $V \leq H \leq G$, $H \in \mathsf{F}$ we have $C_S(H/H_{F(p)}) \leq C_S(V/V_{F(p)})$.

3. The results. For an arbitrary local Fitting class F the formula of an F-injector of a soluble group is described by the following theorem.

Theorem 3.1. Let F be a full integrated Hfunction of a Fitting class F, Σ is a Sylow basis of a soluble group G, $D = N_G(\Sigma)$ and $G_p \in \Sigma$, $D_p = G_p \cap D$, where $p \in \pi(F)$. If W is an F-injector of a group N, $G/N \in \mathbb{N}_p$ and $\Sigma \ WW$, then $Z = W \cdot C_{D_p}(W/W_{F(p)}) = W \cdot C_{G_p}(W/W_{F(p)})$ is an F-injector of G and $\Sigma \ UZ$.

Proof. We show that W is pronormal a G, shortly $W \ pr \ G$. Let $g \in G$. Then by lemma IX.1.3.(b) [2] we have $W^g \in Inj_{\mathsf{F}}(N)$. Consequently, $\langle W, W^g \rangle \leq N$. Hence by theorem 1.5.(c) [2] $W, W^g \in Inj_F(\langle W, W^g \rangle)$ and by theorem 8.2.9 [2] W, W^g are conjugated in $\langle W, W^g \rangle$. Then W pr G.

Since $\Sigma \ \psi W$ and $W \ pr G$, by proposition I.6.8 [2] it follows that $D \subseteq N_G(W)$. Then from $D \subseteq N_G(W_{F(p)})$ it follows that $D \subseteq N_G(W/W_{F(p)})$. Denote $Y = C_{D_p}(W/W_{F(p)})$. By lemma 2.4 we have $Z = WY \in \mathbf{F}$.

Let V be an F-maximal subgroup of G such that $Z \subseteq V$. Because $p \in \pi(F)$ and $W \subseteq V$, lemma IX.1.6 [2] yields $V \in Inj_F(G)$. Let $\Sigma \ UV^g$ for some $g \in G$.

Next we show $W \triangleleft V^g$. By lemma IX.1.3.(a) [2] we have $V \cap N \in Inj_F(N)$ and $W \in Inj_F(N)$. The choice of V implies $W = V \cap N \triangleleft V$. From $W^g \triangleleft V^g$ and $\Sigma \Downarrow V^g$ it follows that $\Sigma \Downarrow W^g$. Because $\Sigma \Downarrow W$ and W pr G, by the theorem I.6.6 [2] we have $W = W^g$ and therefore $W \triangleleft V^g$.

We show $C_{D_p}(V^g/V_{F(p)}^g) \le V^g$. By lemma IX.1.3.(b) [2] we have $V^g \in Inj_F(G)$. Then by lemma 2.4 $V^g \cdot C_{D_p}(V^g/V_{F(p)}^g) \in F$. Because V^g is an F-injector of a group G, $V^g \cdot C_{D_p}(V^g/V_{F(p)}^g) = V^g$ and $C_{D_p}(V^g/V_{F(p)}^g) \le V^g$.

Next we prove that $C_{D_p}(V^g / V_{F(p)}^g) \le Y$. By lemma 2.4 $W \lhd V^g$ implies

$$C_{D_{p} \cap V^{g}}(V^{g}/V^{g}_{F(p)}) \leq C_{D_{p} \cap V^{g}}(W^{g}/W^{g}_{F(p)}).$$

 $\begin{array}{lll} & \operatorname{From} & C_{D_p} \left(V^g \, / V_{F(p)}^g \right) \leq V^g & \text{we} & \text{have} \\ \\ & C_{D_p} \left(V^g \, / V_{F(p)}^g \right) = C_{D_p \cap V^g} \left(V^g \, / V_{F(p)}^g \right) . & \text{Then} & \text{from} \\ & C_{D_p \cap V^g} \left(W^g \, / W_{F(p)}^g \right) \leq Y & \text{it} & \text{follows} & \text{that} \\ & C_{D_p} \left(V^g \, / V_{F(p)}^g \right) \leq Y . \end{array}$

We show $V^g \cap D_p = C_{D_p}(V^g / V_{F(p)}^g)$. Notice that $(V^g \cap D_p)V_{F(p)}^g / V_{F(p)}^g$ is isomorphic to $(V^g \cap D_p)/V^g \cap D_p \cap V_{F(p)}^g$.

Set $a = |V^g \cap D_p : V_{F(p)}^g \cap D_p \models |(V^g \cap D_p)V_{F(p)}^g : V_{F(p)}^g|$. It is clear that *a* is a *p*-number. Besides it is true that $a \cdot |V^g D_p : V_{F(p)}^g D_p \models |V^g : V_{F(p)}^g|$. From $V^g \in \mathsf{F} = LR(F)$ we conclude that $|V^g : V_{F(p)}^g|$ is a *p'*-number. Therefore *a* is a *p'*-number. Notice that *a* is also a *p*-number and thus a = 1. Hence, $V^g \cap D_p = V_{F(p)}^g \cap D_p$ and from $V_{F(p)}^g \triangleleft V^g$ we have $V_{F(p)}^g \cap D_p \leq C_{D_p}(V^g / V_{F(p)}^g)$. Then $V^g \cap D_p \leq C_{D_p}(V^g / V_{F(p)}^g)$ and from $C_{D_p}(V^g / V_{F(p)}^g) \leq V^g$ we conclude $V^g \cap D_p = C_{D_p}(V^g / V_{F(p)}^g)$.

By lemma 2 [13] a local Fitting class F is a Fischer class. Then by theorem IX.3.4 [2] we deduce that F is a permutable Fitting class.

Hence for the F-injector V^g all conditions of the theorem IX.3.19 [2] are true and therefore $V^g \le D(V^g \cap N)$. From $W = V \cap N$ and $W = W^g$ we obtain $W = V^g \cap N$. Thus $V^g \le WD$.

We show $V^g \leq WD_p$. Having $W \triangleleft V^g \leq N_G(W)$, $W \triangleleft N_G(W)$, $D \leq N_G(W)$, $D \in \mathbb{N}$ we conclude $V^g/W \leq WD/W \in \mathbb{N}$. Consequently, $WD_p/W \in Syl_p(WD/W)$. Now from $W = V^g \cap N$ it follows $V^g/W \in \mathbb{N}_p$ and $V^g/W \leq WD_p/W$. Then $V^g \leq WD_p$.

From $V^g \leq WD_p$ and $V^g \cap D_p = C_{D_p}(V^g / V_{F(p)}^g)$, by lemma A.1.3 [2] we obtain $V^g \leq V^g \cap WD_p = W(V^g \cap D_p) = W \cdot C_{D_p}(V^g / V_{F(p)}^g)$.

Then from $C_{D_p}(V^g/V^g_{F(p)}) \leq Y$ it follows that $V^g \leq WY$. The choice of V implies $V^g \leq WY \leq V$. Thus $V = V^g = WY$ and $\Sigma \cup V$.

Since $Z = W \cdot C_{D_p}(W/W_{F(p)}) \le W \cdot C_{G_p}(W/W_{F(p)}) \in \mathsf{F}$ by lemma 2.4. He fact that Z is an F -injector implies $Z = W \cdot C_{G_p}(W/W_{F(p)})$. The theorem is proved. Let F be an arbitrary local Fitting class of finite groups. The formula for an F-injector of a $\pi(F)$ -soluble group is described in the following theorem.

Theorem 3.2. Let π be a set of primes and let F be a local Fitting class with $\pi(\mathsf{F}) \subseteq \pi$. Let F be a full integrated H-function of F . Let G be a π -soluble group, Σ is a Sylow basis of a Hall π -subgroup H of G, $D=N_H(\Sigma)$, $G_p \in \Sigma$, $D_p=G_p \cap D$ for $p \in \pi$. If W is an F -injector of $O^p(H)$ and $\Sigma \ W$, then $Z = W \cdot C_{D_p}(W/W_{\mathsf{F}(p)}) = W \cdot C_{G_p}(W/W_{\mathsf{F}(p)})$ is an F -injector of G if $p \in \pi(\mathsf{F})$; if $p \notin \pi(\mathsf{F})$, then W is an F -injector of G.

Proof. If $p \notin \pi(F)$, then clearly *W* is an F-injector of *G*. If $p \in \pi(F)$ then *Z* is an F-injector of *H* by theorem 3.1. Since F-injector of *H* are F-injector of *G* by lemma 2.3, the assession follows. The theorem is proved.

Remark. Theorem 3.2 yields an algorithm for constructing F-injectors in π -soluble groups for local Fitting classes whose characteristic is containing in π if we consider a series $1 = N_0 \triangleleft N_1 \triangleleft ... \triangleleft N_r = H$ of a Hall π -subgroup Hof G since that $N_i = O^{p_i}(N_{i+1}) \triangleleft N_{i+1}$ for snitesle primes p_i , i = 0, ..., r - 1.

Example. Theorem 3.1 can be applied to obtain explicit descriptions of injectors for cestain Fitting classes F. We demonstrate this for F = N, the class of nilpotent groups. We show how to obtain quickly the well-know description of N-injectors in finite soluble groups (see Theorem IX.4.12 [2].

More preciocly, we show that if $G \in S$, Σ a Sylow basis of G, the $X = \times \{C_{G_q}(O_q(F(G)) | G_q \in \Sigma\}$ is an N-injector of G. Let $X_q = C_{G_q}(O_q(F(G)))$. It is easy W the that $[X_p, X_q] = 1$ for all $p \neq q$, so X is in fact nilpotent (see Theorem IX.4.12.(a) [2]).

We prove that X is a N-injector of G by induction on |G|. This is clearly true if G is nilpotent. So we may assume that G is not nilpotent and choose a maximal normal subgroup N of G and that $F(G) \le N$. Let |G/N| = p. By induction, the correspond holds for N. Since F(G) = F(N), we obtain that $W = \times_{q \ne p} C_{G_q}(O_{q'}(F(G))) \times C_{N_p}(O_{p'}(F(G)))$ is an N-injector of N, where $N_p = N \cap G_p$, $G_p \in \Sigma$. By theorem 3.1, $Z = W \cdot C_{G_p}(W/O_p(W))$ is an N-injector of G (note that the full integrated H-function N is gist $F(p) = N_p$). Since W is nilpotent, $Z = W \cdot C_{G_p}(O_p(W))$. Clearly

$C_{G_p}(O_{p'}(W)) \leq C_{G_p}(F)$	(G)),	thus	$Z \leq X$.	Since	X	is
nilpotent	and		Ζ	is		a
N-injector of G , $Z = X$ follows.						

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