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Инъекторы локальных классов Фиттинга

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Подгруппу V группы G называют $\mathfrak{F}$-инъектором группы G, если $V \cap N$ является $\mathfrak{F}$-максимальной подгруппой в N для всякой субнормальной подгруппы N из G. В настоящей работе найдена формула $\mathfrak{F}$-инъектора π-разрешимой группы для произвольного локального класса Фиттинга $\mathfrak{F}$. В частности, получен метод построения инъектора разрешимой группы для локального класса Фиттинга.

Ключевые слова: инъекторы групп, локальный класс Фиттинга, H-функция.

Injectors of local Fitting classes

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Summary. *A «V» subgroup of a «G» group is called an $\mathfrak{F}$-injector of «G» if $V \cap N$ is an $\mathfrak{F}$-maximal subgroup of «N»« for every subnormal «N» subgroup of «G». In this paper a formula of $\mathfrak{F}$-injector of a π-soluble group is found for voluntary local Fitting «$\mathfrak{F}$» class. This yields a new method of definition of soluble groups for finite injectors for local Fitting classes.*

1. Introduction. In the theory of classes of finite soluble groups the basic result which generalizes fundamental theorems of Sylow and Hall is the theorem of Gaschütz [1] on existence and conjugacy of $\mathfrak{F}$-projectors in groups G for every soluble local formation $\mathfrak{F}$. Recall that a subgroup F is called [2] an $\mathfrak{F}$-projector of G if FN/N is an $\mathfrak{F}$-maximal subgroup in G/N for all normal subgroups N of G.

In the sequel many researchers were interested in finding a formula for an $\mathfrak{F}$-projector. This task was solved by Doerk [3], D'Arcy [4] and N.T. Vorob'ev [5].

In 1967 an importat bright result in the theory of finite soluble groups was established by Gaschütz, Fischer and Hartley in their paper [6], where another generalization of the theorems of Sylow and Hall was presented. It was proved [6] that in every group $G \in \mathfrak{S}$ a unique conjugacy class $\mathfrak{F}$-injectors exist for all soluble Fitting classes. Recall that a subgroup V of a group G is called [2] an $\mathfrak{F}$-injector of G if $V \cap N$ is an $\mathfrak{F}$-maximal subgroup of N for every subnormal subgroup N of G.

In 1975 L.A. Shemetkov [7] extended the Gaschütz-Fischer-Hartley's result on the case of π-soluble groups for all Fitting classes $\mathfrak{F}$ where $\pi = \pi(\mathfrak{F}) = \cup\{\pi(G) \mid G \in \mathfrak{F}\}$. In this connection in the theory of Fitting classes the task of finding an formula for an $\mathfrak{F}$-injector of a finite π-soluble group for an arbitrary local Fitting class $\mathfrak{F}$ is not completed. In the class $\mathfrak{S}$ it was solved by Hartley [8], N.T. Vorob'ev and I.V. Dudkin [9] only for some special cases of a local class $\mathfrak{F}$.

The main result of this paper is a full solution of the task specified above in the class of $\pi(\mathfrak{F})$-soluble groups for a local Fitting class $\mathfrak{F}$. We refer to [2] for the notation and basic results on classes of groups.

2. Preliminaries. All groups considered in this paper are finite.

For the first time a local method of studying of finite soluble groups by means of radicals and Fitting classes was introduced by Hartley [8]. Every map $f : P \rightarrow \{\text{ Fitting classes }\}$ is called a Hartley function or an H-function [10] where P designate the set of primes.

Let $LR(f) = \mathfrak{E}_{\pi} \cap (\cap_{p \in \pi} f(p)\mathfrak{N}_p\mathfrak{E}_{p'})$, where $\pi = \pi(\mathfrak{F})$. A Fitting class $\mathfrak{F}$ is called local [10], if $\mathfrak{F} = LR(f)$ for some H-function f.

A Hartley function f of a Fitting class $\mathfrak{F}$ is called full integrated, if $f(p)\mathfrak{N}_p = f(p)$ and $f(p) \subseteq \mathfrak{F}$ for all prime numbers p. Such a Hartley function exist for every local Fitting class [11].

Let π be a set of primes. A finite group G is said to be π-soluble if every chief factor of G is either an abelian π-group or a π'-group.

A set Σ of pairwise permutable Sylow p-subgroups of G, exactly one for each $p \in \pi(G)$, together with the identity subgroup, is called a Sylow basis of G.

If Σ is a set of subgroups of a group G and if $R \le G$, we denote by $\Sigma \cap R$ the following set of subgroups of G $\Sigma \cap R = \{S \cap R \mid S \in \Sigma\}$.

In general, $\Sigma \cap R$ is not a Sylow basis of R. However, when it happens that $\Sigma \cap R$ is a Sylow basis of R, we say that Σ reduces into R and write $\Sigma \,\psi\, R$. If Σ is a Sylow basis of G, then Σ reduces into R^g for some $g \in G$. In particular, if $R \lhd G$, then $\Sigma \,\psi\, R$. If Σ is a Sylow basis of a group G, the subgroup $N_G(\Sigma) = \{g \in G \mid H = H^g \text{ for all } H \in \Sigma\}$ is called the normalizer of Σ. A system normalizer of G is a subgroup of the form $N_G(\Sigma)$ for some Sylow basis of a group G.

Recall that if H/K is a section of a group G, then the group $N_G(H/K) = \{g \in G \mid (hK)^g \in H/K \text{ for all } h \in H\}$ is called the normalizer of a section H/K in a group G. Let A be a subgroup of a normalizer $N_G(H/K)$ of a section H/K of a group G. Then the group $C_A(H/K)$ is defined as $C_A(H/K) = \{a \in A \mid h^a K = hK \text{ for all } h \in H\}$. If $K \lhd H$ it is easy to see that $A \cap K \le C_A(H/K)$.

Lemma 2.1 (see Theorem 2.2 in [7]). *If F is a Fitting class and G is a π-soluble group ($\pi = \pi(\mathsf{F})$) then G possesses exactly one conjugacy class of F-injectors.*

Lemma 2.2 (see Theorem 2.3 in [7]). *Let F be a Fitting class and V an F-injector of a π-soluble group G, where $\pi = \pi(\mathsf{F})$. If $V \le A \le G$, then V is an F-injector of A.*

Lemma 2.3. *Let F be a Fitting class and $\mathsf{F} \subseteq \mathsf{E}_\pi$. If V is an F-injector of a Hall π-subgroup G_π of a π-soluble group G, then V is also an F-injector of G. If V is an F-injector of a π-soluble group G, then V is an F-injector of some Hall π-subgroup G_π of G.*

Proof. The result follows from lemma 2.1 and lemma 2.2.

Lemma 2.4 (see Theorem 7 in [12]). *Let F be a full integrated H-function of a local Fitting class F and V an F-subgroup of a finite group G. If S is a p-subgroup of G for some $p \in \pi(\mathsf{F})$, then $V \cdot C_S(V/V_{F(p)}) \in \mathsf{F}$. In the case $V \lhd G$, $V \le H \le G$, $H \in \mathsf{F}$ we have $C_S(H/H_{F(p)}) \le C_S(V/V_{F(p)})$.*

3. The results. For an arbitrary local Fitting class F the formula of an F-injector of a soluble group is described by the following theorem.

Theorem 3.1. *Let F be a full integrated H-function of a Fitting class F, Σ is a Sylow basis of a soluble group G, $D = N_G(\Sigma)$ and $G_p \in \Sigma$, $D_p = G_p \cap D$, where $p \in \pi(\mathsf{F})$. If W is an F-injector of a group N, $G/N \in \mathsf{N}_p$ and $\Sigma \,\psi\, W$, then $Z = W \cdot C_{D_p}(W/W_{F(p)}) = W \cdot C_{G_p}(W/W_{F(p)})$ is an F-injector of G and $\Sigma \,\psi\, Z$.*

Proof. We show that W is pronormal a G, shortly $W \; pr \; G$. Let $g \in G$. Then by lemma IX.1.3.(b) [2] we have $W^g \in Inj_{\mathsf{F}}(N)$. Consequently, $< W, W^g > \le N$. Hence by theorem 1.5.(c) [2] $W, W^g \in Inj_{\mathsf{F}}(< W, W^g >)$ and by theorem 8.2.9 [2] W, W^g are conjugated in $< W, W^g >$. Then $W \; pr \; G$.

Since $\Sigma \,\psi\, W$ and $W \; pr \; G$, by proposition I.6.8 [2] it follows that $D \subseteq N_G(W)$. Then from $D \subseteq N_G(W_{F(p)})$ it follows that $D \subseteq N_G(W/W_{F(p)})$. Denote $Y = C_{D_p}(W/W_{F(p)})$. By lemma 2.4 we have $Z = WY \in \mathsf{F}$.

Let V be an F-maximal subgroup of G such that $Z \subseteq V$. Because $p \in \pi(\mathsf{F})$ and $W \subseteq V$, lemma IX.1.6 [2] yields $V \in Inj_{\mathsf{F}}(G)$. Let $\Sigma \,\psi\, V^g$ for some $g \in G$.

Next we show $W \lhd V^g$. By lemma IX.1.3.(a) [2] we have $V \cap N \in Inj_{\mathsf{F}}(N)$ and $W \in Inj_{\mathsf{F}}(N)$. The choice of V implies $W = V \cap N \lhd V$. From $W^g \lhd V^g$ and $\Sigma \,\psi\, V^g$ it follows that $\Sigma \,\psi\, W^g$. Because $\Sigma \,\psi\, W$ and $W \; pr \; G$, by the theorem I.6.6 [2] we have $W = W^g$ and therefore $W \lhd V^g$.

We show $C_{D_p}(V^g/V^g_{F(p)}) \le V^g$. By lemma IX.1.3.(b) [2] we have $V^g \in Inj_{\mathsf{F}}(G)$. Then by lemma 2.4 $V^g \cdot C_{D_p}(V^g/V^g_{F(p)}) \in \mathsf{F}$. Because V^g is an F-injector of a group G, $V^g \cdot C_{D_p}(V^g/V^g_{F(p)}) = V^g$ and $C_{D_p}(V^g/V^g_{F(p)}) \le V^g$.

Next we prove that $C_{D_p}(V^g/V^g_{F(p)}) \le Y$. By lemma 2.4 $W \lhd V^g$ implies

$$C_{D_p \cap V^g}(V^g/V^g_{F(p)}) \le C_{D_p \cap V^g}(W^g/W^g_{F(p)}).$$

From $C_{D_p}(V^g/V^g_{F(p)}) \le V^g$ we have $C_{D_p}(V^g/V^g_{F(p)}) = C_{D_p \cap V^g}(V^g/V^g_{F(p)})$. Then from $C_{D_p \cap V^g}(W^g/W^g_{F(p)}) \le Y$ it follows that $C_{D_p}(V^g/V^g_{F(p)}) \le Y$.

We show $V^g \cap D_p = C_{D_p}(V^g / V^g_{F(p)})$. Notice that $(V^g \cap D_p)V^g_{F(p)} / V^g_{F(p)}$ is isomorphic to $(V^g \cap D_p) / V^g \cap D_p \cap V^g_{F(p)}$.

Set $a = | V^g \cap D_p : V^g_{F(p)} \cap D_p | = | (V^g \cap D_p)V^g_{F(p)} : V^g_{F(p)} |$. It is clear that a is a p-number. Besides it is true that $a \cdot | V^g D_p : V^g_{F(p)} D_p | = | V^g : V^g_{F(p)} |$. From $V^g \in \mathfrak{F} = LR(F)$ we conclude that $| V^g : V^g_{F(p)} |$ is a p'-number. Therefore a is a p'-number. Notice that a is also a p-number and thus $a = 1$. Hence, $V^g \cap D_p = V^g_{F(p)} \cap D_p$ and from $V^g_{F(p)} \lhd V^g$ we have $V^g_{F(p)} \cap D_p \le C_{D_p}(V^g / V^g_{F(p)})$. Then $V^g \cap D_p \le C_{D_p}(V^g / V^g_{F(p)})$ and from $C_{D_p}(V^g / V^g_{F(p)}) \le V^g$ we conclude $V^g \cap D_p = C_{D_p}(V^g / V^g_{F(p)})$.

By lemma 2 [13] a local Fitting class $\mathfrak{F}$ is a Fischer class. Then by theorem IX.3.4 [2] we deduce that $\mathfrak{F}$ is a permutable Fitting class.

Hence for the $\mathfrak{F}$-injector V^g all conditions of the theorem IX.3.19 [2] are true and therefore $V^g \le D(V^g \cap N)$. From $W = V \cap N$ and $W = W^g$ we obtain $W = V^g \cap N$. Thus $V^g \le WD$.

We show $V^g \le WD_p$. Having $W \lhd V^g \le N_G(W)$, $W \lhd N_G(W)$, $D \le N_G(W)$, $D \in \mathfrak{N}$ we conclude $V^g / W \le WD / W \in \mathfrak{N}$. Consequently, $WD_p / W \in Syl_p(WD/W)$. Now from $W = V^g \cap N$ it follows $V^g / W \in \mathfrak{N}_p$ and $V^g / W \le WD_p / W$. Then $V^g \le WD_p$.

From $V^g \le WD_p$ and $V^g \cap D_p = C_{D_p}(V^g / V^g_{F(p)})$, by lemma A.1.3 [2] we obtain $V^g \le V^g \cap WD_p = W(V^g \cap D_p) = W \cdot C_{D_p}(V^g / V^g_{F(p)})$.

Then from $C_{D_p}(V^g / V^g_{F(p)}) \le Y$ it follows that $V^g \le WY$. The choice of V implies $V^g \le WY \le V$. Thus $V = V^g = WY$ and $\Sigma \searrow V$.

Since $Z = W \cdot C_{D_p}(W / W_{F(p)}) \le W \cdot C_{G_p}(W / W_{F(p)}) \in \mathfrak{F}$ by lemma 2.4. He fact that Z is an $\mathfrak{F}$-injector implies $Z = W \cdot C_{G_p}(W / W_{F(p)})$. The theorem is proved.

Let $\mathfrak{F}$ be an arbitrary local Fitting class of finite groups. The formula for an $\mathfrak{F}$-injector of a $\pi(\mathfrak{F})$-soluble group is described in the following theorem.

Theorem 3.2. *Let π be a set of primes and let $\mathfrak{F}$ be a local Fitting class with $\pi(\mathfrak{F}) \subseteq \pi$. Let F be a full integrated H-function of $\mathfrak{F}$. Let G be a π-soluble group, Σ is a Sylow basis of a Hall π-subgroup H of G, $D = N_H(\Sigma)$, $G_p \in \Sigma$, $D_p = G_p \cap D$ for $p \in \pi$. If W is an $\mathfrak{F}$-injector of $O^p(H)$ and $\Sigma \searrow W$, then $Z = W \cdot C_{D_p}(W / W_{F(p)}) = W \cdot C_{G_p}(W / W_{F(p)})$ is an $\mathfrak{F}$-injector of G if $p \in \pi(\mathfrak{F})$; if $p \notin \pi(\mathfrak{F})$, then W is an $\mathfrak{F}$-injector of G.*

Proof. If $p \notin \pi(\mathfrak{F})$, then clearly W is an $\mathfrak{F}$-injector of G. If $p \in \pi(\mathfrak{F})$ then Z is an $\mathfrak{F}$-injector of H by theorem 3.1. Since $\mathfrak{F}$-injector of H are $\mathfrak{F}$-injector of G by lemma 2.3, the assestion follows. The theorem is proved.

Remark. Theorem 3.2 yields an algorithm for constructing $\mathfrak{F}$-injectors in π-soluble groups for local Fitting classes whose characteristic is containing in π if we consider a series $1 = N_0 \lhd N_1 \lhd \ldots \lhd N_r = H$ of a Hall π-subgroup H of G since that $N_i = O^{p_i}(N_{i+1}) < N_{i+1}$ for snitesle primes p_i, $i = 0, \ldots, r-1$.

Example. Theorem 3.1 can be applied to obtain explicit descriptions of injectors for cestain Fitting classes $\mathfrak{F}$. We demonstrate this for $\mathfrak{F} = \mathfrak{N}$, the class of nilpotent groups. We show how to obtain quickly the well-know description of $\mathfrak{N}$-injectors in finite soluble groups (see Theorem IX.4.12 [2]).

More precioсly, we show that if $G \in \mathfrak{S}$, Σ a Sylow basis of G, the $X = \times\{C_{G_q}(O_{q'}(F(G)) \mid G_q \in \Sigma\}$ is an $\mathfrak{N}$-injector of G. Let $X_q = C_{G_q}(O_{q'}(F(G))$. It is easy W the that $[X_p, X_q] = 1$ for all $p \ne q$, so X is in fact nilpotent (see Theorem IX.4.12.(a) [2]).

We prove that X is a $\mathfrak{N}$-injector of G by induction on $|G|$. This is clearly true if G is nilpotent. So we may assume that G is not nilpotent and choose a maximal normal subgroup N of G and that $F(G) \le N$. Let $|G / N| = p$. By induction, the correspond holds for N. Since $F(G) = F(N)$, we obtain that $W = \times_{q \ne p} C_{G_q}(O_{q'}(F(G))) \times C_{N_p}(O_{p'}(F(G)))$ is an $\mathfrak{N}$-injector of N, where $N_p = N \cap G_p$, $G_p \in \Sigma$. By theorem 3.1, $Z = W \cdot C_{G_p}(W / O_p(W))$ is an $\mathfrak{N}$-injector of G (note that the full integrated H-function $\mathfrak{N}$ is gist $F(p) = \mathfrak{N}_p$). Since W is nilpotent, $Z = W \cdot C_{G_p}(O_{p'}(W))$. Clearly $C_{G_p}(O_{p'}(W)) \le C_{G_p}(F(G))$, thus $Z \le X$. Since X is nilpotent and Z is a $\mathfrak{N}$-injector of G, $Z = X$ follows.

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