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INITIAL BOUNDARY VALUE PROBLEM FOR A SYSTEM OF SEMILINEAR PARABOLIC EQUATIONS WITH ABSORPTION AND NONLINEAR NONLOCAL BOUNDARY CONDITIONS#

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Dedicated to Professor A. F. Tedeev on occasion of his 70th birthday

Abstract. In this paper we consider classical solutions of an initial boundary value problem for a system of semilinear parabolic equations with absorption and nonlinear nonlocal boundary conditions. Nonlinearities in equations and boundary conditions may not satisfy the Lipschitz condition. To prove the existence of a solution we regularize the original problem. Using the Schauder–Tikhonov fixed point theorem, the existence of a local solution of regularized problem is proved. It is shown that the limit of solutions of the regularized problem is a maximal solution of the original problem. Using the properties of a maximal solution, a comparison principle is proved. In this case, no additional assumptions are made when nonlinearities in absorption do not satisfy the Lipschitz condition. Conditions are found under which solutions are positive functions. The uniqueness of the solution is established. It is shown that the trivial solution $(0, 0)$ may not be unique.

Keywords: system of semilinear parabolic equations, nonlocal boundary conditions, existence of a solution, comparison principle.

AMS Subject Classification: 35K51, 35K58, 35K61.

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1. Introduction

In this paper we consider the initial boundary value problem for a system of semilinear parabolic equations with absorption and nonlinear nonlocal boundary conditions:

$$\begin{cases} u_t = \Delta u + v^p - au^r, \quad v_t = \Delta v + u^q - bv^s, & x \in \Omega,\ t > 0, \\ \frac{\partial u(x,t)}{\partial \nu} = \int\limits_{\Omega} \phi(x,y,t) u^m(y,t)\,dy, & x \in \partial\Omega,\ t > 0, \\ \frac{\partial v(x,t)}{\partial \nu} = \int\limits_{\Omega} \psi(x,y,t) v^n(y,t)\,dy, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1}$$

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where a, b, p, r, q, s, m, n are positive numbers, Ω is a bounded domain in $\mathbb{R}^N$ for $N \geqslant 1$ with smooth boundary $\partial\Omega$, ν is the unit outward normal vector on $\partial\Omega$.

Throughout this paper we suppose the following conditions:

$$\phi(x,y,t) \in C(\partial\Omega \times \overline{\Omega} \times [0,+\infty)), \quad \phi(x,y,t) \geqslant 0;$$

$$\psi(x,y,t) \in C(\partial\Omega \times \overline{\Omega} \times [0,+\infty)), \quad \psi(x,y,t) \geqslant 0;$$

$$u_0(x) \in C^1(\overline{\Omega}), \quad v_0(x) \in C^1(\overline{\Omega}), \quad u_0(x) \geqslant 0, \quad v_0(x) \geqslant 0 \text{ in } \Omega;$$

$$\frac{\partial u_0(x)}{\partial \nu} = \int_\Omega \phi(x,y,0) u_0^m(y)\,dy, \quad \frac{\partial v_0(x)}{\partial \nu} = \int_\Omega \psi(x,y,0) v_0^n(y)\,dy \text{ on } \partial\Omega.$$

Let $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$, $\Gamma_T = S_T \cup \overline{\Omega} \times \{0\}$, $T > 0$.

Definition 1. We say that a pair of nonnegative functions $(u(x,t), v(x,t))$ is a subsolution of (1) in Q_T, if $u, v \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ and

$$\begin{cases} u_t \leqslant \Delta u + v^p - au^r, \quad v_t \leqslant \Delta v + u^q - bv^s, & (x,t) \in Q_T, \\ \frac{\partial u(x,t)}{\partial \nu} \leqslant \int_\Omega \phi(x,y,t)\, u^m(y,t)\,dy, & (x,t) \in S_T, \\ \frac{\partial v(x,t)}{\partial \nu} \leqslant \int_\Omega \psi(x,y,t)\, v^n(y,t)\,dy, & (x,t) \in S_T, \\ u(x,0) \leqslant u_0(x), \quad v(x,0) \leqslant v_0(x), & x \in \Omega, \end{cases} \tag{2}$$

a pair of nonnegative functions $(u(x,t), v(x,t))$ is a supersolution of (1) in Q_T, if $u, v \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ and it satisfies (2) in the reverse order. We say that $(u(x,t), v(x,t))$ is a solution of problem (1) in Q_T if $(u(x,t), v(x,t))$ is both a subsolution and a supersolution of (1) in Q_T.

Definition 2. We say that a solution $(u_{\max}(x,t), v_{\max}(x,t))$ of (1) in Q_T is a maximal solution if for any other solution $(u(x,t), v(x,t))$ of (1) in Q_T the inequalities $u(x,t) \leqslant u_{\max}(x,t)$, $v(x,t) \leqslant v_{\max}(x,t)$ are satisfied for $(x,t) \in Q_T$.

A lot of articles have been devoted to the investigation of initial boundary value problems for parabolic equations and systems with nonlocal boundary conditions (see, for example, [1–22] and the references therein). In particular, the problem (1) with $a = b = 0$ was considered in [11]. Initial boundary value problem (1) with $a = b = 0$ and Dirichlet nonlocal boundary conditions was studied in [5]. The authors of [23] investigated the existence of global solutions for (1) with zero Dirichlet boundary condition. Blow-up problem for (1) with nonlinear local Neumann boundary conditions was investigated in [24, 25].

This paper is organized as follows. In the next section we prove the existence of a local solution. A comparison principle and the uniqueness of solutions of (1) are established in Section 3.

2. Local Existence

Let $\{\varepsilon_l\}$ be decreasing to 0 sequence such that $0 < \varepsilon_l < 1$, $l \in \mathbb{N}$. For $\varepsilon = \varepsilon_l$ let $u_{0\varepsilon}(x)$ and $v_{0\varepsilon}(x)$ be the functions with the following properties:

$$u_{0\varepsilon}(x), v_{0\varepsilon}(x) \in C^1(\overline{\Omega}), \quad u_{0\varepsilon}(x) \geqslant \varepsilon, \quad v_{0\varepsilon}(x) \geqslant \varepsilon;$$

$$u_{0\varepsilon_i}(x) \geqslant u_{0\varepsilon_j}, \quad v_{0\varepsilon_i}(x) \geqslant v_{0\varepsilon_j} \text{ for } \varepsilon_i > \varepsilon_j;$$

$$0 \leqslant u_{0\varepsilon}(x) - u_0(x) \leqslant 2\varepsilon, \quad 0 \leqslant v_{0\varepsilon}(x) - v_0(x) \leqslant 2\varepsilon;$$

$$\frac{\partial u_{0\varepsilon}(x)}{\partial \nu} = \int_\Omega \phi(x,y,0) u_{0\varepsilon}^m(y)\,dy, \quad \frac{\partial v_{0\varepsilon}(x)}{\partial \nu} = \int_\Omega \psi(x,y,0) v_{0\varepsilon}^n(y)\,dy \ \text{ for } x \in \partial\Omega.$$

Since the nonlinearities in (1), the Lipschitz condition may not be satisfied, and thus we need to consider the following auxiliary problem:

$$\begin{cases} u_t = \Delta u + v^p - au^r + a\varepsilon^r, & x \in \Omega, \ t > 0, \\ v_t = \Delta v + u^q - bv^s + b\varepsilon^s, & x \in \Omega, \ t > 0, \\ \frac{\partial u(x,t)}{\partial \nu} = \int_\Omega \phi(x,y,t) u^m(y,t)\,dy, & x \in \partial\Omega, \ t > 0, \\ \frac{\partial v(x,t)}{\partial \nu} = \int_\Omega \psi(x,y,t) v^n(y,t)\,dy, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_{0\varepsilon}(x), \quad v(x,0) = v_{0\varepsilon}(x), & x \in \Omega, \end{cases} \tag{3}$$

where $\varepsilon = \varepsilon_l$. The notion of a solution $(u_\varepsilon, v_\varepsilon)$ for problem (3) in Q_T can be defined in a similar way as in Definition 1.

Theorem 1. *Problem* (3) *has a unique solution in* Q_T *for small values of* T.

$\lhd$ We start the proof with the construction of a subsolution and a supersolution of (3) in Q_T for some T. Let $\sup_{\bar\Omega} u_{0\varepsilon}(x) \leqslant M$, $\sup_{\bar\Omega} v_{0\varepsilon}(x) \leqslant M$, where $M \geqslant 1$. Denote

$$K = \max\left(\sup_{\partial\Omega \times Q_T} \phi(x,y,t), \sup_{\partial\Omega \times Q_T} \psi(x,y,t) \right)$$

and introduce an auxiliary function $\varphi(x)$ with the following properties:

$$\varphi(x) \in C^2(\bar\Omega), \quad \inf_\Omega \varphi(x) \geqslant 1, \quad \inf_{\partial\Omega} \frac{\partial \varphi(x)}{\partial \nu} \geqslant \bar K \int_\Omega \max\left(\varphi^m(y), \varphi^n(y)\right) dy, \tag{4}$$

where $\bar K = K \max\left(M^{m-1}, M^{n-1}\right) \max\left(1, \exp(m-1), \exp(n-1)\right)$. Let α, β be positive constants such that $\alpha q - \beta = \beta p - \alpha$ and

$$\alpha \geqslant \sup_\Omega \left(\frac{\Delta\varphi(x)}{\varphi(x)} + M^{p-1} \exp(1) \varphi^{p-1}(x) + a \right), \tag{5}$$

$$\beta \geqslant \sup_\Omega \left(\frac{\Delta\varphi(x)}{\varphi(x)} + M^{q-1} \exp(1) \varphi^{q-1}(x) + b \right).$$

It is easy to see that $\alpha q - \beta > 0$ for $pq > 1$ and $\alpha q - \beta \leqslant 0$ for $pq \leqslant 1$. Obviously, $(\varepsilon, \varepsilon)$ is a subsolution of (3) in Q_T for any T. Let us show that

$$f(x,t) = M \exp(\alpha t)\, \varphi(x), \quad g(x,t) = M \exp(\beta t)\, \varphi(x)$$

is a supersolution of (3) in Q_T for $T \leqslant \min\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\alpha q - \beta}\right)$ if $pq > 1$, and for $T \leqslant \min\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$ if $pq \leqslant 1$. Indeed, by (5) we have

$$f_t(x,t) - \Delta f(x,t) - g^p(x,t) + af^r(x,t) - a\varepsilon^r$$
$$= \alpha M \exp(\alpha t)\varphi(x) - \Delta\varphi(x) M \exp(\alpha t) - M^p \exp(\beta p t)\varphi^p(x) + aM^r \exp(\alpha r t)\varphi^r(x) - a\varepsilon^r$$
$$\geqslant M \exp(\alpha t)\varphi(x) \left(\alpha - \frac{\Delta\varphi(x)}{\varphi(x)} - M^{p-1} \exp((\beta p - \alpha)t)\varphi^{p-1}(x) - a \right) \geqslant 0$$

for $(x,t) \in Q_T$. Using (4), we obtain

$$\frac{\partial f(x,t)}{\partial \nu} = M \exp(\alpha t) \frac{\partial \varphi(x)}{\partial \nu} \geqslant M^m \exp(\alpha m t) K \int\limits_{\Omega} \varphi^m(y)\, dy \geqslant \int\limits_{\Omega} \phi(x,y,t) f^m(y,t)\, dy$$

for $(x,t) \in S_T$. In a similar way we show that

$$g_t(x,t) - \Delta g(x,t) - f^q(x,t) + b g^s(x,t) - b\varepsilon^s \geqslant 0 \quad \text{for } (x,t) \in Q_T,$$

$$\frac{\partial g(x,t)}{\partial \nu} \geqslant \int\limits_{\Omega} \psi(x,y,t) g^n(y,t)\, dy \quad \text{for } (x,t) \in S_T.$$

And we have $u_{0\varepsilon}(x) \leqslant f(x,0)$, $v_{0\varepsilon}(x) \leqslant g(x,0)$ for $x \in \Omega$, which implies that $(f(x,t), g(x,t))$ is a supersolution of (3) in Q_T.

To prove the existence of a solution of (3) in Q_T for some T let us define a set

$$B = \Big\{ (h_1(x,t), h_2(x,t)) \in C(\overline{Q_T}) \times C(\overline{Q_T}) :$$
$$\varepsilon \leqslant h_1(x,t) \leqslant f(x,t),\ \varepsilon \leqslant h_2(x,t) \leqslant g(x,t),\ h_1(x,0) = u_{0\varepsilon}(x),\ h_2(x,0) = v_{0\varepsilon}(x) \Big\}.$$

Obviously, B is a nonempty convex subset of $C(\overline{Q_T}) \times C(\overline{Q_T})$. Now we consider the following problem

$$\begin{cases} u_t = \Delta u + v^p - a u^r + a\varepsilon^r, & (x,t) \in Q_T, \\ v_t = \Delta v + u^q - b v^s + b\varepsilon^s, & (x,t) \in Q_T, \\ \frac{\partial u}{\partial \nu} = \int\limits_{\Omega} \phi(x,y,t)\, s_1^m(y,t)\, dy, & (x,t) \in S_T, \\ \frac{\partial v}{\partial \nu} = \int\limits_{\Omega} \psi(x,y,t)\, s_2^n(y,t)\, dy, & (x,t) \in S_T, \\ u(x,0) = u_{0\varepsilon}(x), \quad v(x,0) = v_{0\varepsilon}(x), & x \in \Omega, \end{cases} \tag{6}$$

where $(s_1, s_2) \in B$. Problem (6) has a classical solution, which is bounded in $\overline{Q_T}$ for some T (see, for example, [26]). Let A be a map such that $A(s_1, s_2) = (u, v)$. Denote the set of functions u as U and the set of functions v as V. In order to show that A has a fixed point in B we verify that A is a continuous mapping from B into itself such that AB is relatively compact. Since $(\varepsilon, \varepsilon)$ is a subsolution of (6) in Q_T and $(f(x,t), g(x,t))$ is a supersolution of (6) in Q_T we have that A maps B into itself thanks to a comparison principle for (6) which can be proved in a similar way as Theorem 3 below.

Let $G(x,t;\xi,\tau)$ denote the Green function for the heat equation given by $u_t - \Delta u = 0$ for $x \in \Omega$, $t > 0$ with homogeneous Neumann boundary condition. The Green function has the following properties (see [27, 28]):

$$G(x,t;\xi,\tau) \geqslant 0, \quad x, \xi \in \Omega,\ 0 \leqslant \tau < t, \tag{7}$$

$$\int\limits_{\Omega} G(x,t;\xi,\tau)\, d\xi = 1, \quad x \in \Omega,\ 0 \leqslant \tau < t, \tag{8}$$

$$\sup_{\Omega} \int\limits_{s}^{t} \int\limits_{\partial\Omega} G(x,t;\xi,\tau)\, dS_\xi\, d\tau \leqslant \lambda \sqrt{t-s}, \quad 0 < t - s \leqslant \sigma, \tag{9}$$

for $s \geqslant 0$, $\lambda > 0$ and small $\sigma > 0$.

It is well known that $(u(x,t), v(x,t))$ is a solution of (6) in Q_T if and only if

$$\begin{aligned} u(x,t) = \int_\Omega G(x,t;y,0)u_{0\varepsilon}(y)\,dy + \int_0^t\int_\Omega G(x,t;y,\tau)\left(v^p(y,\tau) - au^r(y,\tau) + a\varepsilon^r\right)\,dy\,d\tau \\ + \int_0^t\int_{\partial\Omega} G(x,t;\xi,\tau)\left(\int_\Omega \phi(\xi,y,\tau)s_1^m(y,\tau)\,dy\right)\,dS_\xi\,d\tau, \end{aligned} \tag{10}$$

$$\begin{aligned} v(x,t) = \int_\Omega G(x,t;y,0)v_{0\varepsilon}(y)\,dy + \int_0^t\int_\Omega G(x,t;y,\tau)(u^q(y,\tau) - bv^s(y,\tau) + b\varepsilon^s)\,dy\,d\tau \\ + \int_0^t\int_{\partial\Omega} G(x,t;\xi,\tau)\left(\int_\Omega \psi(\xi,y,\tau)s_2^n(y,\tau)\,dy\right)\,dS_\xi\,d\tau \end{aligned} \tag{11}$$

for $(x,t) \in Q_T$.

We claim that A is a continuous map. In fact let $\{(s_{1k},\, s_{2k})\}$ be a sequence in B converging to $(s_1, s_2) \in B$ in $C(\overline{Q_T}) \times C(\overline{Q_T})$. Denote $(u_k, v_k) = A(s_{1k}, s_{2k})$. Then by (7), (10), (11) we have

$$\begin{aligned} |u-u_k| + |v-v_k| \leqslant \int_0^t\int_\Omega G(x,t;y,\tau)|v^p - v_k^p|\,dy\,d\tau + \int_0^t\int_\Omega G(x,t;y,\tau)|u^q - u_k^q|\,dy\,d\tau \\ + a\int_0^t\int_\Omega G(x,t;y,\tau)|u^r - u_k^r|\,dy\,d\tau + b\int_0^t\int_\Omega G(x,t;y,\tau)|v^s - v_k^s|\,dy\,d\tau \\ + \sup_{Q_T}|s_1^m - s_{1k}^m|\int_0^t\int_{\partial\Omega} G(x,t;\xi,\tau)\left(\int_\Omega \phi(\xi,y,\tau)\,dy\right)\,dS_\xi\,d\tau \\ + \sup_{Q_T}|s_2^n - s_{2k}^n|\int_0^t\int_{\partial\Omega} G(x,t;\xi,\tau)\left(\int_\Omega \psi(\xi,y,\tau)\,dy\right)\,dS_\xi\,d\tau \\ \leqslant \Phi(\sup_{Q_T}|u-u_k| + \sup_{Q_T}|v-v_k|) \\ + \sup_{Q_T}|s_1^m - s_{1k}^m|\int_0^t\int_{\partial\Omega} G(x,t;\xi,\tau)\left(\int_\Omega \phi(\xi,y,\tau)\,dy\right)\,dS_\xi\,d\tau \\ + \sup_{Q_T}|s_2^n - s_{2k}^n|\int_0^t\int_{\partial\Omega} G(x,t;\xi,\tau)\left(\int_\Omega \psi(\xi,y,\tau)\,dy\right)\,dS_\xi\,d\tau, \end{aligned}$$

where

$$\begin{aligned} \Phi = \Bigg\{p\max\left(\varepsilon^{p-1}, \sup_{Q_T} g^{p-1}(x,t)\right) + q\max\left(\varepsilon^{q-1}, \sup_{Q_T} f^{q-1}(x,t)\right) \\ + ar\max\left(\varepsilon^{r-1}, \sup_{Q_T} f^{r-1}(x,t)\right) + bs\max\left(\varepsilon^{s-1}, \sup_{Q_T} g^{s-1}(x,t)\right)\Bigg\}\sup_{Q_T}\int_0^t\int_\Omega G(x,t;y,\tau)\,dy\,d\tau. \end{aligned}$$

By (8) and (9) there exists T, such that $\Phi < 1$. Then we obtain $(u_k, v_k) \to (u, v)$ in $C(\overline{Q_T}) \times C(\overline{Q_T})$ as $k \to \infty$.

By the definition of B the sets U and V are uniformly bounded.

Now we prove the equicontinuity of the sets U and V. We will consider the set U since the proof for the set V is similar. We show that for any $\epsilon > 0$ there exists $\delta > 0$, such that

$$|u(x_2, t_2) - u(x_1, t_1)| < \epsilon \tag{12}$$

for any $u(x,t) \in U$ and any $(x_1, t_1), (x_2, t_2) \in \overline{Q_T}$ with the property $|(x_2, t_2) - (x_1, t_1)| < \delta$. Applying (10), we obtain

$$\begin{aligned} |u(x_2, t_2) - u(x_1, t_1)| \leqslant & \left| \int_\Omega (G(x_2, t_2; y, 0) - G(x_1, t_1; y, 0) u_{0\varepsilon}(y))\, dy \right| \\ & + \left| \int_0^{t_2} \int_\Omega G(x_2, t_2; y, \tau)(v^p(y,\tau) - a u^r(y,\tau) + a\varepsilon^r)\, dy\, d\tau \right. \\ & \left. - \int_0^{t_1} \int_\Omega G(x_1, t_1; y, \tau)(v^p(y,\tau) - a u^r(y,\tau) + a\varepsilon^r)\, dy\, d\tau \right| \\ & + \left| \int_0^{t_2} \int_{\partial\Omega} G(x_2, t_2; \xi, \tau) \left(\int_\Omega \phi(\xi, y, \tau) s_1^m(y,\tau)\, dy \right) dS_\xi\, d\tau \right. \\ & \left. - \int_0^{t_1} \int_{\partial\Omega} G(x_1, t_1; \xi, \tau) \left(\int_\Omega \phi(\xi, y, \tau) s_1^m(y,\tau)\, dy \right) dS_\xi\, d\tau \right|. \end{aligned} \tag{13}$$

Since $\int_\Omega G(x,t;y,0) u_{0\varepsilon}(y)\, dy$ is a continuous function in $\overline{Q_T}$ (see [29])

$$\left| \int_\Omega (G(x_2, t_2; y, 0) - G(x_1, t_1; y, 0)) u_{0\varepsilon}(y)\, dy \right| < \frac{\epsilon}{3} \tag{14}$$

for small values of δ.

Let us consider the second term in the right hand side of (13). We set $h(y,\tau) = v^p(y,\tau) - a u^r(y,\tau) + a\varepsilon^r$, $H = \sup_{Q_T} |h(y,\tau)|$ and suppose for the definiteness that $t_2 \geqslant t_1$. Using (7), (8) and the continuity of the Green's function for $t > \tau \geqslant 0$, we have

$$\begin{aligned} & \left| \int_0^{t_2} \int_\Omega G(x_2, t_2; y, \tau) h(y,\tau)\, dy\, d\tau - \int_0^{t_1} \int_\Omega G(x_1, t_1; y, \tau) h(y,\tau)\, dy\, d\tau \right| \\ & \leqslant H \left\{ \int_{t_1-\gamma}^{t_2} \int_\Omega G(x_2, t_2; y, \tau)\, dy\, d\tau + \int_0^{t_1-\gamma} \int_\Omega |G(x_2, t_2; y, \tau) - G(x_1, t_1; y, \tau)|\, dy\, d\tau \right. \\ & \left. + \int_{t_1-\gamma}^{t_1} \int_\Omega G(x_1, t_1; y, \tau)\, dy\, d\tau \right\} < \frac{\epsilon}{3} \end{aligned} \tag{15}$$

with an appropriate choice of γ and δ.

Similarly, we estimate the third term in the right hand side of (13)

$$\left| \int_0^{t_2} \int_{\partial\Omega} G(x_2, t_2; \xi, \tau) \left(\int_\Omega \phi(\xi, y, \tau) s_1^m(y, \tau)\, dy \right) dS_\xi\, d\tau \right.$$
$$\left. - \int_0^{t_1} \int_{\partial\Omega} G(x_1, t_1; \xi, \tau) \left(\int_\Omega \phi(\xi, y, \tau) s_1^m(y, \tau)\, dy \right) dS_\xi\, d\tau \right| < \frac{\epsilon}{3} \qquad (16)$$

for small values of δ. From (13)–(16) we derive (12).

The Ascoli–Arzelá theorem guarantees the relative compactness of AB. Thus we are able to apply a corollary of the Schauder–Tikhonov fixed point theorem (see [30]) and conclude that A has a fixed point in B if T is small. Now if $(u_\varepsilon, v_\varepsilon)$ is a fixed point of A then it is a solution of (3) in Q_T. The uniqueness of the solution follows from a comparison principle for (3) which can be proved in a similar way as Theorem 3 in the next section. $\rhd$

Now we are ready to prove the existence of a local solution of (1).

Theorem 2. *For small values of T problem* (1) *has a maximal solution in Q_T.*

$\lhd$ Now, let $\varepsilon_2 > \varepsilon_1$. Then it is easy to show that $(u_{\varepsilon_2}(x,t), v_{\varepsilon_2}(x,t))$ is a supersolution of problem (3) with $\varepsilon = \varepsilon_1$ in Q_T for some T. Applying a comparison principle to problem (3), we have $u_{\varepsilon_2}(x,t) \geqslant u_{\varepsilon_1}(x,t)$ and $v_{\varepsilon_2}(x,t) \geqslant v_{\varepsilon_1}(x,t)$ in Q_T. Using the last inequalities and the continuation principle of solutions we deduce that the existence time of $(u_\varepsilon(x,t), v_\varepsilon(x,t))$ does not decrease as $\varepsilon \searrow 0$. Taking $\varepsilon \to 0$, we get

$$u_{\max}(x,t) = \lim_{\varepsilon\to 0} u_\varepsilon(x,t) \geqslant 0, \quad v_{\max}(x,t) = \lim_{\varepsilon\to 0} v_\varepsilon(x,t) \geqslant 0, \qquad (17)$$

and $(u_{\max}(x,t), v_{\max}(x,t))$ exists in Q_T for some T. By dominated convergence theorem $(u_{\max}(x,t), v_{\max}(x,t))$ satisfies the following equations:

$$u_{\max}(x,t) = \int_\Omega G(x,t;y,0) u_0(y)\, dy + \int_0^t \int_\Omega G(x,t;y,\tau) \big(v_{\max}^p(y,\tau) - a u_{\max}^r(y,\tau)\big)\, dy\, d\tau$$
$$+ \int_0^t \int_{\partial\Omega} G(x,t;\xi,\tau) \left(\int_\Omega \phi(\xi,y,\tau) u_{\max}^m(y,\tau)\, dy \right) dS_\xi\, d\tau,$$

$$v_{\max}(x,t) = \int_\Omega G(x,t;y,0) v_0(y)\, dy + \int_0^t \int_\Omega G(x,t;y,\tau) \big(u_{\max}^q(y,\tau) - b v_{\max}^s(y,\tau)\big)\, dy\, d\tau$$
$$+ \int_0^t \int_{\partial\Omega} G(x,t;\xi,\tau) \left(\int_\Omega \psi(\xi,y,\tau) v_{\max}^n(y,\tau)\, dy \right) dS_\xi\, d\tau.$$

By the properties of the Green function $(u_{\max}(x,t), v_{\max}(x,t))$ is a solution of (1) in Q_T. Let $(u(x,t), v(x,t))$ be another solution of (1) in Q_T. Applying a comparison principle to problem (3), we have $u_\varepsilon(x,t) \geqslant u(x,t)$, $v_\varepsilon(x,t) \geqslant v(x,t)$ in Q_T. Taking $\varepsilon \to 0$ we deduce that $u_{\max} \geqslant u(x,t)$, $v_{\max} \geqslant v(x,t)$ in Q_T. Therefore, $(u_{\max}(x,t), v_{\max}(x,t))$ is a maximal solution of (1) in Q_T. $\rhd$

3. Comparison Principle

We start this section with a comparison principle for problem (1).

Theorem 3. *Let $(\overline{u}(x,t), \overline{v}(x,t))$ and $(\underline{u}(x,t), \underline{v}(x,t))$ be a supersolution and a subsolution of problem* (1) *in Q_T, respectively. Suppose that $\underline{u}(x,t) > 0$ or $\overline{u}(x,t) > 0$ in $Q_T \cup \Gamma_T$ and $\underline{v}(x,t) > 0$ or $\overline{v}(x,t) > 0$ in $Q_T \cup \Gamma_T$ if $\min(p,q,m,n) < 1$. Then $\overline{u}(x,t) \geqslant \underline{u}(x,t)$ and $\overline{v}(x,t) \geqslant \underline{v}(x,t)$ in $Q_T \cup \Gamma_T$.*

$\lhd$ Suppose at first that

$$\min(p,q,m,n) \geqslant 1. \tag{18}$$

Let functions $u_{0\varepsilon}(x)$, $v_{0\varepsilon}(x)$ have the same properties as in the previous section but

$$0 \leqslant u_{0\varepsilon}(x) - \underline{u}(x,0) \leqslant 2\varepsilon, \quad 0 \leqslant v_{0\varepsilon}(x) - \underline{v}(x,0) \leqslant 2\varepsilon. \tag{19}$$

Then problem (1) with $u_0(x) = \underline{u}(x,0)$, $v_0(x) = \underline{v}(x,0)$ has a maximal solution $(u_{\max}(x,t), v_{max}(x,t))$, and, moreover,

$$u_{\max}(x,t) = \lim_{\varepsilon\to 0} u_\varepsilon(x,t), \quad v_{\max}(x,t) = \lim_{\varepsilon\to 0} v_\varepsilon(x,t),$$

where $(u_\varepsilon(x,t)\, v_\varepsilon(x,t))$ is a solution of (3). To establish the theorem it is enough to prove that

$$\underline{u}(x,t) \leqslant u_{\max}(x,t) \leqslant \overline{u}(x,t), \quad \underline{v}(x,t) \leqslant v_{\max}(x,t) \leqslant \overline{v}(x,t) \ \text{ in } Q_{T_0} \cup S_{T_0} \tag{20}$$

for any $T_0 \in (0,T)$. We show only that

$$u_{\max}(x,t) \leqslant \overline{u}(x,t), \quad v_{\max}(x,t) \leqslant \overline{v}(x,t) \ \text{ in } Q_{T_0} \cup S_{T_0} \tag{21}$$

for any $T_0 \in (0,T)$ since the proof of other inequalities in (20) is similar. We set

$$\omega_1(x,t) = u_\varepsilon(x,t) - \overline{u}(x,t), \quad \omega_2(x,t) = v_\varepsilon(x,t) - \overline{v}(x,t). \tag{22}$$

By virtue of (3), (19), (22) and the definition of a supersolution we conclude

$$\begin{cases} \omega_{1t} \leqslant \Delta\omega_1 + p\theta_1^{p-1}\omega_2 - ar\theta_2^{r-1}\omega_1 + a\varepsilon^r, & (x,t) \in Q_{T_0}, \\ \omega_{2t} \leqslant \Delta\omega_2 + q\theta_3^{q-1}\omega_1 - bs\theta_4^{s-1}\omega_2 + b\varepsilon^s, & (x,t) \in Q_{T_0}, \\ \frac{\partial\omega_1(x,t)}{\partial\nu} \leqslant \int\limits_\Omega \phi(x,y,t) m\theta_5^{m-1}\omega_1(y,t)\,dy, & (x,t) \in S_{T_0}, \\ \frac{\partial\omega_2(x,t)}{\partial\nu} \leqslant \int\limits_\Omega \psi(x,y,t) n\theta_6^{n-1}\omega_2(y,t)\,dy, & (x,t) \in S_{T_0}, \\ \omega_1(x,0) \leqslant 2\varepsilon, \quad \omega_2(x,0) \leqslant 2\varepsilon, & x \in \Omega, \end{cases} \tag{23}$$

where θ_i, $i = 1,4,6$, are some continuous functions in $\overline{Q_T}$ between $v_\varepsilon(x,t)$ and $\overline{v}(x,t)$, and θ_i, $i = 2,3,5$, are some continuous functions in $\overline{Q_T}$ between $u_\varepsilon(x,t)$ and $\overline{u}(x,t)$. Based on the assumptions made, we have

$$\begin{gathered} 0 \leqslant \overline{u}(x,t) \leqslant M, \quad 0 \leqslant \overline{v}(x,t) \leqslant M, \quad \varepsilon \leqslant u_\varepsilon(x,t) \leqslant M, \quad \varepsilon \leqslant v_\varepsilon(x,t) \leqslant M \quad \text{in } \overline{Q_{T_0}}, \\ 0 \leqslant \phi(x,y,t) \leqslant M \quad \text{and} \quad 0 \leqslant \psi(x,y,t) \leqslant M \quad \text{in } \partial\Omega \times \overline{Q_{T_0}}, \end{gathered} \tag{24}$$

where M is some positive constant. It follows from (24) that powers of $\theta_i, = 1,\dots,6$ in (23) are positive bounded functions in Q_{T_0}, and, moreover, $\theta_1^{p-1} \leqslant M^{p-1}$, $\theta_3^{q-1} \leqslant M^{q-1}$, $\theta_5^{m-1} \leqslant M^{m-1}$, $\theta_6^{n-1} \leqslant M^{n-1}$. Let us define the functions

$$\omega_1(x,t) = f(x,t) + \varepsilon_1 \exp(\alpha t) h(x), \quad \omega_2(x,t) = g(x,t) + \varepsilon_1 \exp(\alpha t) h(x), \tag{25}$$

where

$$h(x) \in C^2(\overline{\Omega}), \quad h(x) > 1 \quad \text{in } \overline{\Omega}, \quad \frac{\partial h(x)}{\partial \nu} > \{mM^m + nM^n\} \int\limits_{\Omega} h(y)\,dy \quad \text{on } \partial\Omega, \tag{26}$$

$$\varepsilon_1 = 2\varepsilon + \varepsilon^r + \varepsilon^s, \quad \alpha > \max_{\overline{\Omega}} \frac{\Delta h(x)}{h(x)} + pM^{p-1} + qM^{q-1} + a + b. \tag{27}$$

We substitute the functions from (25) into the first inequality of (23) to derive

$$f_t(x,t) + \alpha\varepsilon_1 \exp(\alpha t)h(x) \leqslant \Delta f(x,t) + \varepsilon_1 \exp(\alpha t)\Delta h(x) + p\,\theta_1^{p-1}(x,t)g(x,t)$$
$$+p\,\theta_1^{p-1}(x,t)\varepsilon_1 \exp(\alpha t)h(x) - ar\theta_2^{r-1}(x,t)f(x,t) - ar\theta_2^{r-1}(x,t)\varepsilon_1 \exp(\alpha t)h(x) + a\varepsilon^r \ \text{ in } Q_{T_0}.$$

Hence by (26), (27) we get

$$f_t(x,t) < \Delta f(x,t) + p\,\theta_1^{p-1}(x,t)g(x,t) - ar\,\theta_2^{r-1}(x,t)f(x,t) \quad \text{in } Q_{T_0}. \tag{28}$$

In a similar way we obtain

$$g_t(x,t) < \Delta g(x,t) + q\theta_3^{q-1}(x,t)f(x,t) - bs\,\theta_4^{s-1}(x,t)g(x,t) \quad \text{in } Q_{T_0}.$$

Substituting the functions from (25) into the third and fourth inequalities of (23), we deduce that

$$\frac{\partial f(x,t)}{\partial \nu} < \int\limits_{\Omega} \phi(x,y,t)m\theta_5^{m-1}(y,t)f(y,t)\,dy \quad \text{on } S_{T_0} \tag{29}$$

and

$$\frac{\partial g(x,t)}{\partial \nu} < \int\limits_{\Omega} \psi(x,y,t)n\theta_6^{n-1}(y,t)g(y,t)\,dy \quad \text{on } S_{T_0}.$$

From (23), (25), (27) we have $f(x,0) < 0$, $g(x,0) < 0$ in $\overline{\Omega}$. We prove that

$$f(x,t) < 0, \quad g(x,t) < 0 \quad \text{in } Q_{T_0} \cup S_{T_0}. \tag{30}$$

Let (30) not be true. Then there exists $(x_0,t_0) \in Q_{T_0} \cup S_{T_0}$ such that $t_0 > 0$, $f(x,t) < 0$, $g(x,t) < 0$ for $0 \leqslant t < t_0$ and $f(x_0,t_0) = 0$ or $g(x_0,t_0) = 0$ for some $x_0 \in \overline{\Omega}$. Suppose that $f(x_0,t_0) = 0$. If $x_0 \in \Omega$, then $f_t(x_0,t_0) = 0$, $\Delta f(x_0,t_0) \leqslant 0$ and by (28) we get

$$0 = f_t(x_0,t_0) < \Delta f(x_0,t_0) + p\,\theta_1^{p-1}(x_0,t_0)g(x_0,t_0) \leqslant 0.$$

If $x_0 \in \partial\Omega$, then (29) yields

$$0 \leqslant \frac{\partial f(x_0,t_0)}{\partial \nu} < \int\limits_{\Omega} \phi(x_0,y,t_0)m\theta_5^{m-1}(y,t_0)f(y,t_0)\,dy \leqslant 0.$$

If $g(x_0,t_0) = 0$ we can obtain a contradiction in a similar way.

Taking $\varepsilon \to 0$ in (30) and using (22), (25)–(27), we deduce (21).

If (18) doesn't hold we can introduce $\omega_1 = \underline{u}(x,t) - \overline{u}(x,t)$, $\omega_2(x,t) = \underline{v}(x,t) - \overline{v}(x,t)$ and prove the theorem in a similar way using the positiveness of some functions in a subsolution and a supersolution. ▷

Remark. Under the condition $\min(r,s) < 1$ we don't suppose the positiveness of a subsolution or a supersolution in Theorem 3. A comparison principle for problem (1) with zero Dirichlet boundary condition is proved in [23] for the case $\min(r,s) < 1$ under the conditions $\overline{u}(x,t) > 0$ and $\overline{v}(x,t) > 0$ in $\overline{Q_T}$.

To prove the uniqueness of a solution of problem (1) we need the following statement.

Lemma. *Let (u, v) be a solution of (1) in Q_T. If $\min(r, s) \geqslant 1$ and $u_0(x) \not\equiv 0$ or $v_0(x) \not\equiv 0$ in Ω, then $u(x, t) > 0$ and $v(x, t) > 0$ in $Q_T \cup S_T$. If $u_0(x) > 0$ and $v_0(x) > 0$ in $\overline{\Omega}$ and either $p < r < 1$, $q < s < 1$ or $\max(r, s) \geqslant 1$, then $u(x, t) > 0$ and $v(x, t) > 0$ in $Q_T \cup \Gamma_T$.*

$\lhd$ Let $\min(r, s) \geqslant 1$ and $u_0(x) \not\equiv 0$ in Ω. Denote

$$M = \sup_{Q_{T_0}} u(x, t),$$

where $T_0 \in (0, T)$. We set $h(x, t) = u(x, t) \exp(\lambda t)$ with $\lambda > aM^{r-1}$. Then we have in Q_{T_0}

$$h_t - \Delta h = \exp(\lambda t)(\lambda u + u_t - \Delta u) \geqslant \exp(\lambda t) u \left(\lambda - a u^{r-1}\right) \geqslant 0.$$

Since $h(x, 0) = u_0(x) \geqslant 0$ and $u_0(x) \not\equiv 0$ in Ω, by the strong maximum principle $h(x, t) > 0$ in Q_{T_0}. Let $h(x_0, t_0) = 0$ at some point $(x_0, t_0) \in S_{T_0}$. Then according to Theorem 3.6 of [31] it yields $\partial h(x_0, t_0)/\partial\nu < 0$, which contradicts the boundary condition for u in (1). Hence $u(x, t) > 0$ in $Q_T \cup S_T$ since T_0 may be any in $(0, T)$.

We show that

$$v(x, t) > 0 \quad \text{in } Q_T \cup S_T. \tag{31}$$

If either $v_0(x) \not\equiv 0$ or $v_0(x) \equiv 0$ and there is no $\tau \in (0, T_0)$, such that

$$v(x, t) \equiv 0 \quad \text{in } Q_\tau, \tag{32}$$

then we prove (31) as above. If there exists $\tau \in (0, T_0)$, such that (32) holds, then we have a contradiction in Q_τ with the second equation in (1).

Suppose now that $u_0(x) > 0$ and $v_0(x) > 0$ in $\overline{\Omega}$ and $p < r < 1$, $q < s < 1$. Let

$$\varepsilon_2 = \min\left\{\min_{\overline{\Omega}} u_0(x), \min_{\overline{\Omega}} v_0(x), \left(\frac{1}{a}\right)^{\frac{1}{r-p}}, \left(\frac{1}{b}\right)^{\frac{1}{s-q}}\right\}.$$

It is easy to see that $(\varepsilon_2, \varepsilon_2)$ is a subsolution of (1) in Q_T and by Theorem 3 $u(x, t) \geqslant \varepsilon_2$ and $v(x, t) \geqslant \varepsilon_2$ in $Q_T \cup \Gamma_T$.

If $u_0(x) > 0$ and $v_0(x) > 0$ in $\overline{\Omega}$ and $s \geqslant 1$, then arguing as above, we obtain

$$v(x, t) \geqslant \varepsilon_3 \quad \text{in } Q_T \cup \Gamma_T$$

for some $\varepsilon_3 > 0$. Set

$$\varepsilon_4 = \min\left\{\min_{\overline{\Omega}} u_0(x), \left(\frac{\varepsilon_3^p}{a}\right)^{\frac{1}{r}}\right\}.$$

Then $\underline{u}(x, t) = \varepsilon_4$ is a subsolution of the following problem

$$\begin{cases} u_t = \Delta u + v^p - a u^r, & (x, t) \in Q_T, \\ \frac{\partial u(x,t)}{\partial\nu} = \int\limits_{\Omega} \phi(x, y, t) u^m(y, t)\, dy, & (x, t) \in S_T, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \tag{33}$$

Now by the comparison principle for (33) we conclude that $u(x, t) \geqslant \varepsilon_4$ in $Q_T \cup \Gamma_T$. The proof is similar in the remaining case. $\rhd$

As a simple consequence of Theorem 3 and Lemma, we get the following uniqueness result for problem (1).

Theorem 4. *Let (u, v) be a solution of (1) in Q_T. Suppose that at least one of the following conditions holds:*

1) $u(x,t) > 0$ *and* $v(x,t) > 0$ *in* $Q_T \cup \Gamma_T$;

2) $\min(p,q,m,n) \geqslant 1$;

3) $\min(p,q,m,n) < 1$, $u_0(x) > 0$ *and* $v_0(x) > 0$ *in* $\overline{\Omega}$ *and either* $p < r < 1$, $q < s < 1$ *or* $\max(r,s) \geqslant 1$.

Then a solution of problem (1) is unique in Q_T.

Now we show that problem (1) may have a nonunique solution in Q_T.

Theorem 5. *Let* $u_0(x) = v_0(x) \equiv 0$ *and at least one of the following conditions hold:*

1) $pq < 1$, $r > \lambda p$ *and* $s > \frac{q}{\lambda}$ *for* $\lambda \in \left[q, \frac{1}{p}\right]$;

2) $\min(1,r) > m$ *and* $\phi(x,y_1,t_1) > 0$ *for any* $x \in \partial\Omega$ *and some* $y_1 \in \partial\Omega$ *and* $t_1 \in [0,T)$;

3) $\min(1,s) > n$ *and* $\psi(x,y_2,t_2) > 0$ *for any* $x \in \partial\Omega$ *and some* $y_2 \in \partial\Omega$ *and* $t_2 \in [0,T)$.

Then problem (1) has a nonunique solution in Q_T.

$\lhd$ We note that problem (1) with trivial initial datum $u_0(x) = v_0(x) \equiv 0$ has the trivial solution $(0,0)$. As we showed in Theorem 2 a maximal solution $(u_{\max}(x,t), v_{\max}(x,t))$ satisfies (17), where $(u_\varepsilon(x,t), v_\varepsilon(x,t))$ is some positive in $\overline{Q}_T$ supersolution of (1). To prove the theorem we construct a nontrivial nonnegative subsolution $(\underline{u}(x,t)), \underline{v}(x,t)$ of (1) with trivial initial datum. By Theorem 3 then we have $u_\varepsilon(x,t) \geqslant \underline{u}(x,t)$, $v_\varepsilon(x,t) \geqslant \underline{v}(x,t)$ and therefore a maximal solution is nontrivial.

Let the conditions 1) hold. We put

$$\underline{u}(x,t) = (Ctw(x,t))^\alpha, \quad \underline{v}(x,t) = (Ctw(x,t))^\beta, \tag{34}$$

where positive constants C, α, β will be chosen later and $w(x,t)$ is a solution of the following problem

$$\begin{cases} w_t = \Delta w, & (x,t) \in Q_T, \\ w(x,t) = 0, & (x,t) \in S_T, \\ w(x,0) = w_0(x), & x \in \Omega. \end{cases}$$

Here $w_0(x)$ is a bounded nontrivial nonnegative continuous function, which satisfies the boundary condition. By the strong maximum principle

$$0 < w(x,t) < M = \sup_{x\in\Omega} w_0(x) \quad \text{for } (x,t) \in Q_T. \tag{35}$$

We note that

$$\underline{u}(x,0) = \underline{v}(x,0) = 0 \text{ for } x \in \Omega \quad \text{and} \quad \frac{\partial \underline{u}(x,t)}{\partial \nu} \leqslant 0, \ \frac{\partial \underline{v}(x,t)}{\partial \nu} \leqslant 0 \text{ for } (x,t) \in S_T. \tag{36}$$

Suppose at first that

$$\beta = \alpha q + 1, \tag{37}$$

where $\alpha \geqslant \frac{p+1}{1-pq}$. Then

$$\alpha \geqslant \beta p + 1. \tag{38}$$

We put $\lambda = \frac{\alpha q + 1}{\alpha}$. It is easy to check that λ takes all values in $\left(q, \frac{q+1}{p+1}\right]$ if α takes all values in $\left[\frac{p+1}{1-pq}, \infty\right)$. Since $r > \lambda p$ and $s > \frac{q}{\lambda}$ for $\lambda \in \left(q, \frac{q+1}{p+1}\right]$ we have

$$\alpha r - \beta p > 0, \quad \beta s - \alpha q > 0. \tag{39}$$

By virtue of (34)–(39), after simple calculations we obtain

$$\begin{aligned}\underline{u}_t - \Delta\underline{u} - \underline{v}^p + a\underline{u}^r &\leqslant \alpha t^{\alpha-1}(Cw(x,t))^{\alpha} - (Ctw(x,t))^{\beta p} + a(Ctw(x,t))^{\alpha r} \\ &\leqslant (Ctw(x,t))^{\beta p}\Big\{\alpha T^{\alpha-\beta p-1}(CM)^{\alpha-\beta p} + a(CTM)^{\alpha r-\beta p} - 1\Big\} \leqslant 0, \qquad (40)\end{aligned}$$

$$\begin{aligned}\underline{v}_t - \Delta\underline{v} - \underline{u}^q + b\underline{v}^s &\leqslant \beta t^{\beta-1}(Cw(x,t))^{\beta} - (Ctw(x,t))^{\alpha q} + b(Ctw(x,t))^{\beta s} \\ &\leqslant (Ctw(x,t))^{\alpha q}\Big\{\beta CM + b(CTM)^{\beta s-\alpha q} - 1\Big\} \leqslant 0 \qquad (41)\end{aligned}$$

for $(x,t) \in Q_T$ if C is sufficiently small. It is easy to see that (40), (41) hold for $r > qp$ and $s > 1$ under suitable choice of α and C. Thus by (36), (40), (41) we conclude that $(\underline{u}(x,t), \underline{v}(x,t))$ is a nontrivial subsolution of (1) with trivial initial datum and the theorem is proved for $\lambda \in \big[q, \frac{q+1}{p+1}\big]$. To prove the theorem for $\lambda \in \big(\frac{q+1}{p+1}, \frac{1}{p}\big]$ we put $\alpha = \beta p + 1$ with $\beta \geqslant \frac{q+1}{1-pq}$ and argue in a similar way.

Now we suppose that the conditions 2) hold. Let us consider the following problem

$$\begin{cases} u_t = \Delta u - au^r, & (x,t) \in Q_T, \\ \frac{\partial u(x,t)}{\partial \nu} = \int\limits_{\Omega} \phi(x,y,t)u^m(y,t)\,dy, & (x,t) \in S_T, \\ u(x,0) = 0, & x \in \Omega. \end{cases} \qquad (42)$$

It is proved in [9] that (42) has a nontrivial nonnegative solution $u_n(x,t)$. Then a pair of functions $(u_n(x,t), 0)$ is a nontrivial subsolution of (1) with trivial initial datum.

The remaining case can be treated similarly. ⊳

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НАЧАЛЬНО-КРАЕВАЯ ЗАДАЧА ДЛЯ СИСТЕМЫ ПОЛУЛИНЕЙНЫХ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ С ПОГЛОЩЕНИЕМ И НЕЛИНЕЙНЫМИ НЕЛОКАЛЬНЫМИ ГРАНИЧНЫМИ УСЛОВИЯМИ

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Аннотация. В работе мы рассматриваем классические решения начально-краевой задачи для системы полулинейных параболических уравнений с поглощением и нелинейными нелокальными граничными условиями. Нелинейности в уравнениях и граничных условиях могут не удовлетворять условию Липшица. Для доказательства существования решения мы регуляризуем исходную задачу. Используя теорему Шаудера — Тихонова о неподвижной точке, доказывается существование локального решения регуляризованной задачи. Показано, что предел решений регуляризованной задачи является максимальным решением исходной задачи. Используя свойства максимального решения, доказывается принцип сравнения. При этом не делается дополнительных предположений, когда нелинейности в поглощении не удовлетворяют условию Липшица. Найдены условия, при выполнении которых решения являются положительными функциями. Устанавливается единственность решения. Показано, что нулевое решение может быть неединственным.

Ключевые слова: система полулинейных параболических уравнений, нелокальные граничные условия, существование решения, принцип сравнения.

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