Ministry of Education of the Republic of Belarus
Educational Establishment
“Vitebsk State University named after P.M. Masherov”
Department Mathematics of the Educational

M.N. Podoksenov, L.V. Kazhekina

DESCRIPTIVE GEOMETRY HANDS-ON TRAINING

Practical course for self-study activities

Vitebsk
VSU named after P.M. Masherov
2025

UDC 514.18(075.8)
LBC 22.151.34я13
P78

Published by the decision of the Research and Methodology Council of the Educational Establishment “Vitebsk State University named after P.M. Masherov”. Minutes № 3 d/d 26.02.2025.

Authors: Associate Professor of the Department of Mathematics of the Educational Establishment “Vitebsk State University named after P.M. Masherov”, Candidate of Physical and Mathematical Sciences, Associate Professor **M.N. Podoksenov;** Senior Lecture of the World Languages Department of the Educational Establishment “Vitebsk State University named after P.M. Masherov” **L.V. Kazhekina**

R e v i e w e r :
Associate Professor of the Department of Mathematics and Computer Security of Polotsk State University named after Euphrosyne of Polotsk, Candidate Pedagical Sciences, Associate Professor *A.P. Matylenok*

Podoksenov, M.N.
P78 Descriptive Geometry. Hands-on Training : practical course for self-study activities / M.N. Podoksenov, L.V. Kazhekina. – Vitebsk : VSU named after P.M. Masherov, 2025. – 43 p.

This study guide is intended for students majoring in general higher education, studying in English as part of an experimental project and mastering the academic discipline “Descriptive Geometry and Engineering Graphics” as an additional program. Theoretical material, examples of problem solutions are presented, individual versions of problems and examples of test assignments are attached.

UDC 514.18(075.8)
LBC 22.151.34я13

TABLE OF CONTENTS

Descriptive geometry is the branch of geometry which allows the representation of three-dimensional objects in two dimensions by using a specific set of procedures. The resulting techniques are important for engineering, architecture, design and in art. [https://en.wikipedia.org/wiki/Descriptive_geometry]

CHAPTER 1. POINTS, LINES AND PLANES

1.1 *Parallel projection*

Definition. Let us choose some plane σ in space and a vector $\vec{\mathbf{p}}$ that is not parallel to σ (*Figure* 1.1). Let $\bar{A}$ be an arbitrary point in space. Let us draw a line through A that is parallel to $\vec{\mathbf{p}}$. This line will intersect the plane σ at a point A_o, which is called *the parallel projection of the point A onto the plane σ in the direction of the vector* $\vec{\mathbf{p}}$.

The set of projections of all points of the figure Φ make up the figure Φ_o, which is called the projection of the figure Φ. If the vector $\vec{\mathbf{p}}\perp\sigma$, then the projection is called orthogonal.

In what follows we will assume that all lines and segments under consideration are not parallel to the vector $\vec{\mathbf{p}}$.

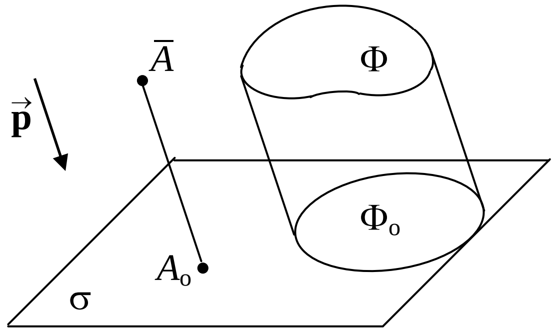

*Fig.*1.1

Properties of parallel projection

1. The projection of a line is a line. The projections of parallel lines are parallel or coincide.

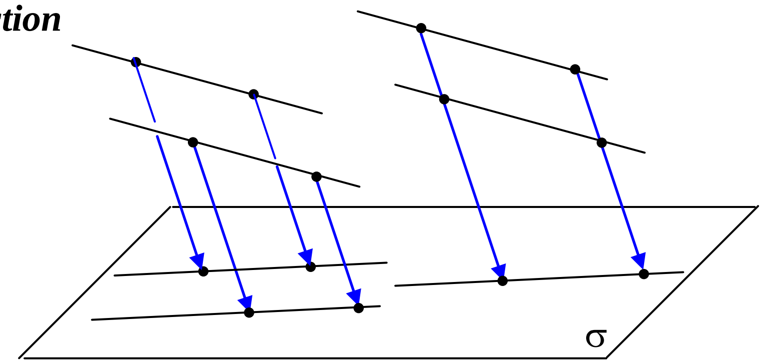

*Fig.*1.2

2. The projection of the segment *AB* is the segment A_oB_o, where A_o, is the projection of point *A*, B_o, is the projection of point *B*.

3. If a segment is parallel to the direction of projection, then it is projected to a point.

4. In parallel projection, the simple relationship of three points is preserved. In particular, the projection of the midpoint of the segment *AB* is the midpoint of the segment A_oB_o.

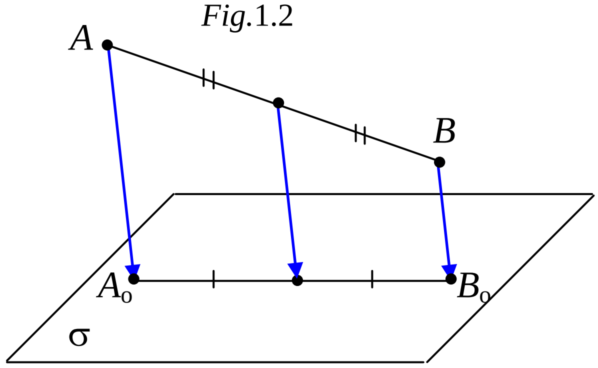

*Fig.*1.3

6. The projections of parallel segments, or segments lying on the same line, are parallel or lie on the same line.

7. The projections of parallel segments, or segments lying on the same line (*Fig*. 1.4) are proportional to these segments:

$$\frac{A_oB_o}{C_oD_o} = \frac{\overline{A}\overline{B}}{\overline{C}\overline{D}}. \qquad (1)$$

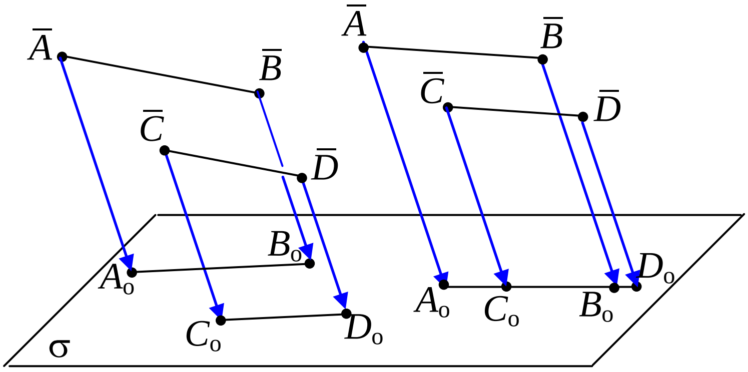

Fig.1.4

The proof of these statements does not go beyond the school mathematics course. You can prove them yourself.

Proportion (1) can also be rewritten as follows:

$$\frac{A_oB_o}{AB} = \frac{C_oD_o}{CD} = k = \text{const}. \qquad (2)$$

This means that the ratio of the lengths of the projections to the lengths of the segments themselves is a constant value (if the segments lie on parallel lines or on the same line), and it is called the distortion coefficient. It is easy to prove that $k = \cos\alpha$, where α is the angle of inclination of the line to the plane of projections. In particular, the following property is true.

8. If a segment is parallel to the projection plane, then it is projected in full size.

Although, according to the definition, the projection vector $\vec{\mathbf{p}}$ can be located at any angle to the projection plane, the orthogonal projection is most often used.

1.2 *Central projection*

Let the plane of projections α be given and the point $S \notin \alpha$ be the projection center. *The central projection* of an arbitrary point M is the point $M' = SM \cap \alpha$. In this case, the line is called the projecting line. The projection of the figure F is the figure F', consisting of the projections of all points of the figure F.

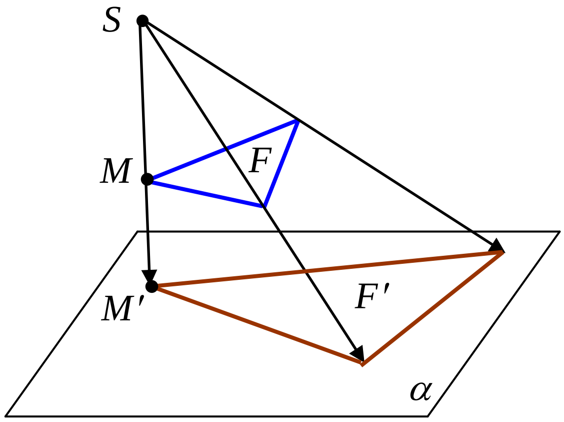

Fig.5

To construct a projection of any figure, it is not necessary to construct projections of all its points. Thus, to construct a projection of a line, it is enough to construct projections of two of its points; to construct a projection of a polygon, it is enough to construct projections of its vertices.

It is the central projection that is used in artistic graphics, since the image that appears on the retina of the eye is the central projection of the original. Here, the clarity of the image in the drawing is important, and the central projection makes it possible to evaluate which of the depicted objects are closer and which are further away.

Central projection is inconvenient in technical graphics, where the main requirement is the ability to obtain an accurate representation of the shape and size of an object from an image. For example, central projections of parallel lines may turn out to be intersecting lines, and projections of some points may be completely absent.

If the line SM is parallel to the projection plane, then the point M has no projection (*Fig*. 1.6).

Central projection distorts the proportionality of segments lying on the same line or on parallel lines. For example, in *Figure* 1.7, segments *AB*, *BC*, and *CD* are all different, and their projections are equal.

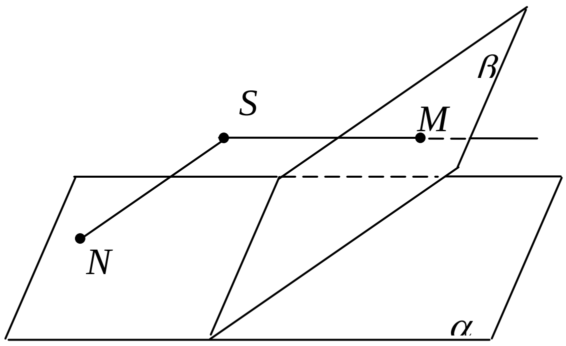

Fig.1.

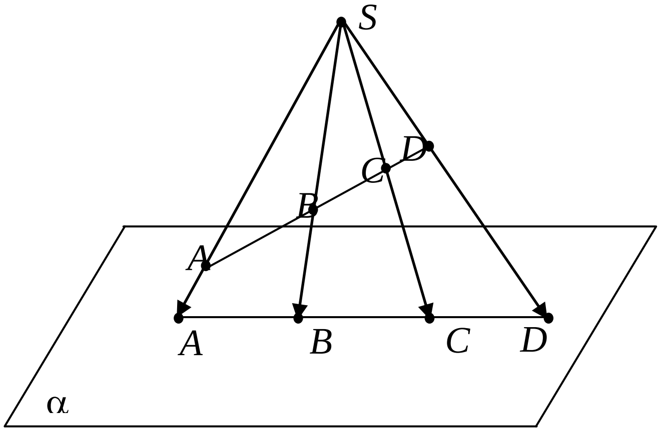

Fig.1.

1.3 *Reversibility of a drawing. A point in a system of several projection planes*

Definition. Reversibility of a drawing is the ability to uniquely determine the position of a point in space based on its projections.

For example, in *Figure* 8 the projections of two different points *A* and *B* are one point $A_o = B_o$. Thus, one projection of a point does not determine the position of this point in space.

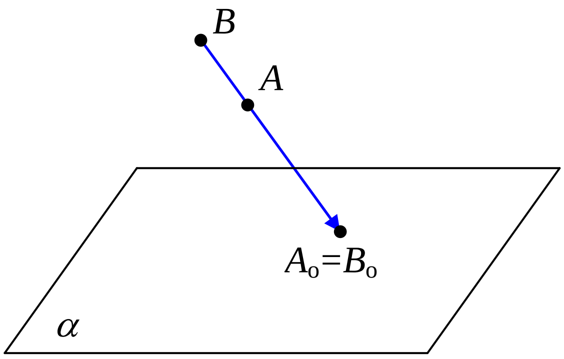

Fig.1.8

To obtain a reversible drawing, the figure must be projected onto two or more planes. It is convenient to carry out projections onto three mutually perpendicular planes in space, which can be considered coordinate planes (*Fig*. 1.9).

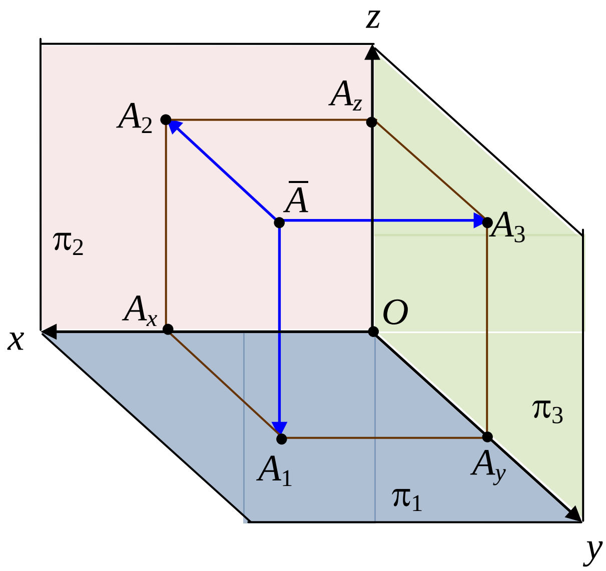

Fig.1.9

Three planes divide the space into 8 octants. Their numbering is not so important; what is important is that the projected object is usually placed in the first octant, in which point $\bar{A}$ is located. Plane π_1 is called horizontal, π_2 – frontal, π_3 – profile. Accordingly, the projections of point $\bar{A}$ are called by these names: A_1 – horizontal projection, A_2 – frontal projection, A_3 – profile projection.

In order to obtain a flat image, we rotate plane π_1 around the Ox axis downwards, and plane π_3 around the Oz axis to the right until it coincides with plane π_2 (*Fig*. 1.10). In this case, axis Oy is depicted twice. Line A_1A_2 is called the vertical line of communication, A_1A_3 is called the horizontal line of communication, A_2A_3 is called the broken line of communication.

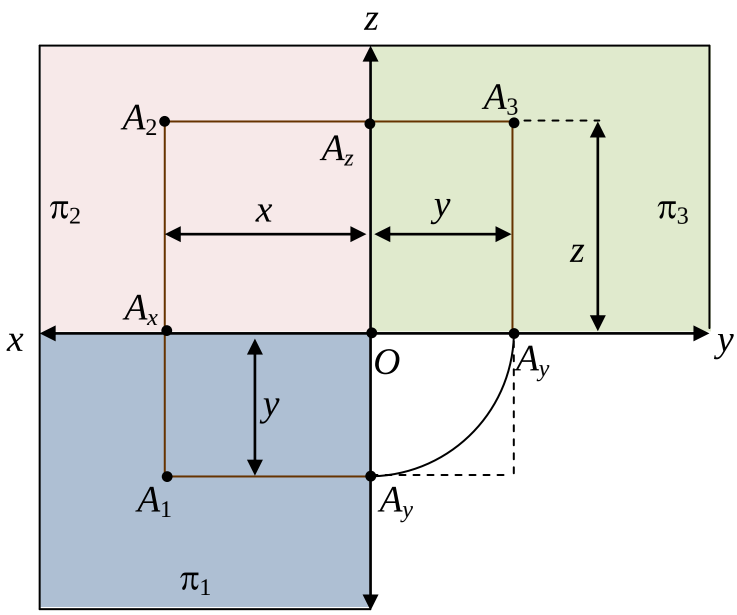

Fig.1.10

Having only two of the three projections of point $\bar{A}$, it is possible to reconstruct the missing third projection using the connection lines. Therefore, it is sufficient to have images of only the horizontal and vertical projection planes and the projections of the point onto these planes in order to be able to uniquely determine the position of the point in space (*Fig*. 1.11).

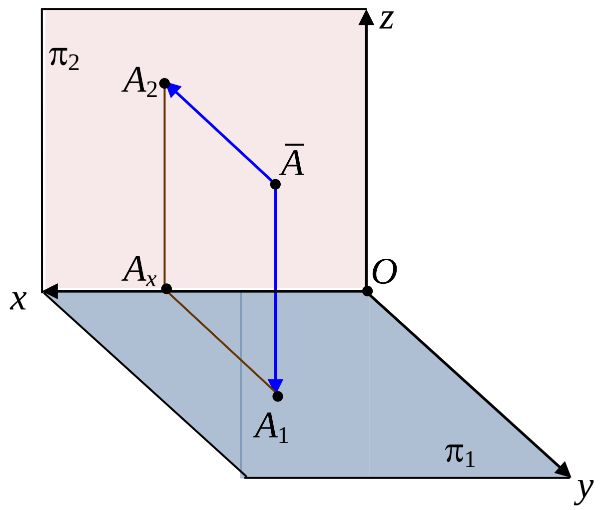

Fig.1.11a

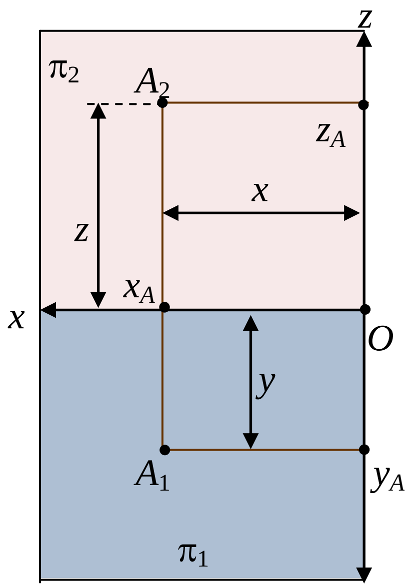

Fig.1.11b

If we remove the designation of the planes, we finally obtain an image (*Fig*. 1.12), which is called the "*Monge diagram*". If a point $\bar{A}$ is defined by a horizontal projection A_1 and a frontal A_2, we will designate it (A_1, A_2).

When depicting complex objects, sometimes projections onto only two planes are not enough.

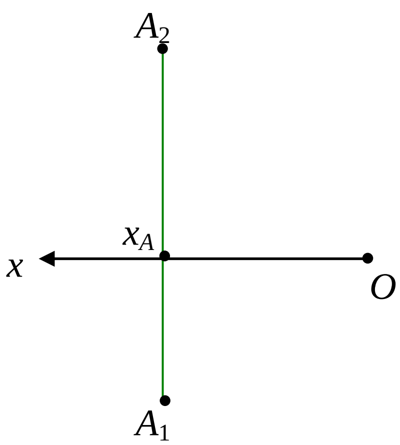

Fig.1.12

1.4 *Drawing of a straight line*

A straight line is defined by two of its points. Let us choose two points $\bar{A}$, $\bar{B}$ on the straight line and construct their projections A_0, B_0 on the image plane α. Then the straight line A_0B_0 is a projection of the straight line $\bar{A}\bar{B}$. This projection can be obtained as follows. Let us draw the plane β through the straight line AB perpendicular to the plane α (Fig. 1.13). This plane will also pass through the projecting rays $\bar{A}A_0$ and $\bar{B}B_0$. The line obtained at the intersection of planes α and β is the projection of line $\bar{A}\bar{B}$.

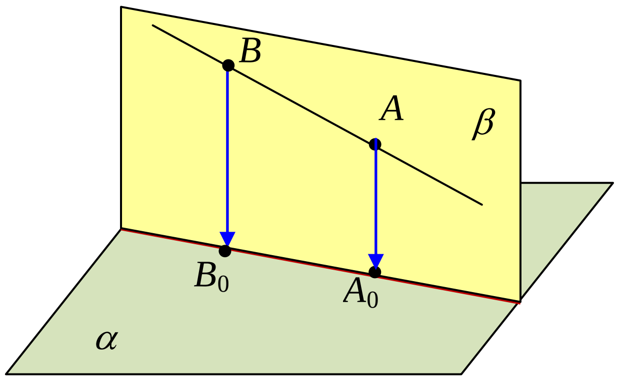

Fig.1.13

Just as in the case of the image of a point, one projection does not determine the position of a line in space. We need to have at least two projections. If a line $\bar{a}$ is defined by a horizontal projection a_1 and a frontal projection a_2, we denote it by (a_1, a_2).

Definition. *A line of general position* is a line that is not parallel to any of the projection planes.

Points $\bar{A}$, $\bar{B}$, which define a straight line of general position, are located at different distances from all projection planes (*Fig.* 1.14). The result of the projection is shown in *Figure* 1.15.

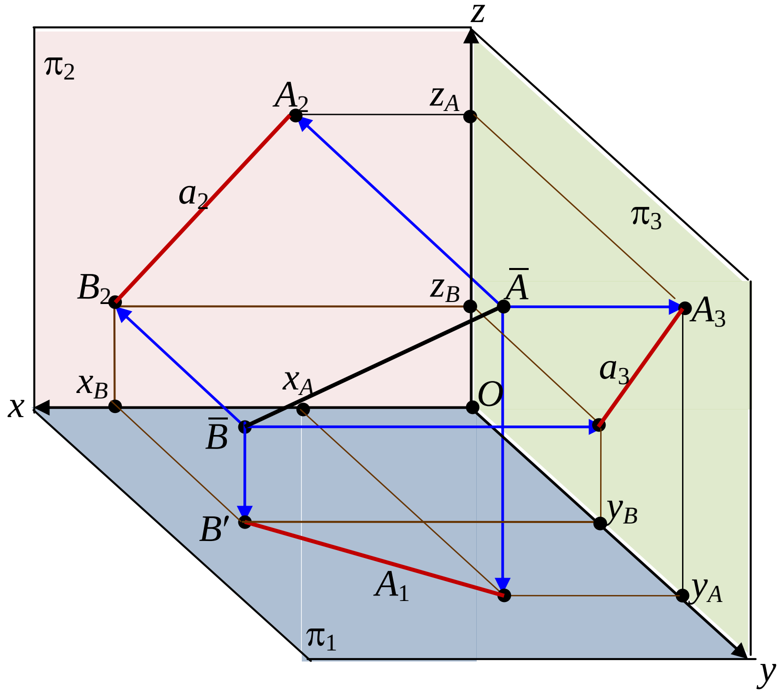

Fig.1.14

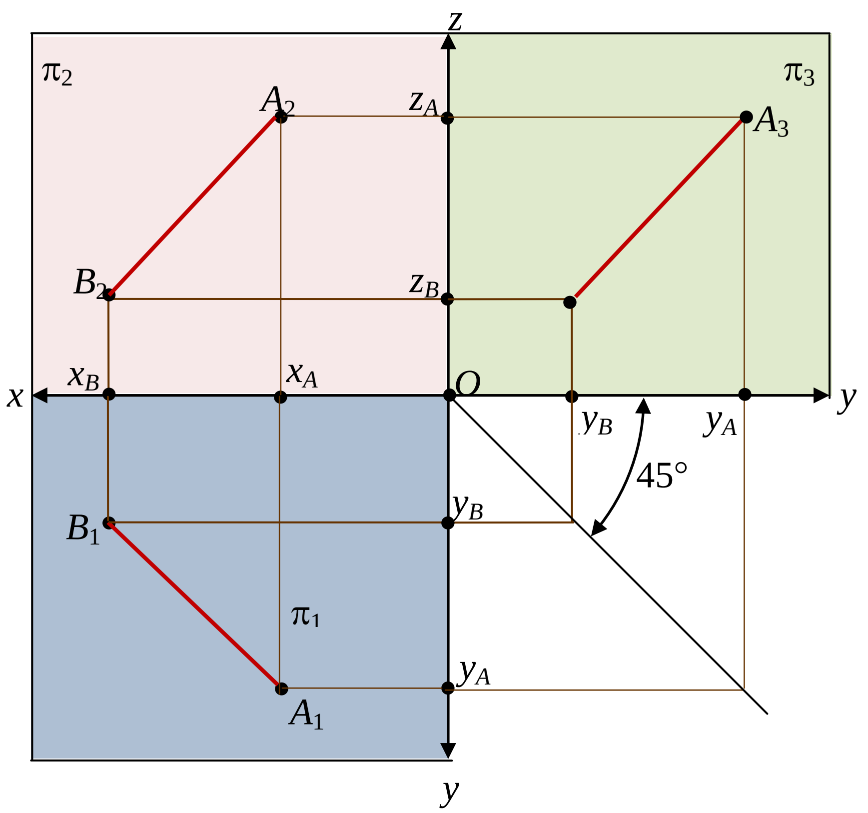

Fig.1.15

Definition. *Straight lines of particular position* are straight lines that are parallel or perpendicular to the projection planes.

Straight lines of particular position are subdivided into

1. *level lines* that are parallel to one of the projection planes;
2. *projecting lines*, which are parallel to two projection planes, i.e. they are perpendicular to one of the projection planes.

In turn, level lines are subdivided into

1а) *horizontal lines*, which are parallel to the horizontal plane;

1б) *frontal lines*, which are parallel to the frontal plane;

1в) *profile lines*, which are parallel to the profile plane.

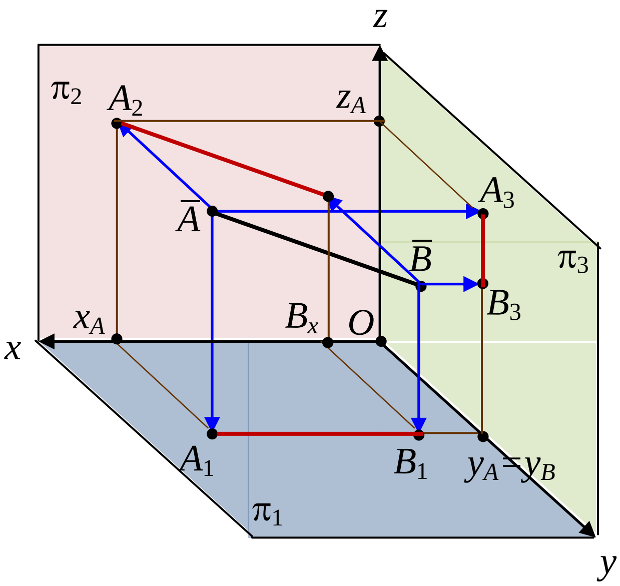

Fig.1.16a

Figures 1.16a, 1.16b, 1.17 show in detail the peculiarity of the frontal line image. We will analyze the remaining cases in laboratory classes.

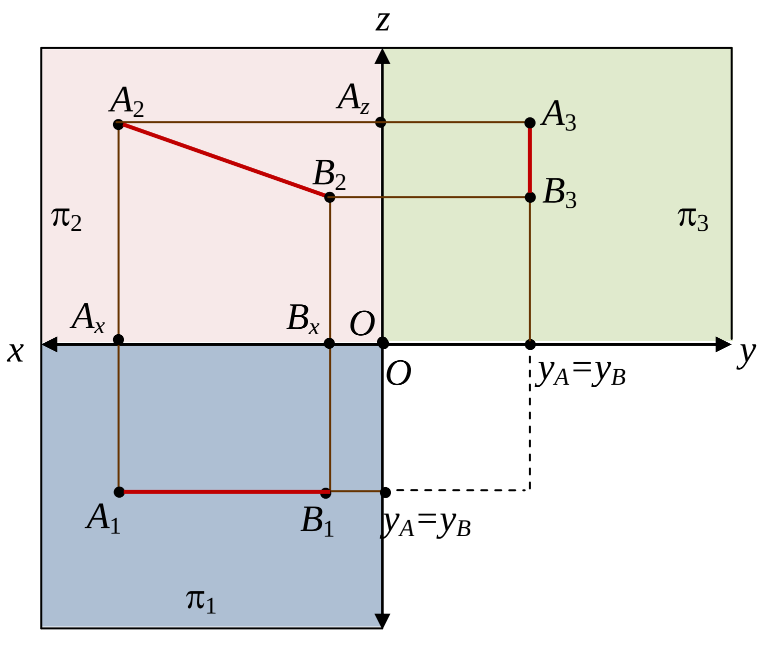

Fig.1.16b

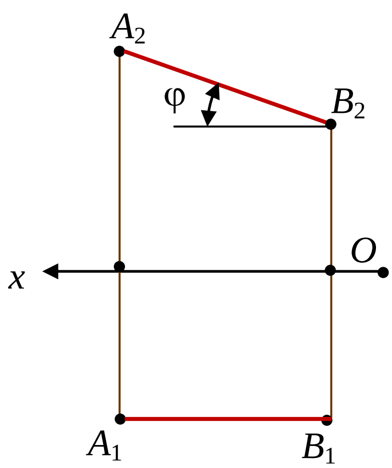

Fig.1.17

In *Figure* 1.17 φ is the natural value of the angle between the line and the Ox axis or between the line and the plane π_1; A_2B_2 is the natural value of the depicted segment $\overline{AB}$.

Figures 1.18 and 1.19 show a horizontal and a profile line. Accordingly, A_1B_1 and A_3B_3 are the natural values of the segments.

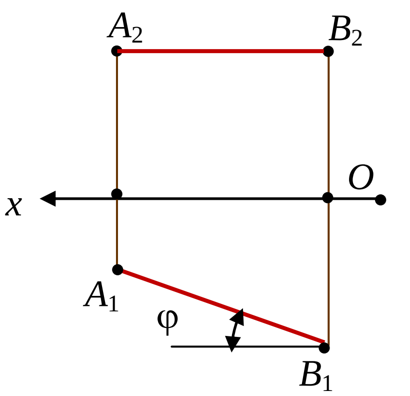

Fig.1.18

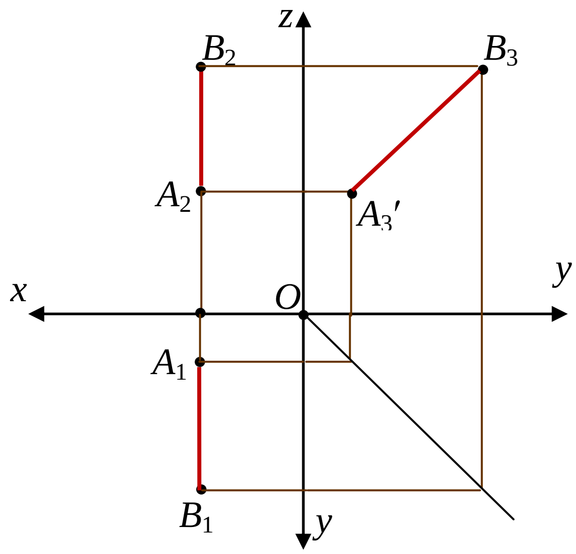

Fig.1.19

Projecting straight lines of the level are divided into

1a) *horizontally projecting*, which are perpendicular to the horizontal plane;

1б) *frontally projecting*, which are perpendicular to the frontal plane;

1в) *profile-projecting*, which are perpendicular to the profile plane.

The projections of these lines onto the corresponding planes represent a point.

Exercise. Indicate the name of the line shown in each of the following drawings.

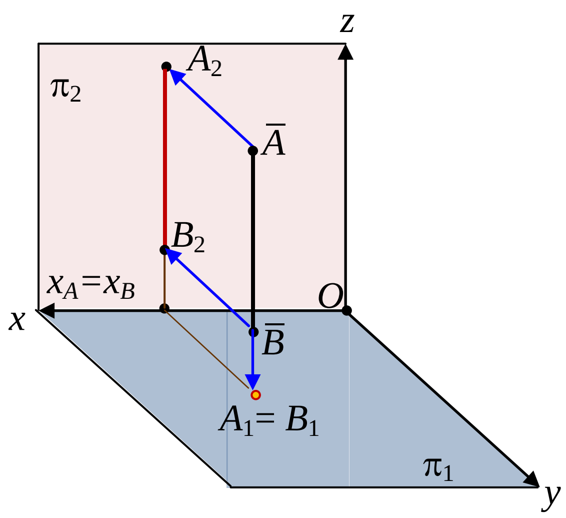

Fig.20a

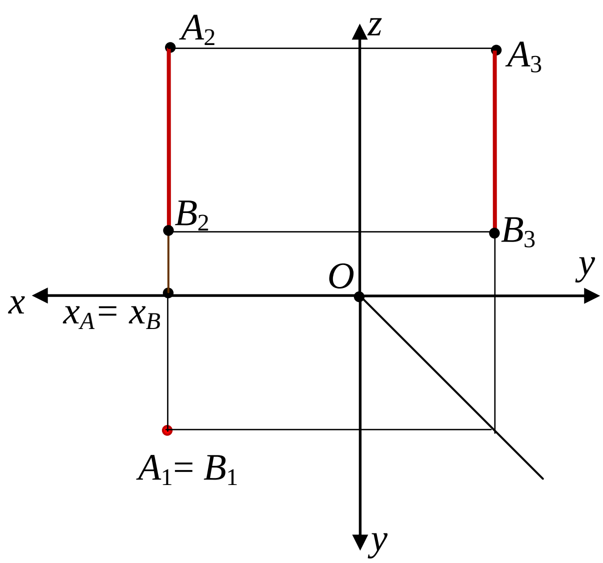

Fig.20b

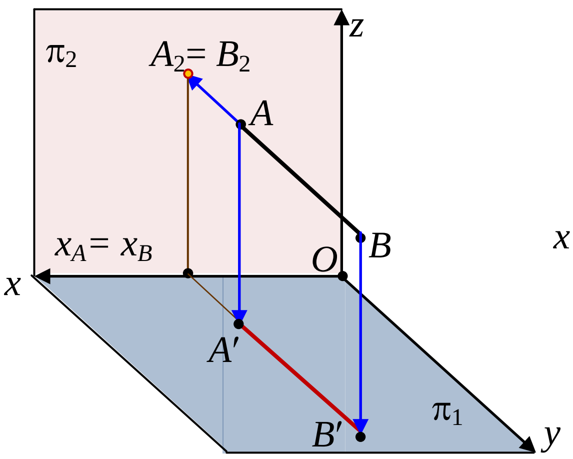

*Fig.*21a

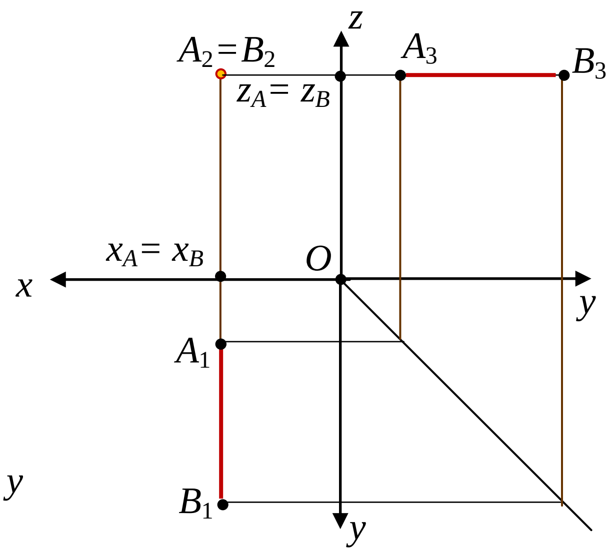

*Fig.*21б

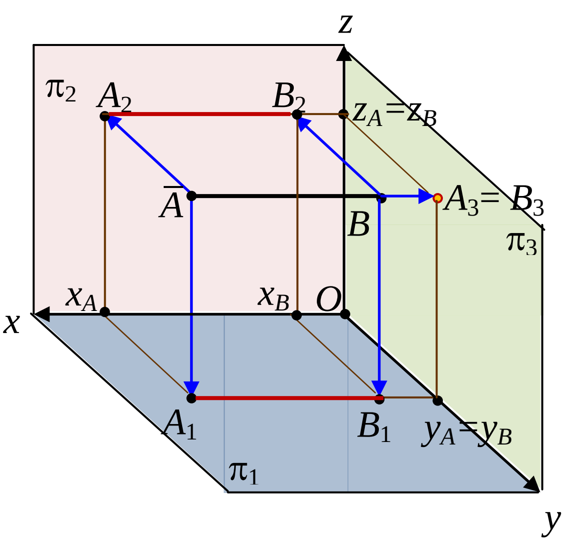

*Fig.*22a

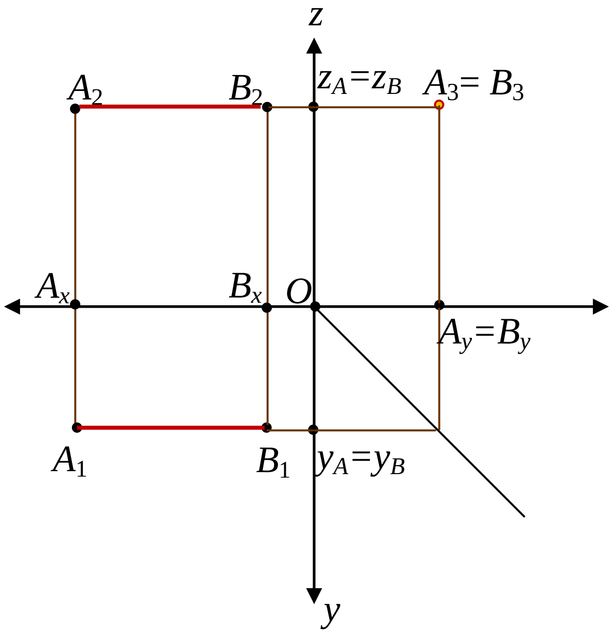

*Fig.*22b

1.5 *Traces of a straight line*

Definition. *Trace of a line* is the point of its intersection with one of the projection planes.

A line of general position has three traces – horizontal, frontal and profile. Level lines have two traces each, and projecting lines have one trace each.

In order to find a trace on a spatial image, we continue the line itself and its projection onto the corresponding plane until the intersection. In *Figure* 1.23a, $\bar{K}$ is the trace of the line on plane π_1, and $\bar{L}$ is the trace on plane π_2.

In order to find the trace of a line on the plane π_1 in the system of two projection planes, we extend A_2B_2 to the intersection with Ox, find the point K_2, and from this point draw a line parallel to Oy to the intersection with A_1B_1.

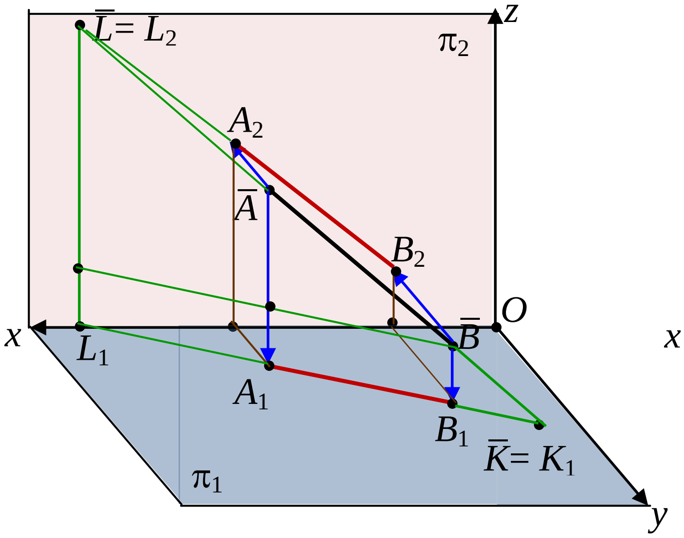

рис.1.23а

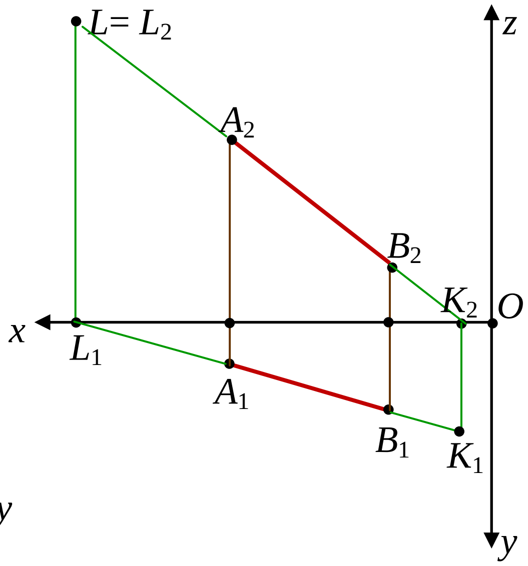

рис.1.23б

1.6 *Determining the natural size of a segment and the angles of inclination of a straight line*

The natural length of a line segment in a particular position coincides with the length of one of its projections. Therefore, we are interested in the problem of how, using a drawing in a system of two projection planes, we can determine the natural length of a line segment in a general position.

From the drawing (Figure 1.24a) we can determine the segments

$$\Delta x = |x_B - x_A|, \;\; \Delta y = |y_B - y_A|, \Delta z = |z_B - z_A|.$$

Then

$$|\bar{A}\bar{B}| = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}.$$

At the same time, we have the following segments on the drawing:

$$|A_1B_1| = \sqrt{\Delta x^2 + \Delta y^2}, \; |A_2B_2| = \sqrt{\Delta x^2 + \Delta z^2}.$$

Therefore

$$|\bar{A}\bar{B}| = \sqrt{|A_1B_1|^2 + \Delta z^2} = \sqrt{|A_2B_2|^2 + \Delta y^2}.$$

From here we obtain methods for constructing a segment equal to the natural value of a given segment. From any of the vertices A_2 or B_2 we set aside a segment equal to Δy at a right angle to the segment A_2B_2 (Figure 1.24b). We obtain a right triangle in which the hypotenuse is the desired segment.

Similarly, we can set aside a segment equal to Δz at a right angle to the segment A_1B_1, and the length of the hypotenuse of the resulting right-angled triangle will also represent the desired segment.

The same figure shows the method for finding the angles φ_1 and φ_2 of inclination of a given line to the horizontal and frontal planes.

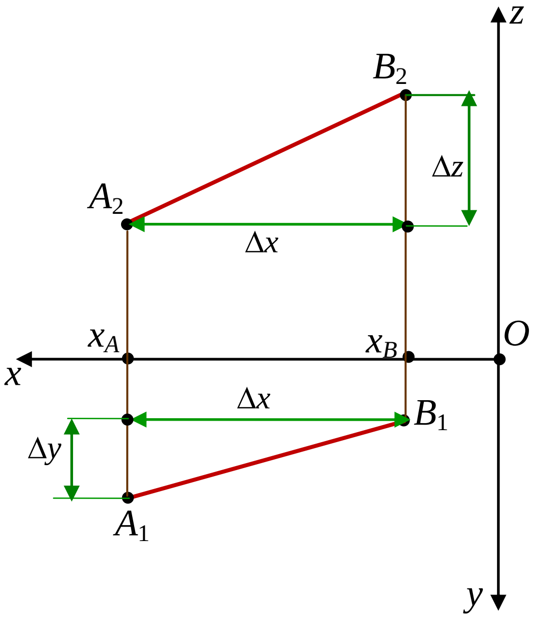

рис.1.24a

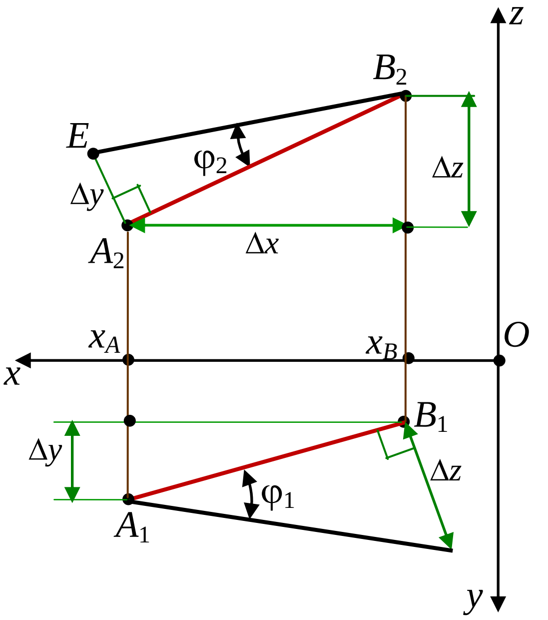

рис.1.24b

1.7 *Relative position of two lines in space*

Two lines in space can be
a) parallel;   b) coincide;
c) intersect; d) cross.

The projections of parallel lines onto any of the projection planes are parallel or lie on the same line (*Fig*. 1.25). Conversely, if all three pairs of projections of two lines are parallel or lie on the same line (but not all three pairs), then the lines are parallel.

For lines of general position, parallelism of two pairs of projections automatically results in parallelism of third projections. But for lines of particular position this is not the case. If the projections of lines onto one or two planes are parallel or coincide, then the lines themselves can intersect (*Figure* 1.26).

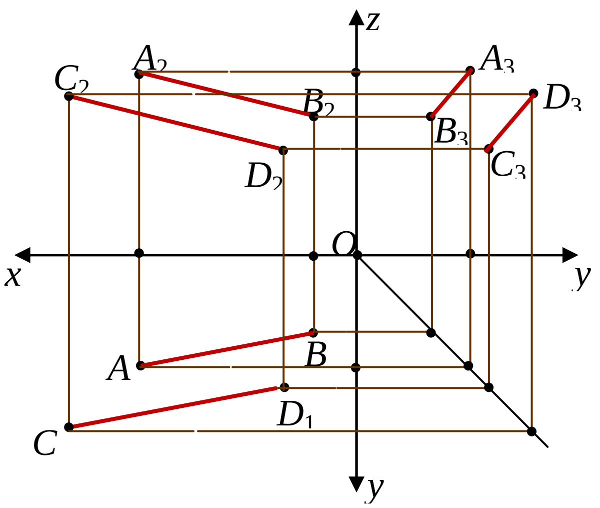

Fig.1.25

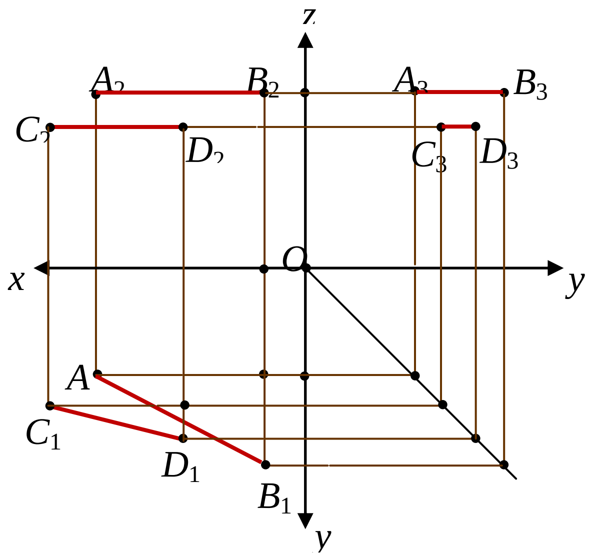

Fig.1.26

If lines intersect, then their projections onto the same planes also intersect, and the intersection points are projections of the same point (*Fig*. 1.27a).

In Figure 1.27b, the intersection points of the horizontal projections and the intersection point of the frontal projections are projections of different points. This means that intersecting lines are depicted.

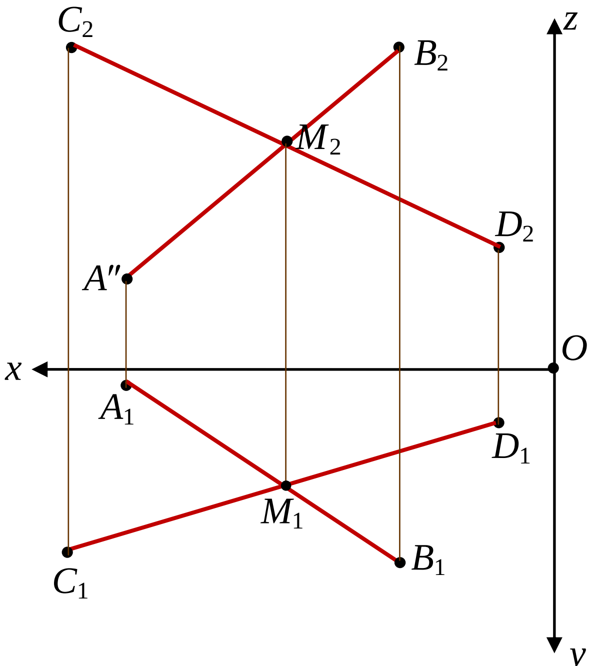

Fig.1.27a

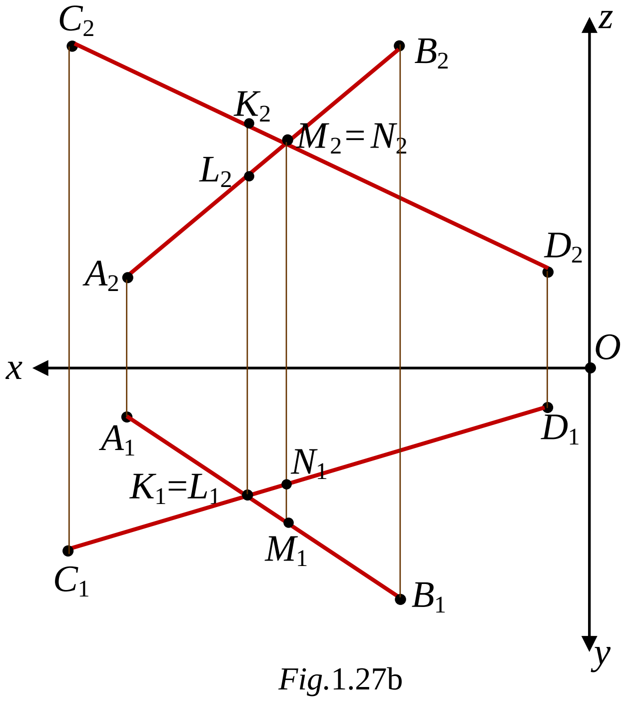

Fig.1.27b

1.8 *Representation of a plane on a drawing*

Definition. *A plane of particular position* is a plane perpendicular to one of the projection planes. All other planes are called *planes of general position*.

In order to depict a plane, it is necessary to specify some elements defining it on the drawing. For example, a plane will be specified if are specified on the drawing

a) three of its points that do not lie on one line;

b) a line and a point that does not lie on it (*Fig*. 1.28 a);

c) two different lines (parallel or intersecting) that lie on a plane (*Fig*. 1.28 b, c);

d) any flat figure that lies in this plane but does not lie on one line.

But all these methods can be reduced to one: the plane is defined by a triangle (Fig. 1.29).

But the most convenient method is considered to be the representation of a plane using its traces.

Definition. *The trace of a plane* is the line of its intersection with one of the projection planes (*Fig*.1.30 a,b).

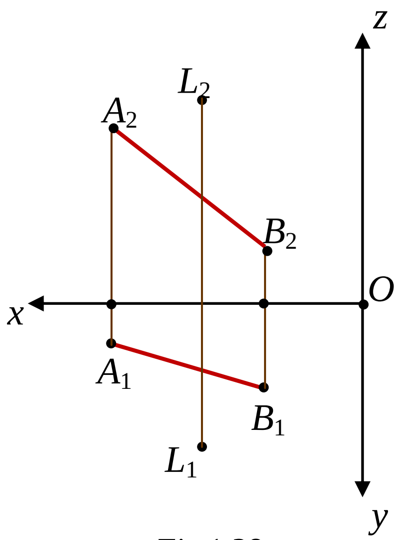

*Fig.*1.28 a

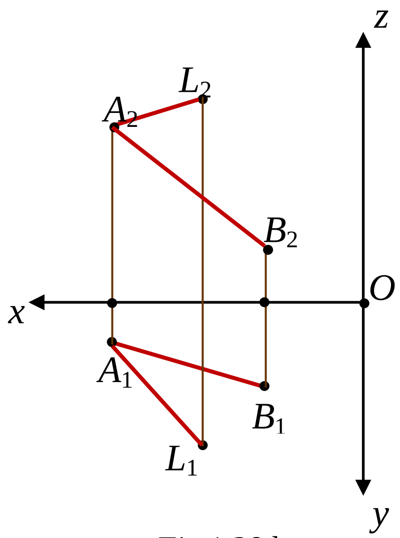

*Fig.*1.28 b

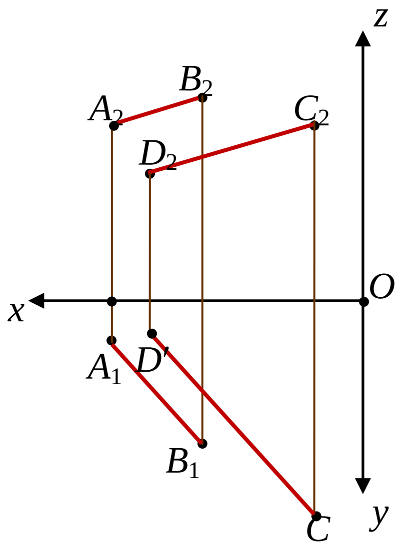

*Fig.*1.28 c

The traces of the plane α on the horizontal, frontal and profile plane will be designated h_α, f_α, p_α, respectively. The points of intersection of the plane with the coordinate axes will be designated x_α, y_α, z_α. These are the so-called points of convergence of the traces.

Having two traces of a plane on the drawing, we can find a third trace along the lines of connection. The key to solving many problems is the statement: "a straight line lies on a plane, then and only then do its traces lie on the same traces of the plane".

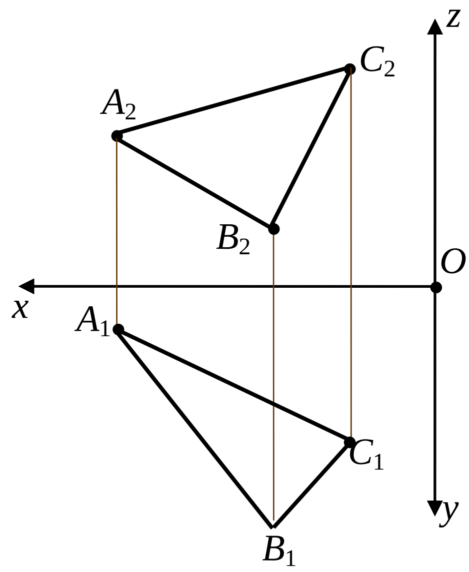

*Fig.*1.29

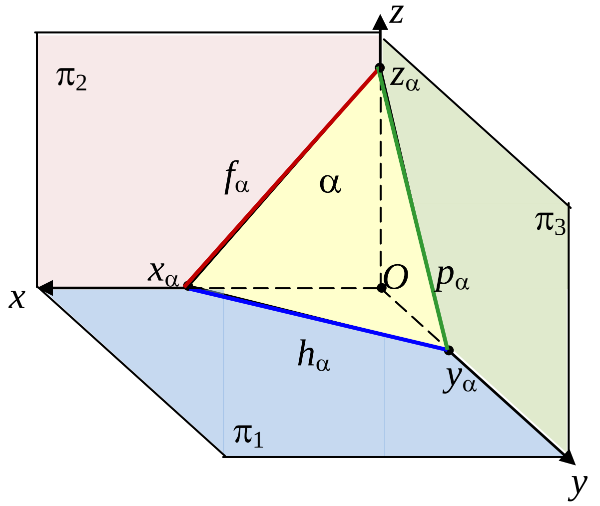

*Fig.*1.30a

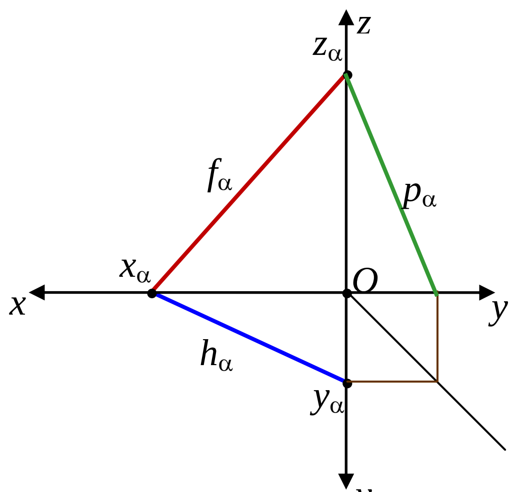

*Fig.*1.30b

The planes of particular position are divided into two types:

1) projecting, which are perpendicular to only one of the projection planes;

2) level planes, which are perpendicular to two projection planes, or, what is the same, perpendicular to the coordinate axes.

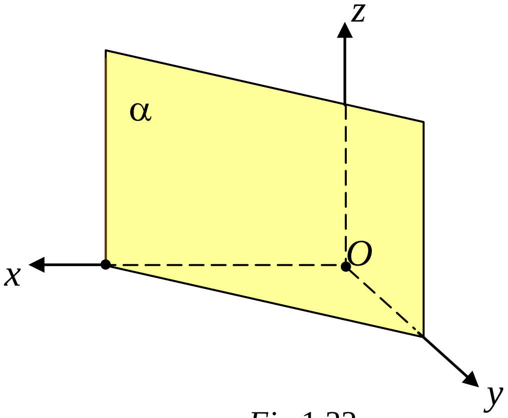

*Fig.*1.32a

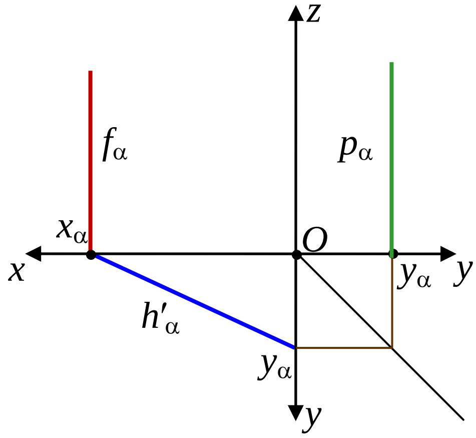

*Fig.*1.32b

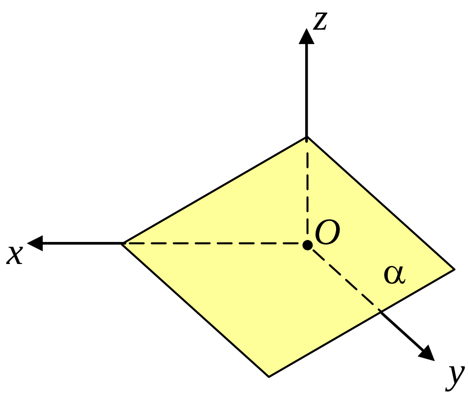

*Fig.*1.33a

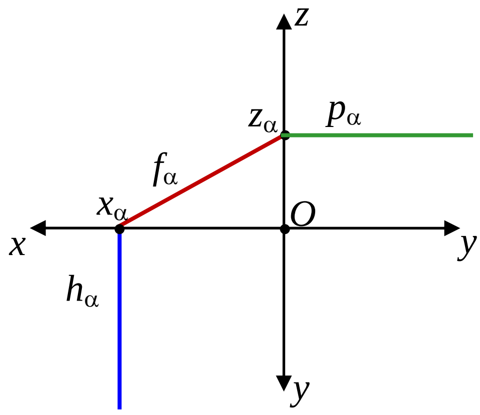

*Fig.*1.33b

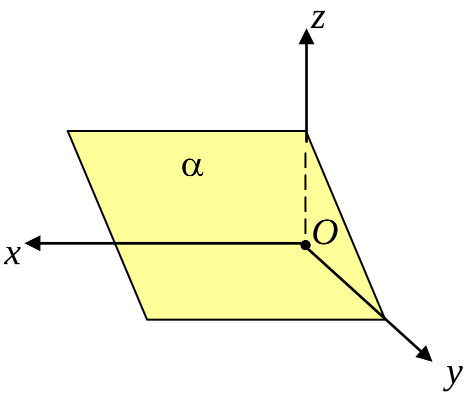

*Fig.*1.34a

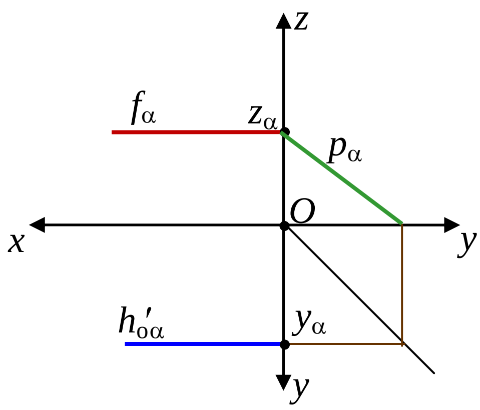

*Fig.*1.34b

In turn, projecting planes are divided into horizontally projecting, frontally projecting and profile-projecting, in accordance with the name of the planes to which they are perpendicular (*Fig*. 1.32 – 1.34).

Level planes are divided into horizontal, frontal and profile, in accordance with the name of the planes to which they are parallel (Fig. 1.35 – 1.37).

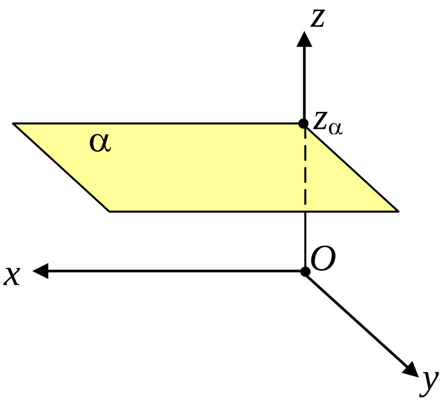

рис.1.35а

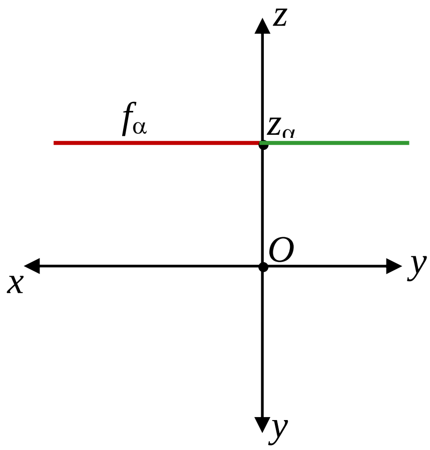

рис.1.35б

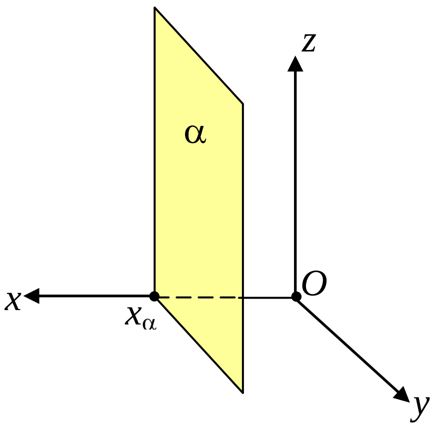

рис.36а

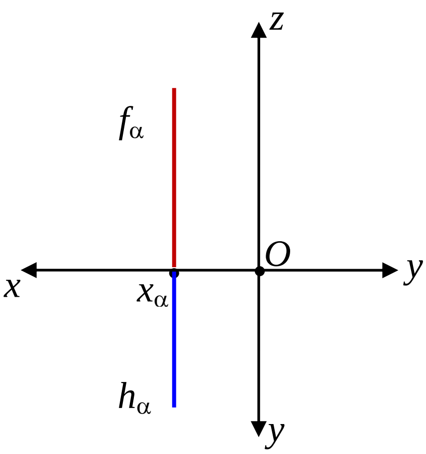

рис.36б

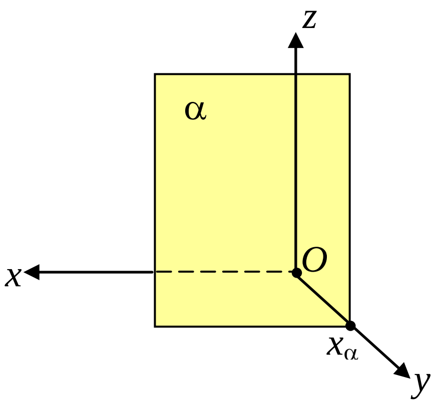

рис.1.37а

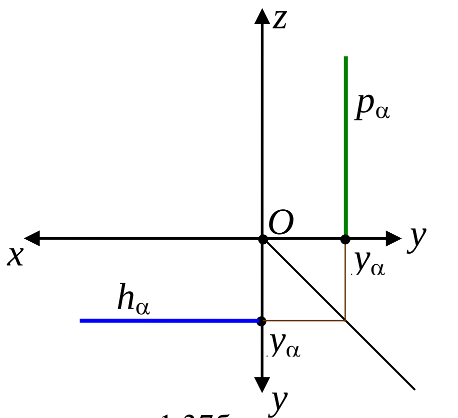

рис.1.37б

We see that each of the projecting planes has one trace that coincides with its projection. Each of the level planes has two such traces. They are called “trace-projection”.

If any object (point, segment, triangle) lies in a plane, then the projection of this object lies on the trace-projection of the plane.

Figure 1.38 shows an image of the horizontally projecting plane β and point C. Since C_1 lies on the horizontal trace-projection h_β of plane β, then point C lies in plane β. It is more difficult to verify that a point belongs to a plane of general position. We will consider this problem in the next chapter.

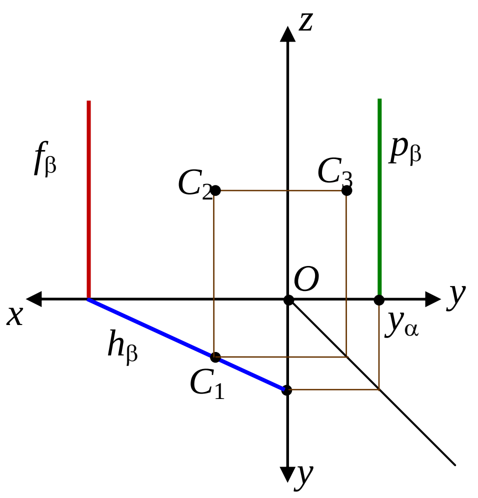

*Fig.*1.38

CHAPTER 2. AXONOMETRY

2.1 *Axonometry. Image of points*

Let a Cartesian coordinate system $\bar{O}\bar{x}\bar{y}\bar{z}$ be given in space, $\bar{O}\bar{E}_1$, $\bar{O}\bar{E}_2$, $\bar{O}\bar{E}_3$ are unit directed segments on the coordinate axes. Let $\bar{M}$ be an arbitrary point in space with coordinates (x_1, x_2, x_3). The projection of the point $\bar{M}$ onto the horizontal plane $\bar{O}\bar{E}_1\bar{E}_2$ parallel to the $\bar{O}\bar{z}$ axis is denoted by $\bar{M}_1$, and let $\bar{M}_x$ be the projection of the point $\bar{M}_1$ onto the $\bar{O}\bar{x}$ axis parallel to the $\bar{O}\bar{y}$ axis (*Figure* 2.1). Then the broken line $\bar{O}\bar{M}_x\bar{M}_1\bar{M}$ is called *the coordinate broken line* of the point $\bar{M}$. For its links holds

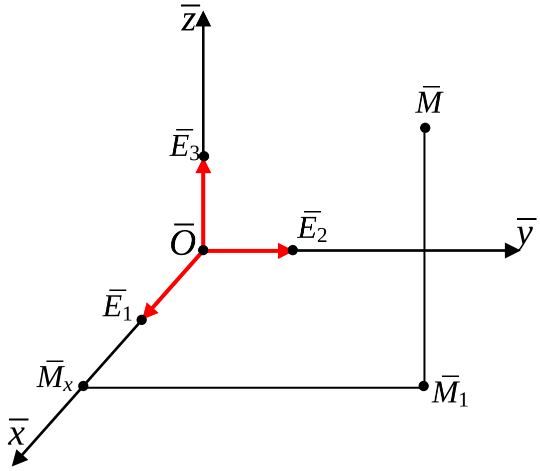

Fig.2.1

$$|\bar{O}\bar{M}_x| = |x_1|,\ |\bar{M}_x\bar{M}_1| = |x_2|,$$

$$|\bar{M}_1\bar{M}| = |x_3|.$$

Let us choose the image plane σ and the projection direction not parallel to the coordinate planes. Let us project the frame together with the coordinate polyline onto the plane and apply the similarity transformation. We obtain the image $\mathcal{R} = (O, E_1, E_2, E_3)$ of the frame and the image OM_xM_1M of the coordinate polyline (*Figure* 2.2). The image preserves the ratio of the segments belonging to the parallel lines. Therefore

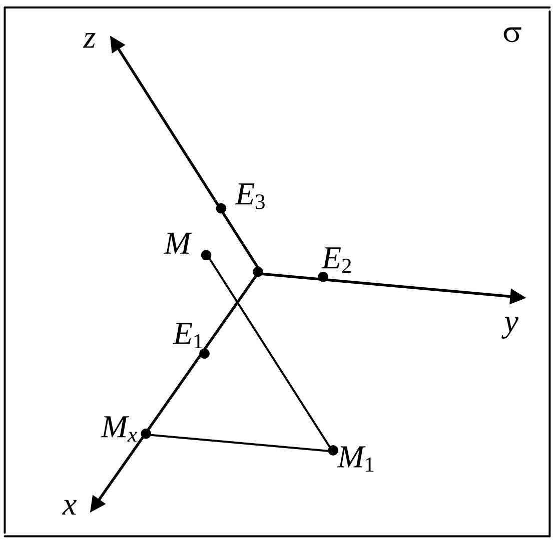

Fig.2.2

$$|OM_x| = |x_1||OE_1|,\ |M_xM_1| = |x_2||OE_2|,\ |M_3M| = |x_3||OE_3|.$$

The following statement follows from this.

If an image of an affine frame is given on the plane σ, then we can construct an image M of a given point $\bar{M}$ by its coordinates. If images of an affine frame and a coordinate broken line are given, then we can determine the coordinates of the point $\bar{M}$.

Let us emphasize that the image of the point M itself does not $\bar{M}$ give the possibility to find the coordinates of this point. If two points M and M_1 are given on the image, then we can restore the image of the entire coordinate broken line, and thus find the coordinates of $\bar{M}$.

If we can construct images of points in a coordinate system, then we can also construct images of spatial figures. This method is called the axonometric projection method. Point O is called *the origin of the axonometric coordinate system*, and the axes *Ox*, *Oy*, *Oz* are called *axonometric axes*. Images of coordinate planes are called *axonometric planes*.

Let $\bar{M}_2$, $\bar{M}_3$ be projections of point M onto the frontal and profile planes, respectively, parallel to the coordinate axes $\bar{O}\bar{y}$ and $\bar{O}\bar{x}$ (*Figure* 2.3). Let M_2, M_3 be images of these points. Then point M is called an axonometric projection of point $\bar{M}$, and M_1, M_2, M_3 are called its secondary projections.

In order to determine the coordinates of point $\bar{M}$ from its image, it is sufficient to have its axonometric projection and any of the secondary ones on the drawing. But if it is not specified which secondary projection is being discussed, then it is assumed that this is point M_1.

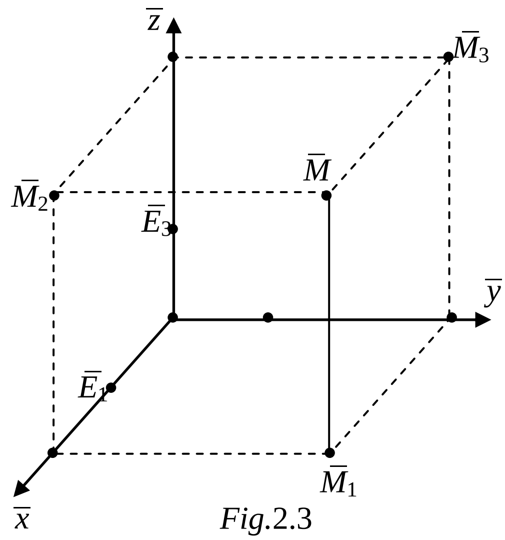

*Fig.*2.3

Instead of the statement "*a point* $\bar{M}$ *is given in space, the axonometric projection of which is* M*, and the secondary projection is* M_1", we will say "*a point* (M, M_1) *is given*". We will call the line M_1M *the connection line*.

Questions for self-control

1. What is called the original, and what is called the image of the original?
2. What is a "coordinate polyline"?
3. What is an axonometric projection, and what is a secondary projection of a point? On which axonometric plane is the secondary projection of a point selected by default?
4. What does the phrase "*a point* (M, M_1) *is given*" mean?
5. What line do we call the connection line?

2.2 *Representation of lines and planes in axonometric projection*

We will assume that the direction of projection is not parallel to the considered lines and planes. Then the image of a line will be a line, and the image of a plane will cover the entire plane of images σ.

The line $\bar{a}$ on the image plane is defined by its two points (M, M_1) and (N, N_1) or by the axonometric projection a and the secondary projection a_1. Then we say that the line (MN, M_1N_1) or the line (a, a_1) is given. If the line $\bar{a}$ is not parallel to the $\bar{O}\bar{z}$ axis, then its secondary projection is a line (*Figure*

2.4). If $\overline{a} \| \overline{Oz}$, then its secondary projection is a point. In the latter case, we still sign this point as a_1 (*Figure* 2.5). The $\overline{Oz}$ axis itself is defined on the image as (OE_3, O) or (Oz, O).

If the straight line $\overline{b}$ lies in a horizontal plane, then its axonometric and secondary projections coincide: $b = b_1$ (*Figure* 2.5).

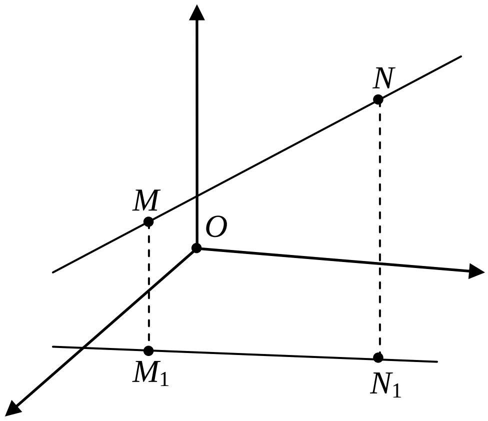

*Fig.*2.4

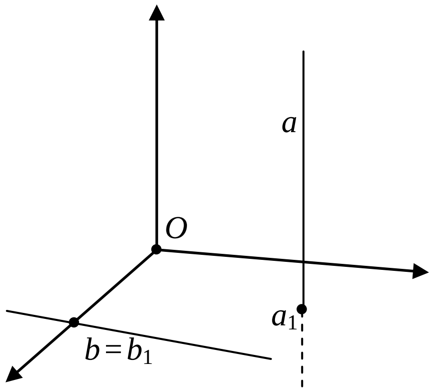

*Fig.*2.5

Let us consider possible variants of the arrangement of two lines (a, a_3) and (b, b_3). Both intersecting lines (*Figures* 2.6 *a-c*) and parallel lines (*Figures* 2.7 *a-c*) may have coincident axonometric or secondary projections. If both projections coincide, then the lines themselves coincide.

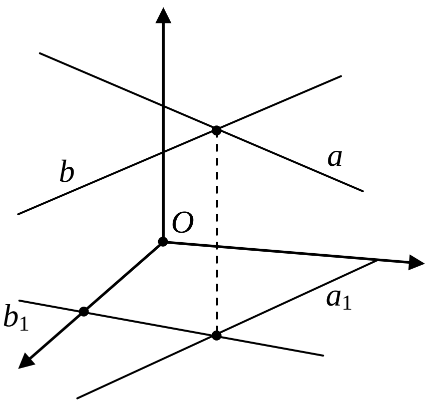

*Fig.*2.6*a*

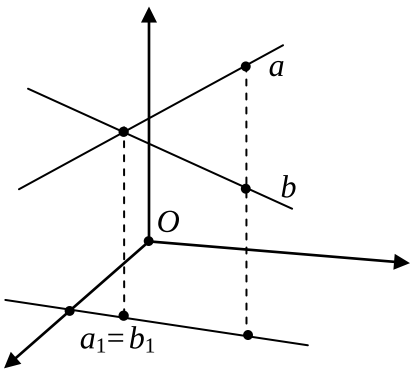

*Fig.*2.6*b*

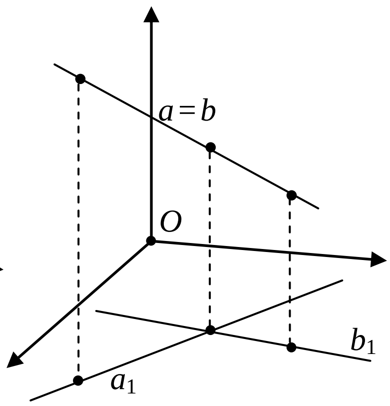

*Fig.*2.6*c*

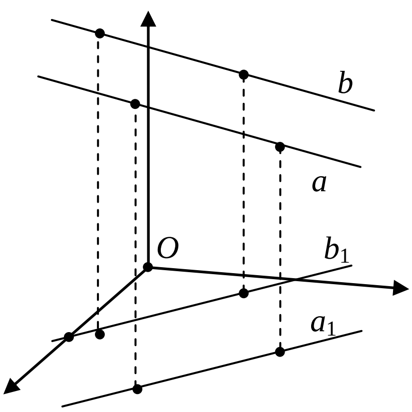

*Fig.*2.7*a*

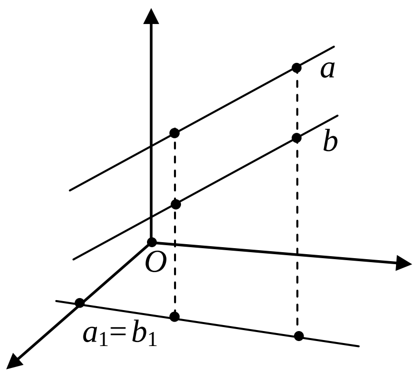

*Fig.*2.7*b*

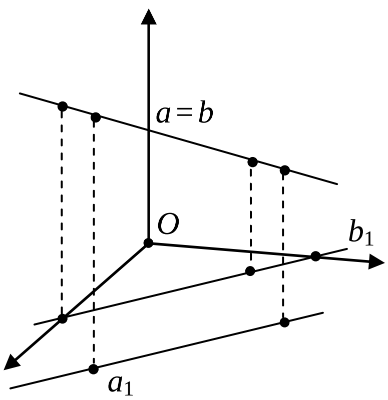

*Fig.*2.7*c*

A plane $\overline{\alpha}$ can be defined by three of its points, or by two of its lines, or by a line and a point not lying on it. The most convenient way to represent a plane is by means of its traces (*Fig*.1.30a) or a horizontal trace and the point of intersection with the Oz axis. In the latter case, we can easily draw the other two traces (Fig. 8). By default, if it is not stated on which plane the trace is given, we assume that a horizontal trace is given.

The plane can be located so that there is no horizontal trace or point (P, O) (*Figures* 2.9 *a*, *b*).

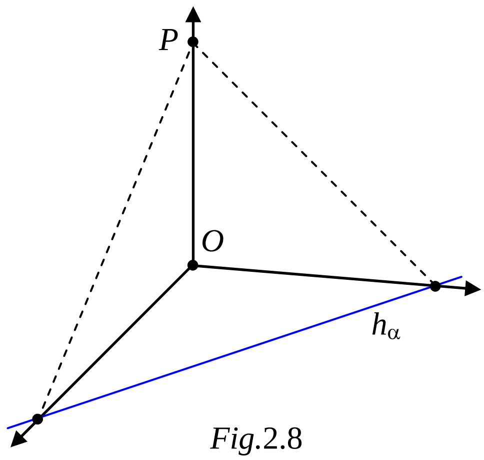

Fig.2.8

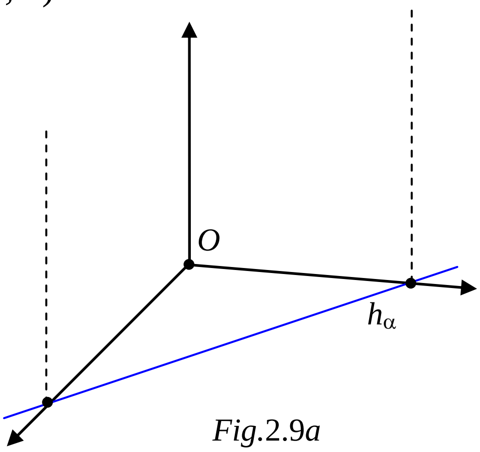

Fig.2.9*a*

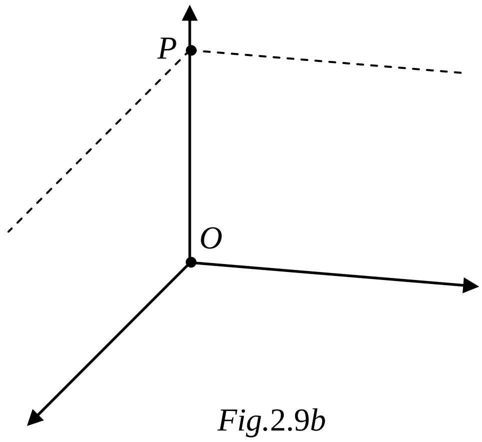

Fig.2.9*b*

In Chapter 1 we said what is called a trace of a line. By default, if it is not said on which axonometric plane the trace of a line is given, then it is assumed that it is given just on a horizontal plane (*Figure* 2.10).

The key to solving many construction problems is the following obvious statement: "*If a line lies on a plane, then its trace lies on the trace of the plane* (*Figure* 2.11)".

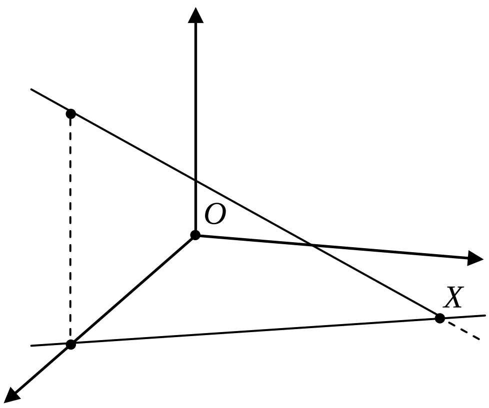

Fig.2.10

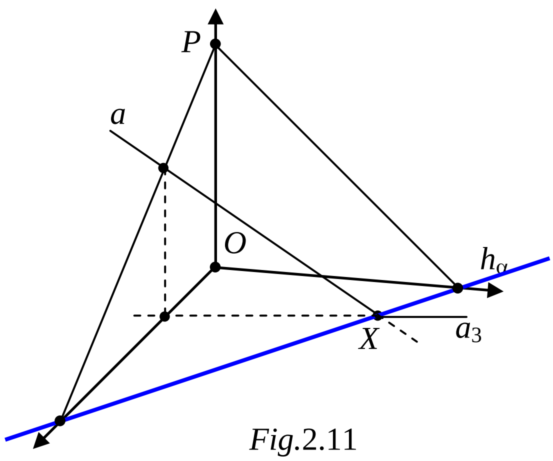

Fig.2.11

Questions for self-control

1. What are the methods for defining a line and a plane in an axonometric projection? Which method of defining a plane is considered the most convenient?

2. Can axonometric or secondary projections of different lines coincide if these lines are: a) intersecting; b) parallel? Can both axonometric and secondary projections coincide for these lines?

3. What is a line trace and what is a plane trace? On which axonometric plane is the line or plane trace selected by default?

4. Which phrase serves as the key to solving many problems involving construction in axonometric projection?

2.3 *Problems on construction in axonometric projection*

We will use the following problems to construct sections of polyhedra.

***Task* 1.** *The line* (a, a_1) *lies in the plane defined by three points* (A, A_1), (B, B_1), (C, C_1), *which do not lie on the same line. Given a line* a, *construct* a_1.

Solution. We construct lines AB, A_1B_1, AC, A_1C_1, BC, B_1C_1. We agreed that the direction of projection is not parallel to the lines and planes under consideration. Therefore, points A, B, C do not lie on one line and lines AB, AC, BC do not coincide. Line a intersects two of these lines at points M and N. From these points we can construct secondary projections M_1 и N_1 (for this, it is necessary to draw the connection lines parallel to Oz). Then $a_1 = M_1N_1$ (*Figure* 2.12).

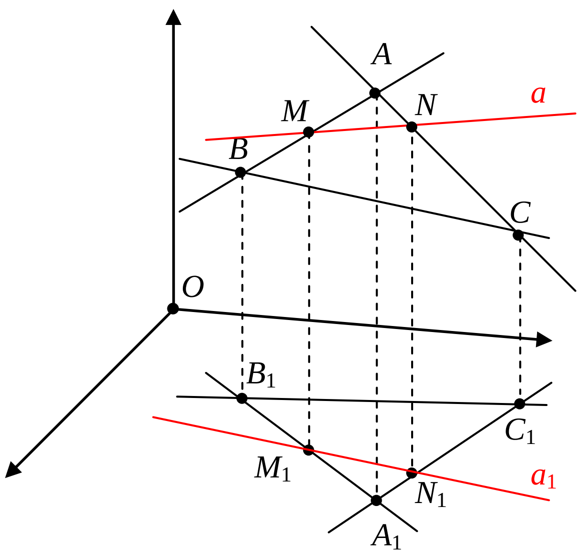

*Fig.*2.12

And, conversely, if a line a_1 is given, we can find M_1 и N_1, and along the connection line find M and N. Then $a = MN$. But here it is possible that A_1, B_1, C_1 lie on one line. Then the problem has no solution.

***Task* 2.** *Point* (X, X_1) *lies in the plane defined by three points* (A, A_1), (B, B_1), (C, C_1), *which do not lie on the same line. Given point* X_1, *construct* X.

Solution. Point (X, X_1) lies in the same plane as points (A, A_1), (B, B_1), (C, C_1). Therefore, lines (XC, X_1C_1) and (AB, A_1B_1) lie in the same plane. Let them intersect at point (M, M_1) (if these lines do not intersect, then lines (XA, X_1A_1) and (BC, B_1C_1) intersect, and we will consider them.

We construct (*Figure* 2.13):

1. $M_3 = X_3C_3 \cap A_3B_3$;
2. $m \| OE_3,\ M_3 \in m$;
3. $AB \cap m = M$;
4. $l \| OE_3,\ X_3 \in l$;
5. $l \cap CM = X$.

Similarly, we given point X we can find X_1. Independently analyze the case when A_1, B_1, C_1 lie on the same line.

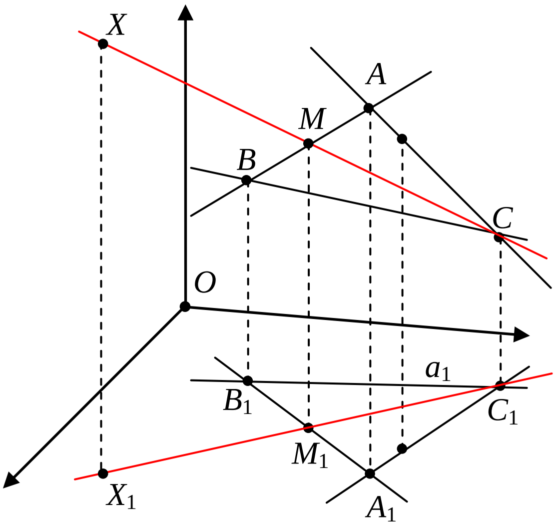

Fig.2.13

***Task* 3.** *Construct traces of the line* $(a,\ a_1)$ *on all coordinate planes.*

Solution. It is obvious that $X = a \cap a_1$ is the trace of a line on the horizontal plane (*Figure* 2.14). Let (C, C_1) be the intersection point of the line with the frontal plane. Then $C_1 \in OE_3$ and $C_1 \in a_1$ $\Rightarrow$ $C_1 \in a_1 \cap Ox$. In order to find C, the line of communication is parallel to OE_3.

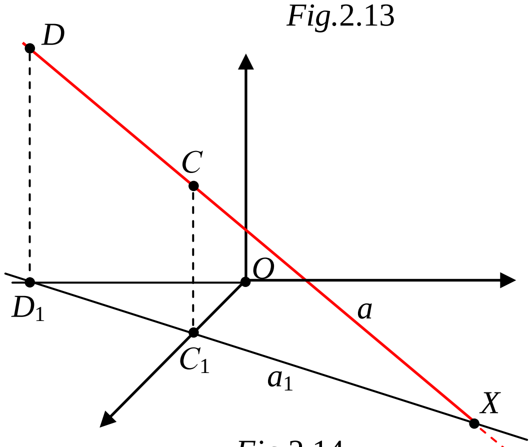

Fig.2.14

The trace of a straight line on a profile plane is constructed in a similar manner.

***Task* 4.** *A plane is defined by three points* $(A,\ A_1)$, $(B,\ B_1)$, $(C,\ C_1)$ *that do not lie on the same line. Construct its trace.*

Solution. Lines (AB, A_1B_1) and (AC, A_1C_1) lie on the plane $\Rightarrow$ their traces lie on the trace of the plane. We construct (*Figure* 2.15):

1. $X = AB \cap A_1B_1$,

   $Y = AC \cap A_1C_1$;
2. $h_\alpha = XY$ is the trace.

If any of the lines has no trace, then instead of it we consider the line $(BC,\ B_1C_1)$.

Exercise. Using Figure 12, construct the trace of the plane.

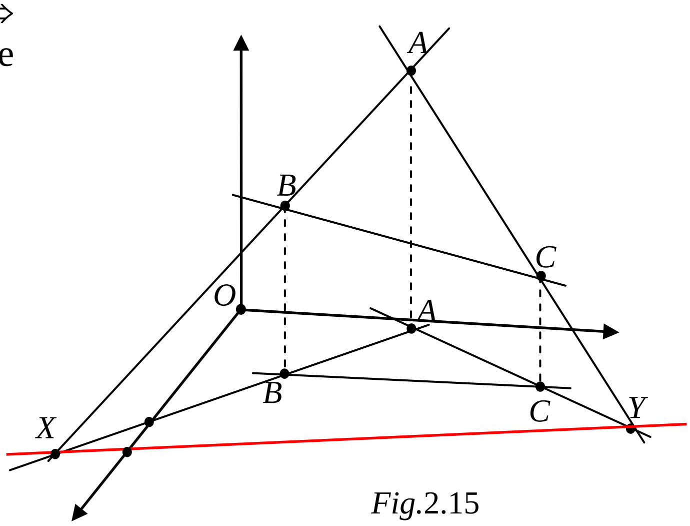

Fig.2.15

***Task* 5.** *The plane is defined by its trace* h_α *and the point* (P, O). *The point* (M, M_1) *lies on the plane* (*Figure 2.16*). *The point M is given in the drawing. Construct* M_1.

Solution. Points (P, O) and lie on a plane. Therefore, the line (M, M_1) also lies on a plane. Consequently, its trace X lies on the trace of the plane, and $X = PM \cap OM_1$. We construct:

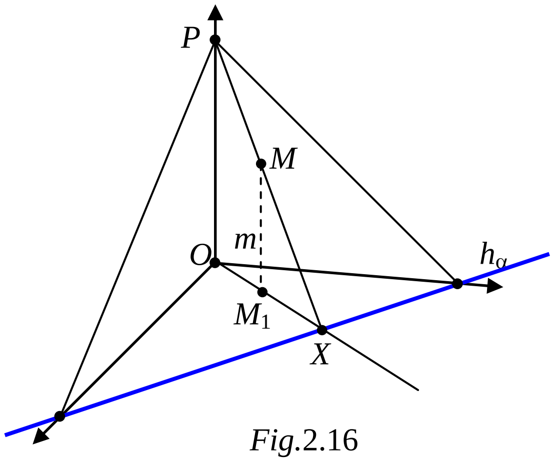

Fig.2.16

1. $X = PM \cap p$;
2. $m || Oz,\ M \in m$;
3. $m \cap OX = M_1$.

It is possible that the lines PM and h_α do not intersect (*Figure* 2.17). This means that the line $\bar{P}\bar{M}$ has no trace. Then it is parallel to the plane $\bar{O}\bar{E}_1\bar{E}_2$. In this case $OM_3||PM$ and we can construct it.

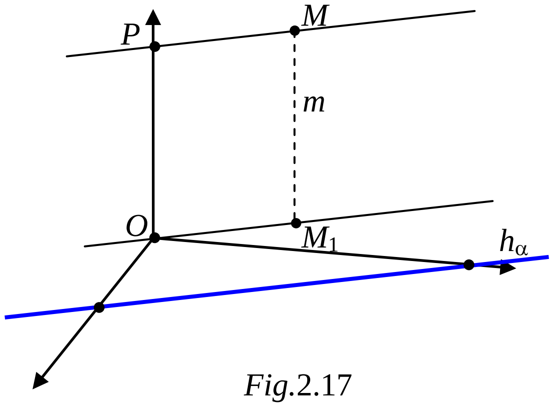

Fig.2.17

Similarly, given a point M_1, we can construct M.

Self-control exercises

1.1. Using Figure 2.16, understand how to solve the following problem. Given a trace h_α of a plane and a point (M, M_1) belonging to the plane. Find the point (P, O) on the axonometric axis Oz. Perform this construction in Figure 2.18.

1.2. In Figure 15, find the point representing the intersection of the given plane with the $\bar{O}\bar{z}$ axis.

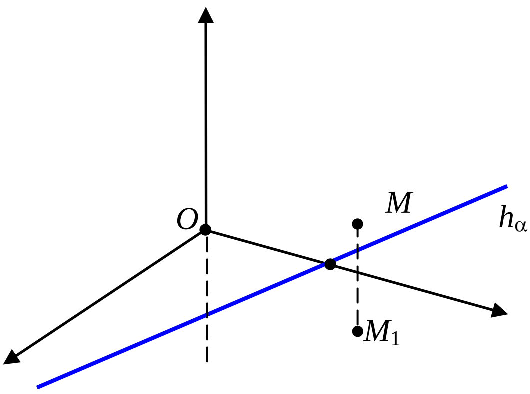

рис.2.18

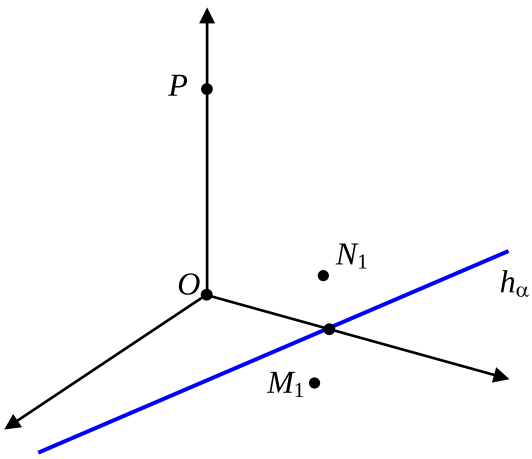

рис.2.19

1.3. In Figure 2.20, find the trace of the plane and the point representing the intersection of the given plane with the $\overline{O}\overline{z}$ axis.$\overline{O}\overline{z}$.

2.1. In Figure 2.19, construct points M and N if (M, M_1) and (N, N_1) lie in the plane defined by the trace and the point (P, O).

3.2. In Figure 2.21, find the trace of the plane passing through the points given in the drawing.

3.3. In Figure 2.21, find the trace of the plane and the point representing the intersection of the given plane with the $\overline{O}\overline{z}$ axis.

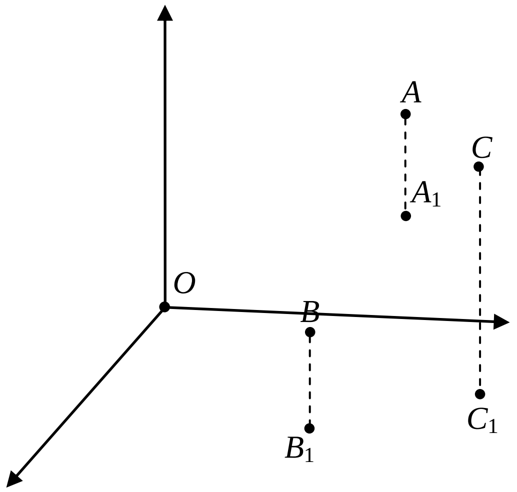

*рис.*2.20

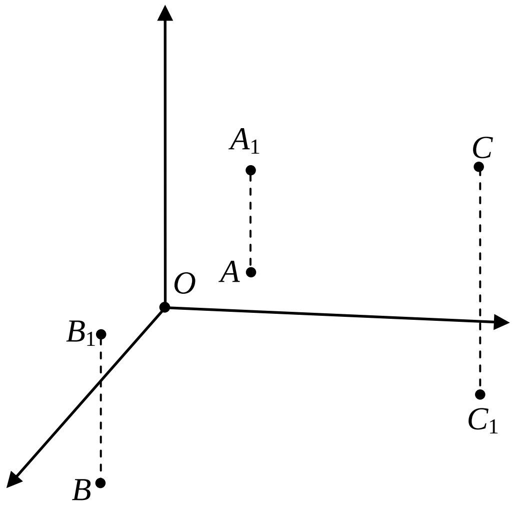

*рис.*2.21

2.4. *Image of polyhedra in parallel projection*

When depicting polyhedrons, we assume that none of the polyhedron's faces are parallel to the projection direction. Then each face will be depicted as a polygon, and the polyhedron's image consists of several polygons.

1. We accept without proof that the vertices of any quadrilateral can be chosen as the image of the vertices of a triangular pyramid. If, for clarity, the invisible lines are depicted as dotted lines, then the following possible options will be obtained (*Figure* 2.22).

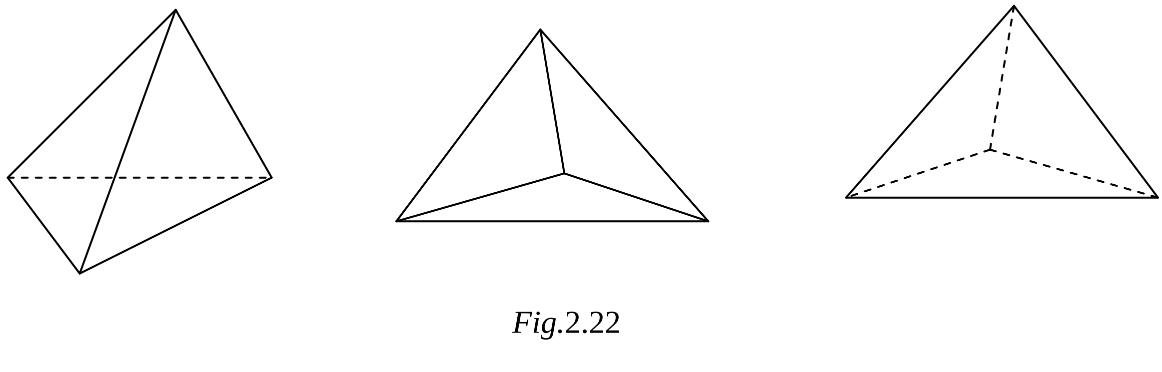

*Fig.*2.22

2. Each face of the parallelepiped is represented by a parallelogram and the opposite faces are represented by equal parallelograms. Thus, the image of the parallelepiped consists of three pairs of parallelograms, and the parallelograms in every pair are obtained from each other by parallel translation (*Fig*. 2.23).

As an images of three vertices of the lower base and one vertex of the upper base, we can choose the vertices of an arbitrary quadrilateral (for example, *ABDA′*). Then, the images of the remaining vertices can be completed uniquely.

3. The image of an *n*-gonal prism consists of two identical *n*-gons, which are obtained from each other by parallel translation, and *n* parallelograms. As an images of three vertices of the lower base and one vertex of the upper base, we can choose the vertices of an arbitrary quadrangle (in *Figure* 2.24, we have highlighted these points by making them bold). After this, the remaining vertices are completed uniquely.

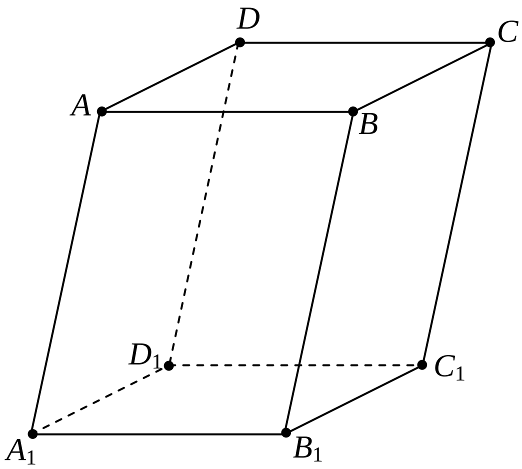

рис.2.23

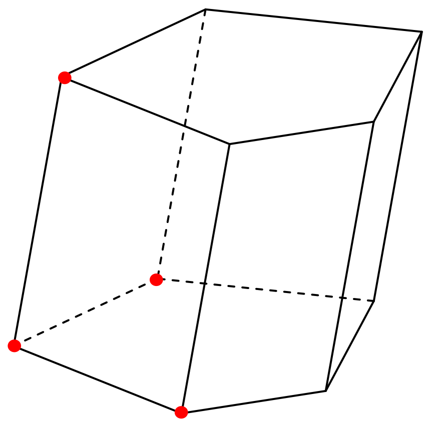

рис.2.24

4. The image of an *n*-angle pyramid consists of a polygon representing the base and triangles with a common vertex representing the lateral faces (*Figure* 2.25). We can choose any four points as an image of the three vertices of the lower base and one vertex of the upper base, of which no three lie on the same line. The remaining vertices of the base are constructed according to the rules for constructing images of flat polygons.

If the pyramid is regular, then it is also customary to depict the height falling into the center of the base (*Figure* 2.26).

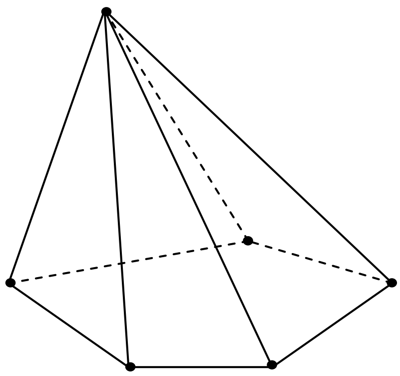

Fig.2.25

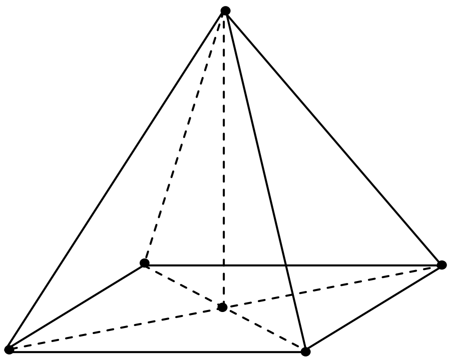

Fig.2.26

Questions for self-control

1. What can be chosen as an image of the vertices of a triangular pyramid?

2. How many points can be chosen arbitrarily when constructing an arbitrary: a) prism; b) pyramid? What is the only requirement that the chosen points must satisfy?

3. What should be added to the image of the pyramid to emphasize that it is correct?

2.5. *Construction of sections of polyhedra. Elementary techniques*

In what follows, for the sake of convenience of presentation, we will not distinguish the original from the image. For example, if A is the image of the vertex of a polyhedron, then we call it a vertex. We will consider the plane of the lower base of a prism or pyramid to be a horizontal plane and secondary projections of points are constructed precisely on this plane.

The image of a polyhedron is called *complete* if we can uniquely construct a secondary projection for each of its vertices.

Definition. A plane is called *a cutting plane* for a polyhedron if there are points of this polyhedron on both sides of this plane. Each face of the polyhedron is intersected by the secant plane along a segment. The union of these segments forms a polygon, which is called a *section of the polyhedron*.

***Example* 1.** A section of a tetrahedron can be a triangle or a quadrangle. Since there are only 4 faces, there can be no other options (*Figures* 2.27 a, b).

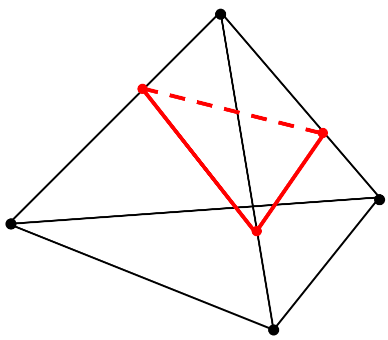

*Fig.*2.27 *a*

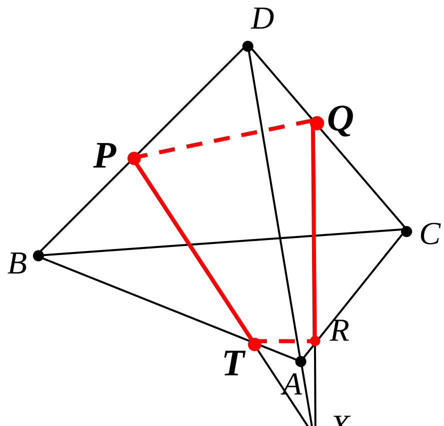

*Fig.*2.27 *b*

In Figure 2.27 b), we demonstrated one of the elementary techniques used in constructing sections. Suppose that we are given three points P, Q, R, which define a cutting plane (we have highlighted them in bold). Then we can immediately draw two sides of the section: PQ and PT. How to complete the construction of the section?

Lines PT and DA lie in the plane of one lateral face. Therefore, they intersect. Let us lengthen these lines on the drawing until they intersect at point X. Then and also lie in the plane of one lateral face, and we can connect these points.

At the intersection with side AC, we obtain another vertex of section R. $PQRT$ is the desired section. Segments TX and RX are not included in the section.

***Example* 2.** The cross-section of a cube can be a triangle, quadrangle, pentagon or hexagon (*Figures* 28 a, b). Since there are only 6 faces, there are no other options.

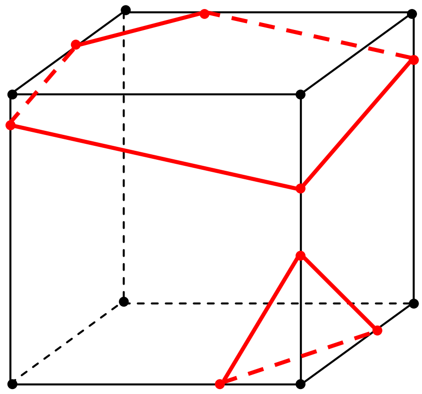

Fig. 2.28 a

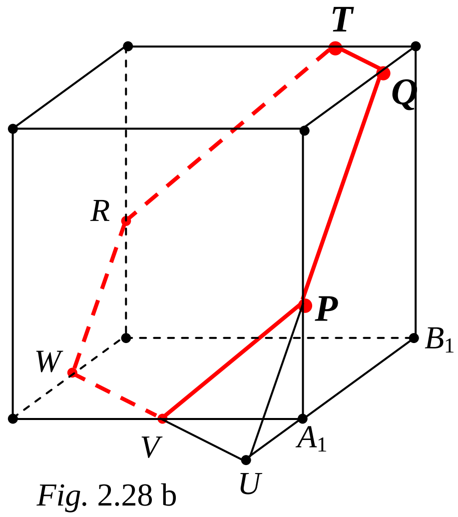

Fig. 2.28 b

Также на рисунке 2.28 б) показано применение ещё одного элементарного приема. Предположим, что нам даны три точки P, Q, R, which define a cutting plane (we have highlighted them in bold). Then we can immediately draw two sides of the section: PQ and QT. How to complete the construction of the section?

We extend lines PQ and A_1B_1 until they intersect at point U. Then we draw line UW parallel to TQ and find two more vertices of the section: V and W. This action is explained as follows: the opposite faces of the cube are parallel, and therefore the cutting plane intersects them along segments of parallel lines. Similarly, we draw line TR parallel to PV. We got the section $PQTRWV$.

The application of the first method in constructing a section of a triangular prism is also shown in *Figure* 2.29. Here the points M, N, P are given. In order to clearly understand the position of the point N, the secondary projection N_1 is also shown.

***Exercise*.** Construct a drawing on your own in which point Y is not on edge CC_1, but on its extension.

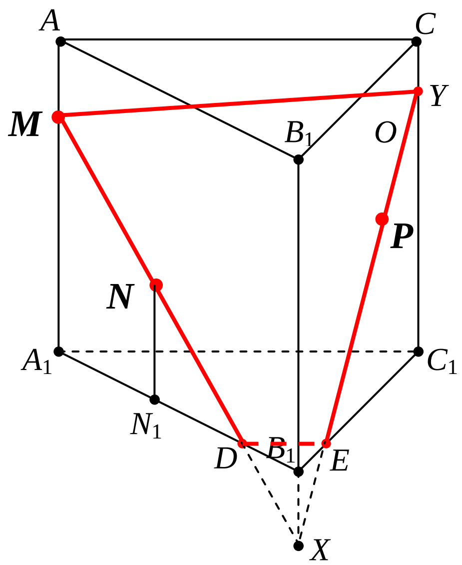

Fig. 2.29

2.6. *Internal projection method*

Before continuing reading, study problem 2 from section 2.3 carefully again.

In this section, we will consider the situation when among the given points there are no two that lie on the same face, and we cannot draw one of the sides of the section at once. Therefore, our goal will be to find a point on one of the lateral edges of the prism or pyramid (or its extension) that belongs to the cutting plane.

Task 2. *Given an image of a triangular prism $ABCA_1B_1C_1$ and three points M, N, P, which lie respectively on the edge CC_1 и гранях ABB_1A_1, BCC_1B_1. Construct a section of the prism by a plane passing through M, N, P.*

Solution. We will look for a point X where the cutting plane intersects the edge BB_1. We know its secondary projection $X_1 = B_1$ and are given three points through which the cutting plane passes. First, we construct secondary projections M_1, N_1, P_1 of the given points (*Figure* 2.30). As a rule, we will not include this obvious action in the record of construction process. Then, just as in problem 2 from section 2.30, we construct:

1. $M_1B \cap N_1P_1 = Y_1$;
2. on the connection line find $Y \in NP$;
3. $MY \cap BB_1 = X$.

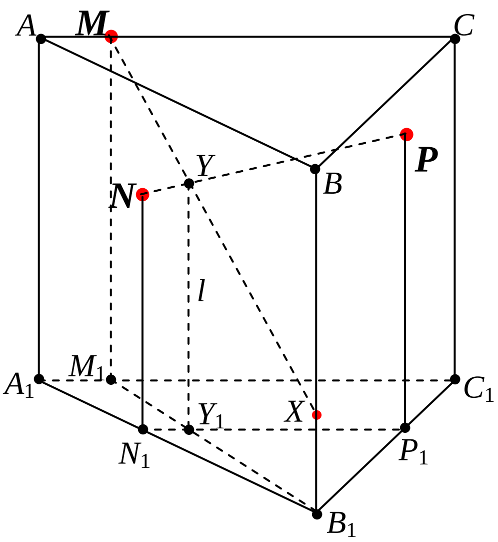

Fig. 2.30

The point X found is the main element of the construction. We will not record further actions: they are similar to what we did in the previous section. The result is shown in Figure 2.31.

The method of constructing a section described in this problem will be called the *matching method* or the *internal projection method*. Its main advantage is that all constructions are carried out inside the image of the polyhedron (i.e., they do not require additional space).

This method allows you to find the intersection of a cutting plane not only with one edge, but with several vertical edges (or their extensions) at once, as shown in the following example.

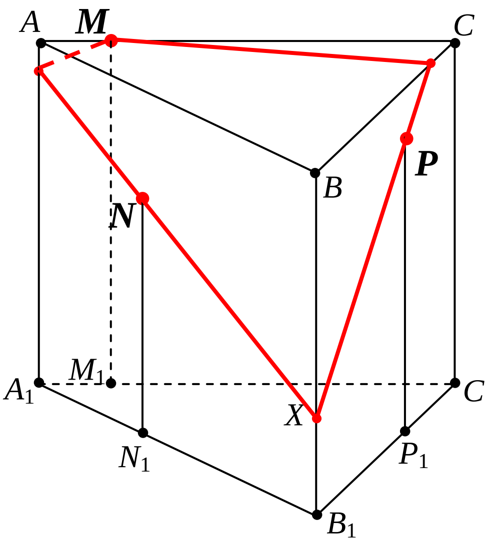

Fig. 2.31

***Task* 2.** *Given an image (Figure 2.32) of a pentagonal prism $ABCDEA_1B_1C_1D_1E_1$ and three points M, N, P, which lie respectively on the edges EE_1, DD_1 and face ABB_1A_1. Construct a section of the prism by a plane passing through M, N, P.*

Solution. We will look for the intersection of the cutting plane with the edges BB_1 and CC_1. We construct (*Figure* 2.32):

1. $BE \cap P_1D = Q_1$,
   $CE \cap P_1D = R_1$;
2. on the connection line we find $Q, R \in PN$;
3. $MQ \cap BB_1 = X$, $MR \cap CC_1 = Y$;
4. $XP \cap AA_1 = F$, $XY \cap BC = G$,
   $NY \cap DC = H$;
5. *MNHGXF* is the required section.

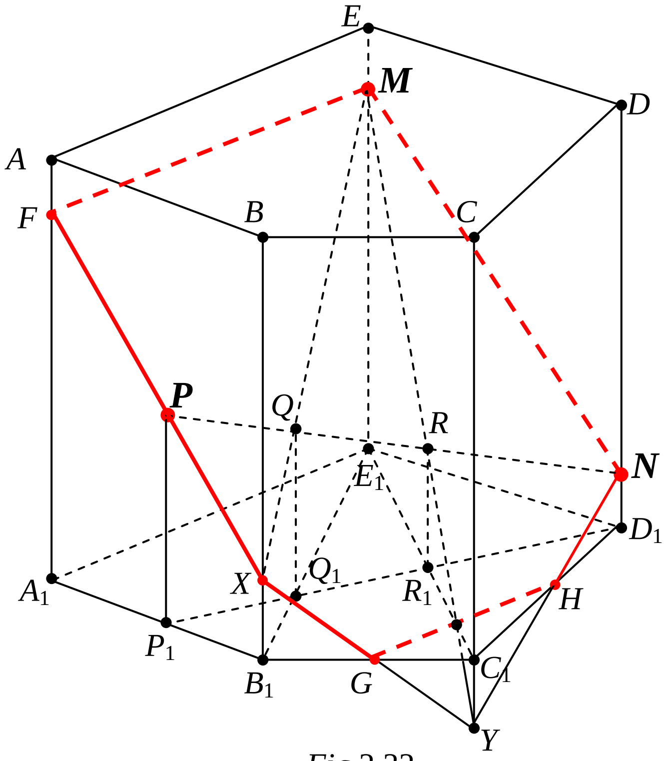

*Fig.*2.32

This solution is not the only one. Figure 2.33 shows how we could find the intersection of the cutting plane with edge AA_1. Then, using lines *FN* and *AD*, we can find the intersection with edge CC_1. Do it yourself.

When constructing sections of a pyramid, all constructions are performed similarly. The only difference is that the secondary projections of points are constructed from the apex (and not using vertical lines). Therefore, we designate them differently: the projection of point *P* onto the base is designated P_o.

***Task* 4.** *Given an image of a quadrangular pyramid SABCD and three points M, N, P, which lie respectively on the edge SC and faces SAB, SAD. Construct a section of the pyramid by a plane passing through M, N, P.*

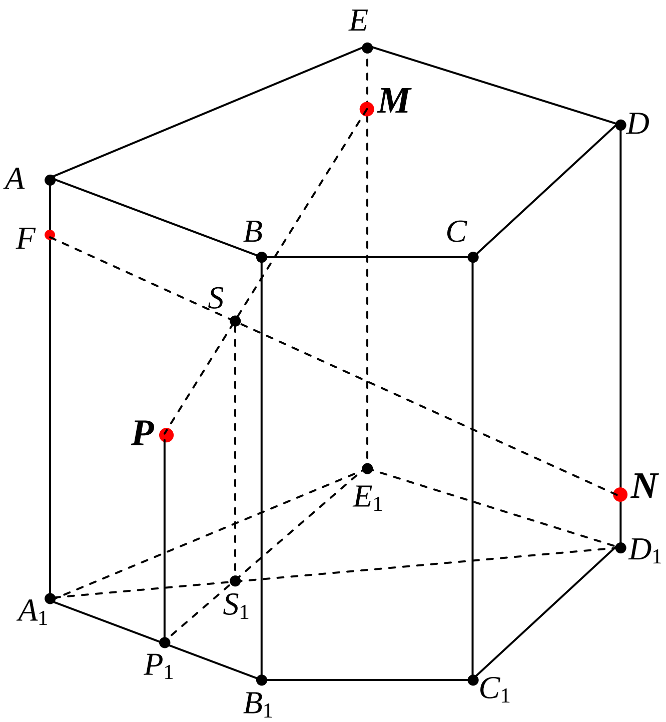

*Fig.*2.33

Solution. First, we construct secondary projections of the given points: $SP \cap AD = P_o$, $SN \cap AB = N_o$ (*Figure* 2.34 *a*). The secondary projection of point M already exists – it is C. In the future, we will not include this action in the description of the construction.

Next we look for the intersection of the cutting plane with the edge SA. We construct (*Figure* 2.34 b):

1. $CA \cap P_oN_o = Y_o$; 2. $SY_o \cap PN = Y$;

3. $MY \cap SA = X$;

4. $XP \cap SD = F$, $XN \cap SB = E$;

5. $XEMF$ is the required section.

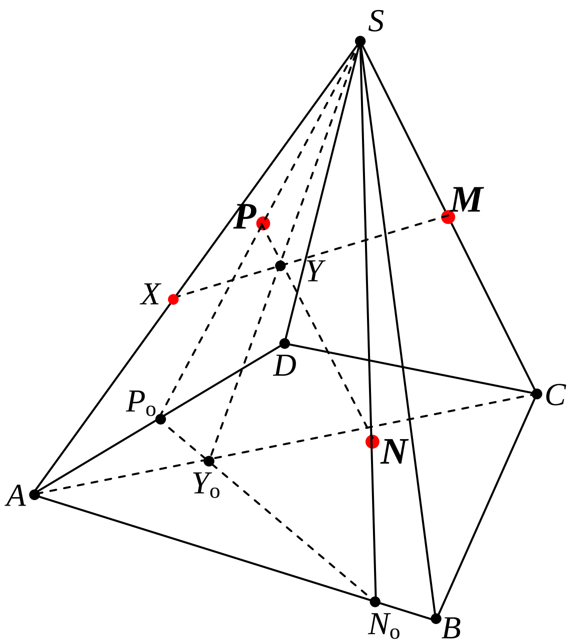

Fig.2.34 *a*

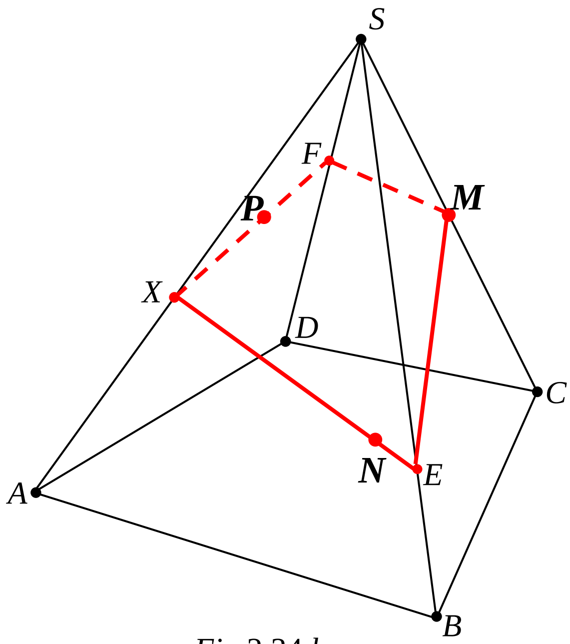

Fig.2.34 *b*

Questions and exercises for self-control

1. What plane is called a cutting plane for a polyhedron? What is called a section of a polyhedron?

2. How many sides can have a section of: a) a triangular pyramid; b) a cube?

3. What is the main advantage of the matching method?

4. How are secondary projections of points constructed in problems on a pyramid?

5.1. Figure 35 shows an image of a triangular prism and three points M, N, P on its surface. Find the point X representing the intersection of the plane MNP with the edge CC_1. In this case, all the necessary lines, except two, have already been drawn on the drawing, and the point Y_1 has been found.

5.2. Figure 36 shows an image of a quadrangular prism and three points M, N, P on its surface. Find the points X and W representing the intersection of the plane MNP with the edges CC_1 and DD_1 respectively (or with the extensions of these edges). In this case, all the necessary lines, except for four, have already been drawn on the drawing, and the points Y_1 and Z_1 have been found.

5.3. Complete task 5.2 and complete the construction of the section on a separate drawing, transferring only points M, N, P and already constructed points (without auxiliary lines) to the new drawing.

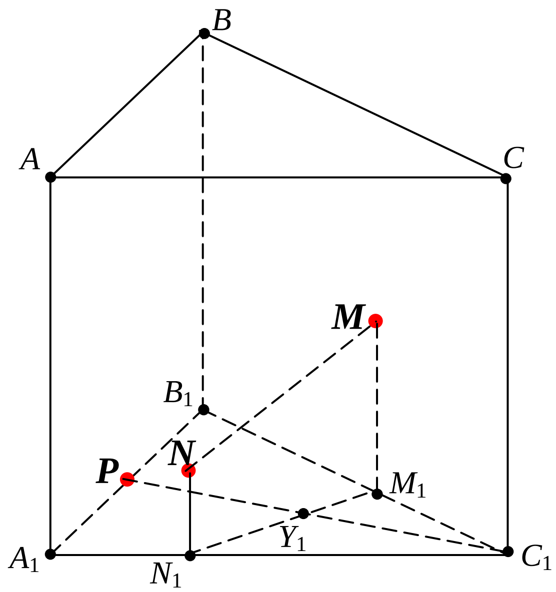

*Fig.*2.35

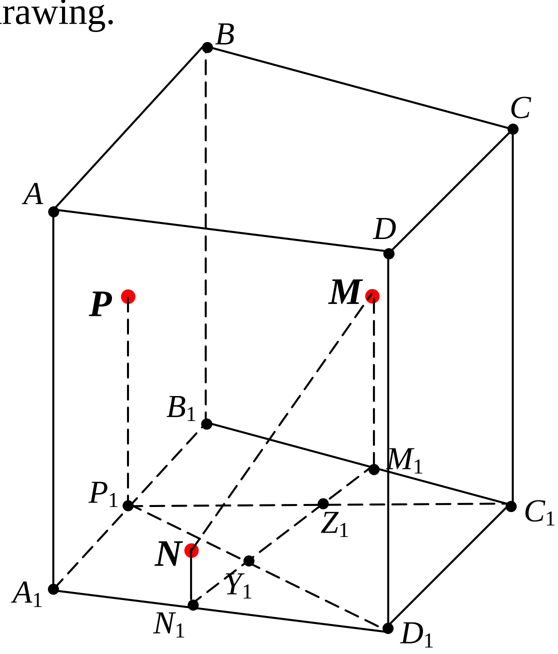

Fig. 2.36

6.1. Figure 37 shows a picture of a triangular pyramid and three points M, N, P on its surface. Find the point X representing the intersection of the plane MNP with the edge SC. In this case, all the necessary lines, except two, have already been drawn on the drawing, and the point Y_o has been found.

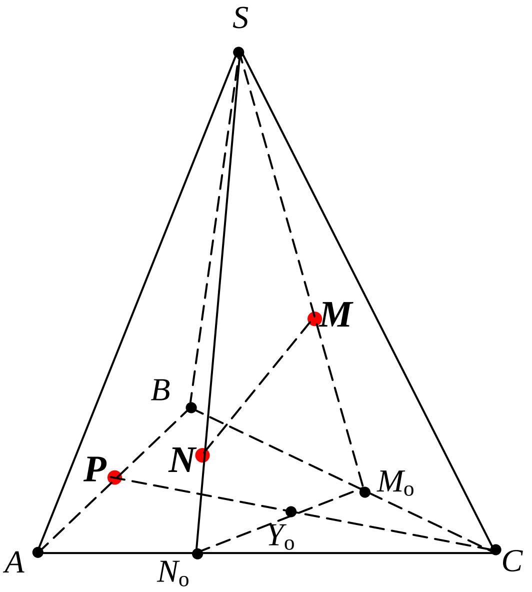

*Fig.*2.37

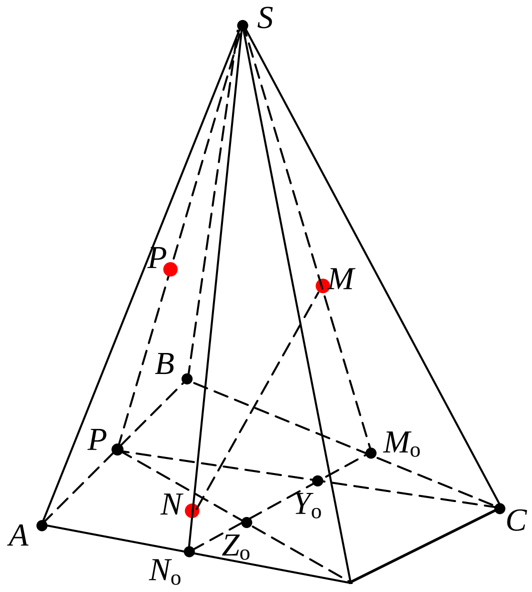

Fig. 2.38

6.2. Figure 38 shows a picture of a quadrangular pyramid and three points M, N, P on its surface. Find the point X and W, depicting the intersection of the plane MNP with the edges SC and OD respectively (or with the extensions of these edges). In this case, all the necessary lines, except two, have already been drawn on the drawing, and the points Y_o and Z_o have been found. .

6.3. Complete task 6.2 and complete the construction of the section on a separate drawing, transferring to the new drawing only points M, N, P and already constructed points (without auxiliary lines).

2.7 Construction of sections of polyhedra by the trace method

In this method, our first step (after finding the secondary projections of the given points) is to construct the trace of the cutting plane on the plane of the upper or lower base of the prism or truncated pyramid or on the base of the pyramid. As examples, we will consider the same problems as in Section 2.6.

In some cases, it is more convenient to consider the plane of the upper base as the horizontal plane.

Task* 1. *Solution. We already have one point on the upper base of the prism, therefore we consider the horizontal plane to be the plane of the upper base and we will construct the trace on this plane.

We construct secondary projections N_1 and P_1 of points N and P onto the upper base (*Figure* 2.39).

Then:

1. $NP \cap N_1P_1 = X$;
2. $MX = h_\alpha$ is the trace;
3. $h_\alpha \cap B_1C_1 = D$.

The further steps have already been shown in *Figure* 2.31.

Task* 3. *Solution. We will construct a trace of the cutting plane on the lower base of the prism (Figure 2.40).

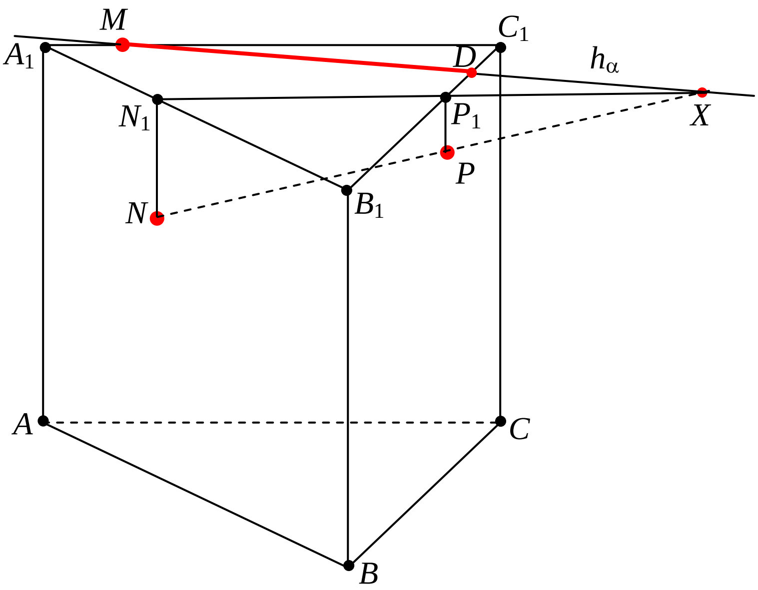

Fig. 2.39

1. $MN \cap ED = X$, $MP \cap EP_3 = Y$;
2. $h_\alpha = XY$ is the trace;
3. $h_\alpha \cap BC = G$, $h_\alpha \cap DC = H$.

The points G and H found are not enough to complete the construction. We need to find a point on the edge BB_1 or on the edge AA_1. Therefore, we will use the following method, which we will call the *main method of working with a trace*.

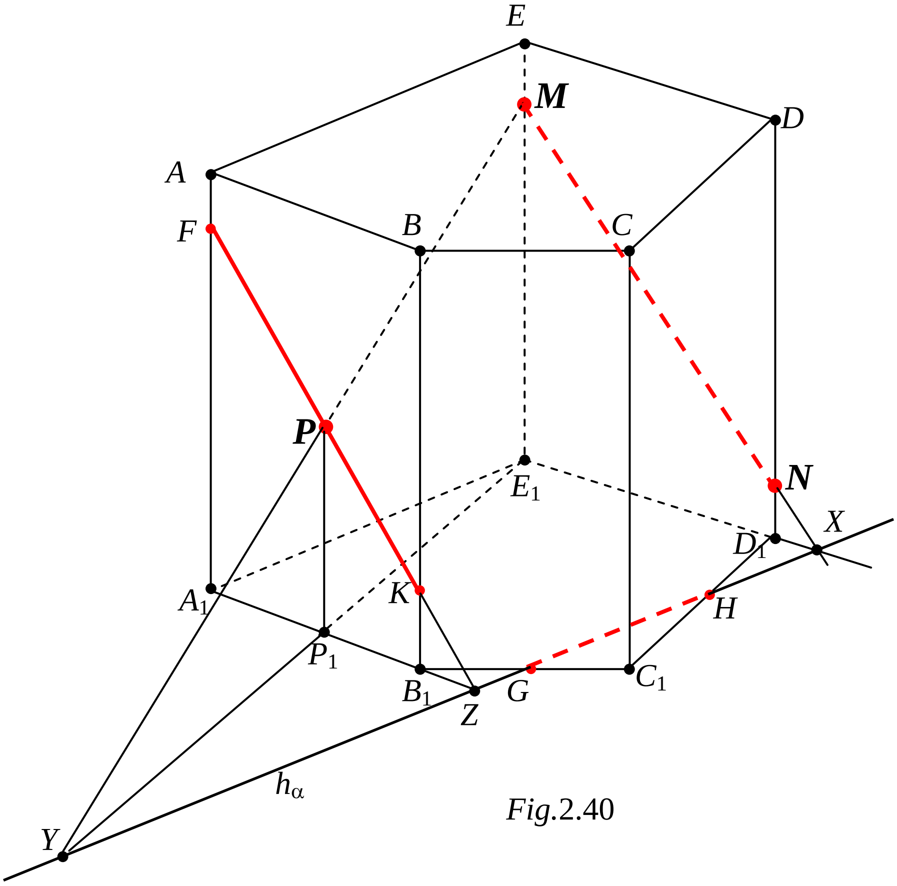

Fig.2.40

In the face ABB_1A_1 we already have one point P. Therefore, we lenghten the lower edge of this face, i.e. AB, until it intersects with the trace. These two lines must intersect, since they lie in the same plane:

4. $AB \cap h_\alpha = Z$.

Points P and Z lie simultaneously in the plane of one face and in the secant plane. Therefore, they lie on the intersection line of the planes. We construct:

5. $PZ \cap AA_1 = F$; $PZ \cap BB_1 = K$.

The further steps have already been shown above. If it turns out that line AB does not intersect with the trace, then the sought line FK will also be parallel to the trace.

The disadvantage of the trace method is that the construction often goes far beyond the drawing and does not fit on a sheet of paper.

Task* 3. *Solution. We construct (Figure 2.41):

1. $PN \cap P_oN_o = X$;
2. $MN \cap CN_o = Y$;
3. $h_\alpha = XY$ – is the trace;
3. $CB \cap p = Z$;
4. $ZM \cap SB = E$;
5. $EN \cap SA = G$
6. $GEMF$ is the required section.

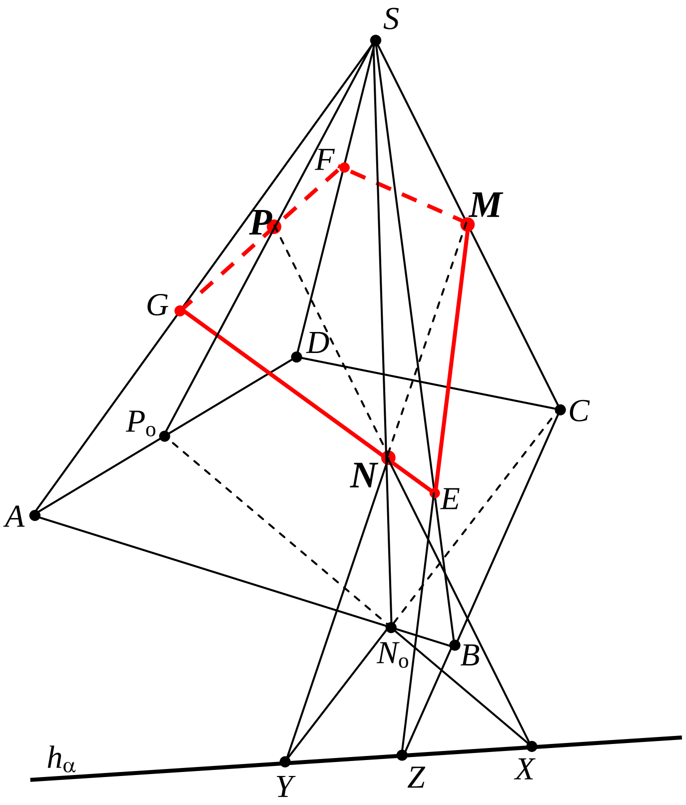

Fig.2.41

When constructing sections of a prism and a truncated pyramid, it is sometimes useful to use the following fact: the traces of the cutting plane on the lower and upper bases are parallel. In order to demonstrate this technique with an example, we will slightly change the data in problem 3.

Point F is not on edge AA_1,, but on edge A_1B_1 (*Figure* 2.42). Next, we draw a straight line $h'_\alpha || h_\alpha$ through F and find point L on edge A_1E_1. It remains to connect all the found vertices of the section.

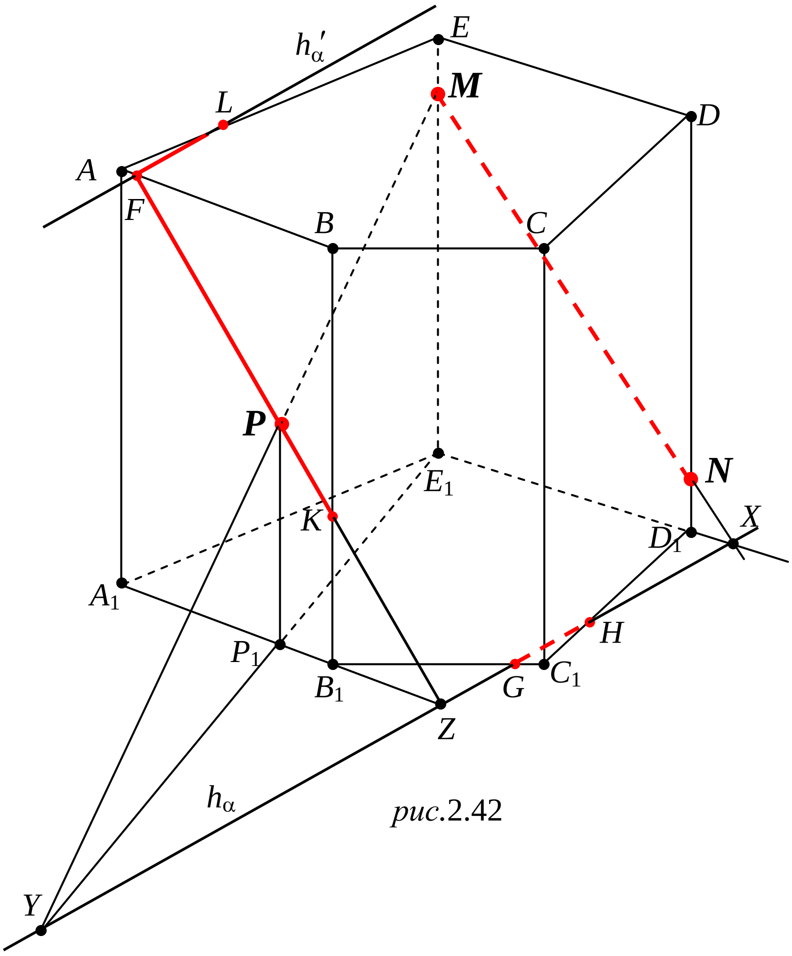

рис.2.42

Similarly, when constructing a section of a cube or a regular quadrangular prism, one can use the fact that the opposite lateral faces are parallel, and therefore the cutting plane intersects them along parallel segments (Figures 28 a, b).

Exercises for self-control

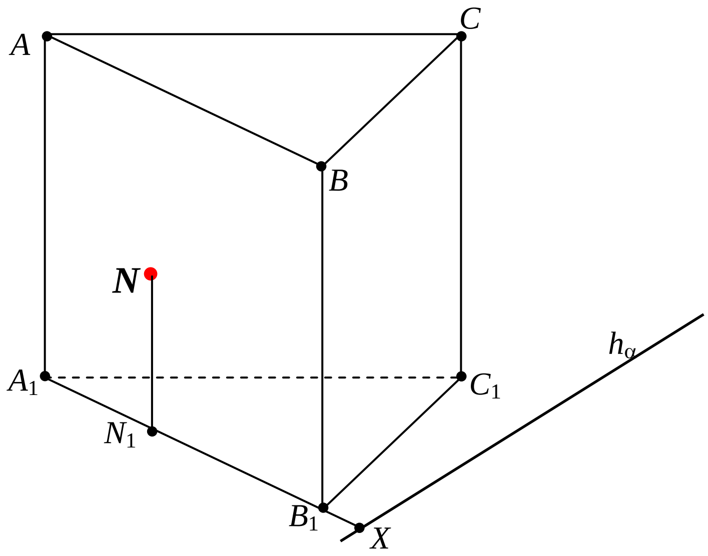

Fig.2.43

1.1. Given an image of a triangular prism and a trace of a cutting plane. Given a point on the lateral face. Construct a section of the prism (*Figure* 2.43).

As a hint, the first action has already been performed (point X has been found on the trace).

1.2. For the figure shown in *Figure* 2.36, construct the trace of the plane passing through the given points M, N, P on the lower base of the prism and find the intersection of the secant plane with any of the lateral faces.

1.3. For the figure shown in Figure 2.36, construct a section by a plane passing through the given points M, N, P, using the trace method.

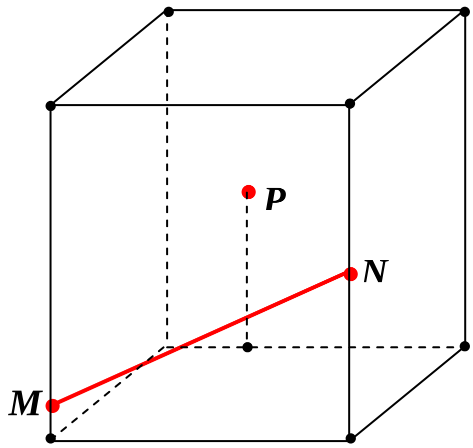

Fig.2.44

2.1. An image of a regular quadrangular prism is given. A section side has already been constructed on one of the lateral faces. A point belonging to the opposite lateral face is given (*Figure* 2.44). Complete the construction of the section.

2.2. For the figure shown in Figure 2.38, construct a section by a plane passing through the given points M, N, P, on the base of the pyramid and find the intersection of the section plane with any of the lateral faces.

2.3. For the figure shown in Figure 2.38, construct a section by a plane passing through the given points M, N, P, using the trace method.

CHAPTER 3. POLYHEDRA IN THE SYSTEM OF SEVERAL PROJECTION PLANES

3.1 *Construction of projections of lines and points*

***Task* 1.** An image of a quadrangular prism is given on the horizontal and frontal projection planes and a frontal projection M_2 of a point M lying on its lateral face $ABBA$ is given (Figure 3.1). Construct an image of the prism on the profile plane and find the two missing projections of the point.

The solution to the problem is shown in the figure.

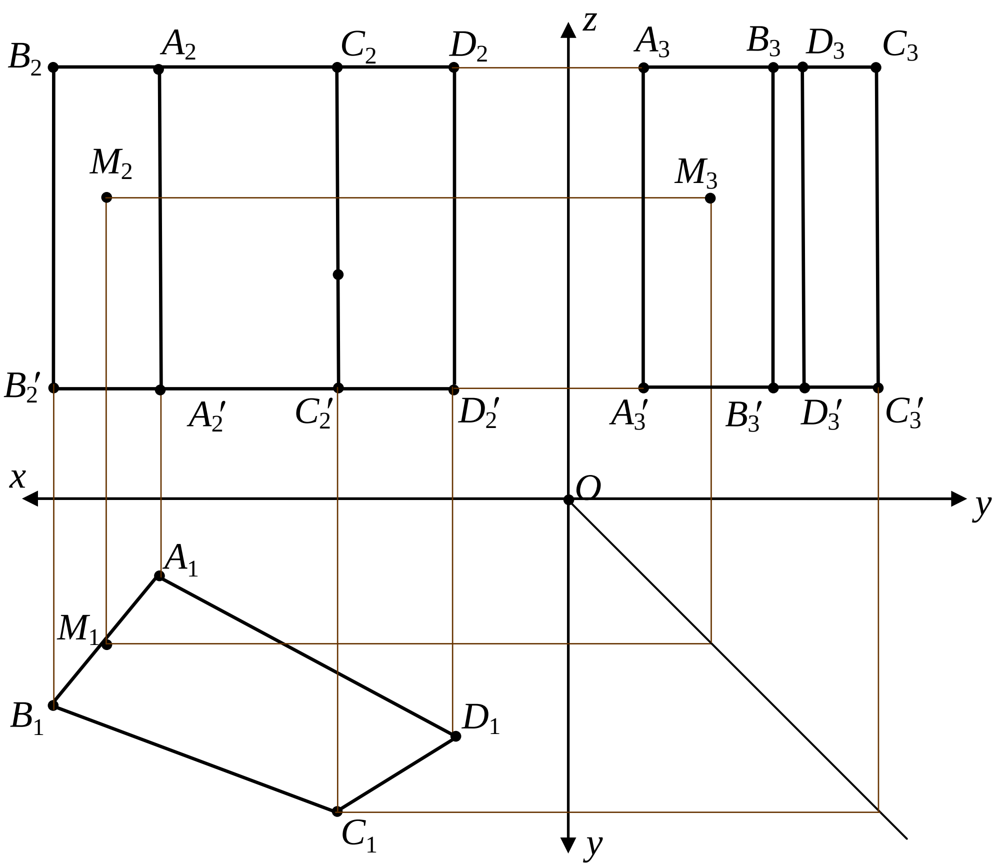

Fig. 3.1

***Task* 2.** Given an image of a triangular pyramid on the horizontal and frontal projection planes and given a frontal projection M_2 of a point M lying on its lateral face SAB (*Figure* 3.2). Construct an image of the pyramid on the profile plane and find the two missing projections of the point.

Solution. The construction of the third projection of the pyramid is shown in the drawing. To construct the missing projections of the point, we draw the segment S_2F_2 through the given point M_2, where $F_2 \in A_2B_2$, using the connection lines we find F_1 and F_3, and then the sought points M_1 and M_3 lie on the segments S_1F_1 and S_3F_3, respectively.

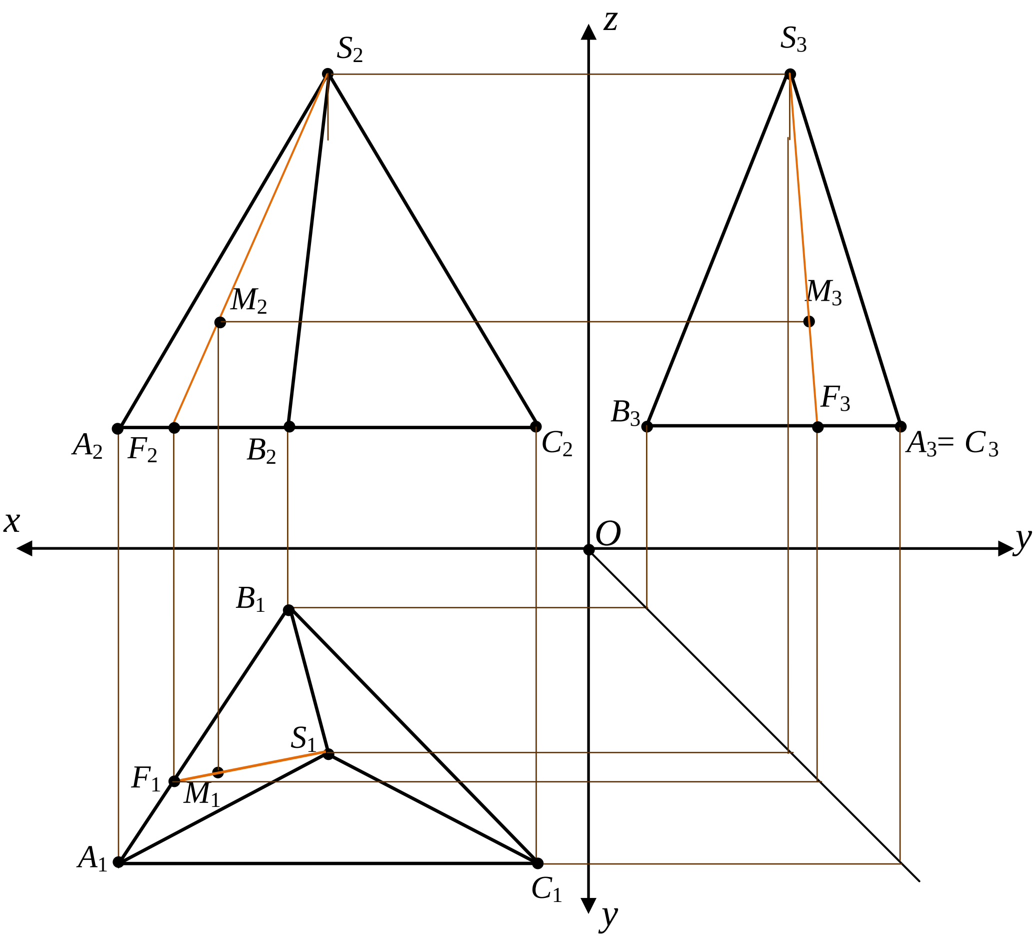

Fig. 3.2

3.2 *Sections of polyhedra. Developments*

***Task* 3.** An image of a triangular prism is given on three projection planes, and a section by the frontal-projecting plane is depicted on the frontal plane (*Figure* 3.3).

1) Construct an image of the section on the remaining projection planes.

2) Find the natural form of this section.

Solution. The solution to the first point is shown in the drawing. Note that we do not need broken connection lines to construct the section image on the horizontal ($M_1N_1P_1$) and profile ($M_3N_3P_3$) planes.

We omit the theoretical justification for the solution of point 2 and apply a purely practical approach. We draw perpendiculars to the segment $M_2N_2P_2$, and on them we lay off segments equal to the distance from the Ox axis to the points M_1, N_1, P_1. We obtain the triangle $M_2N_2P_2$, which represents the natural form of the section.

We will consider another way of constructing a natural cross-section shape in practical classes.

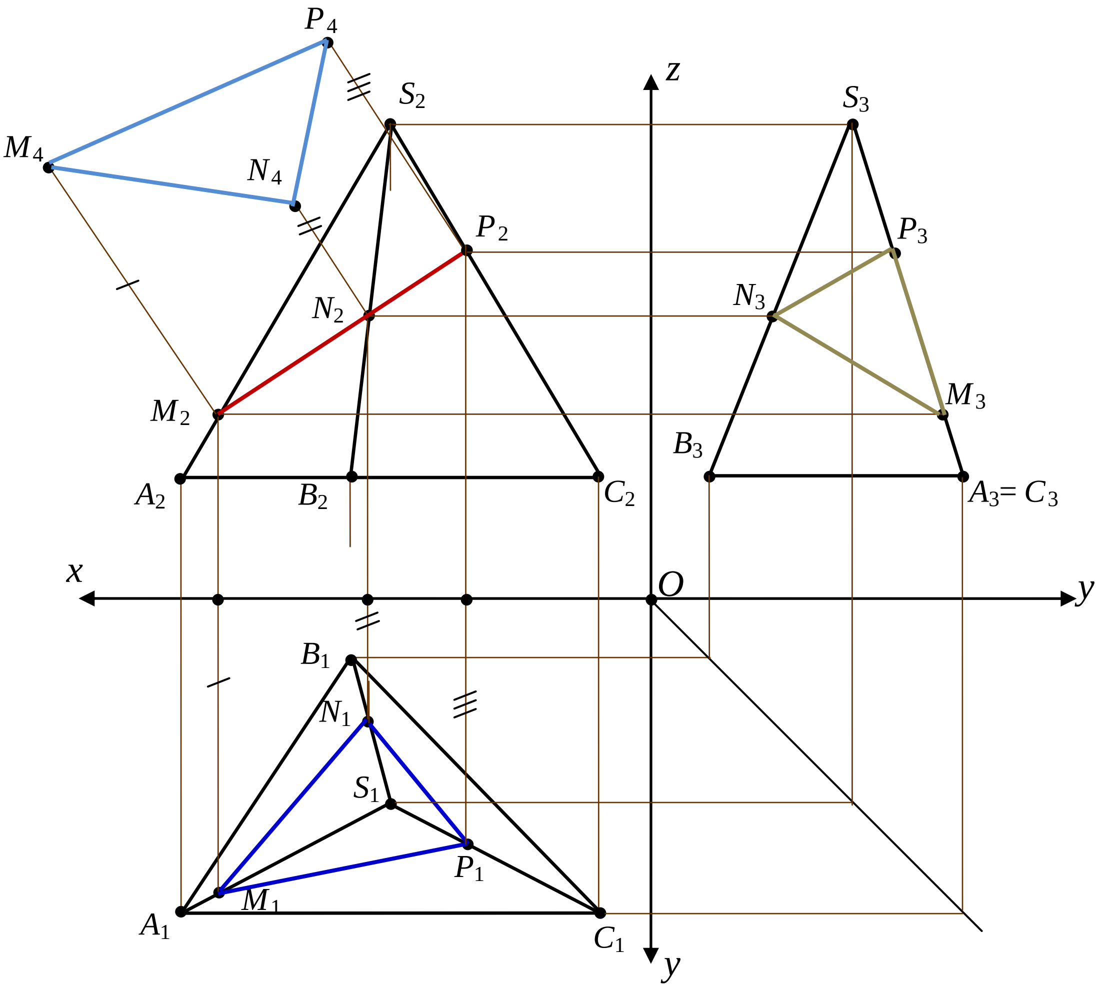

Fig. 3.3

***Task* 4.** An image of a triangular prism on the horizontal and frontal projection planes is given (*Figure* 3.4). Construct a development of this pyramid.

Solution. The base edges are shown in the drawing in natural size. Therefore, to construct the development, we need to find only the natural size of the side edges. For this, we use the rotation method.

We rotate the points A_1, B_1, C_1 around the point S_1 so that the new segments S_1A_1', S_1B_1', S_1C_1' are parallel to the Ox-axis. In this case, the points A_2, B_2, C_2 are moved parallel to the Ox-axis and take a new positions A_2', B_2', C_2'. The segments S_2A_2', S_2B_2', S_2C_2' represent the natural length of the lateral edges.

Then, using a compass and ruler, we construct triangles that represent the natural shape of all the faces together on one drawing (*Figure* 3.5).

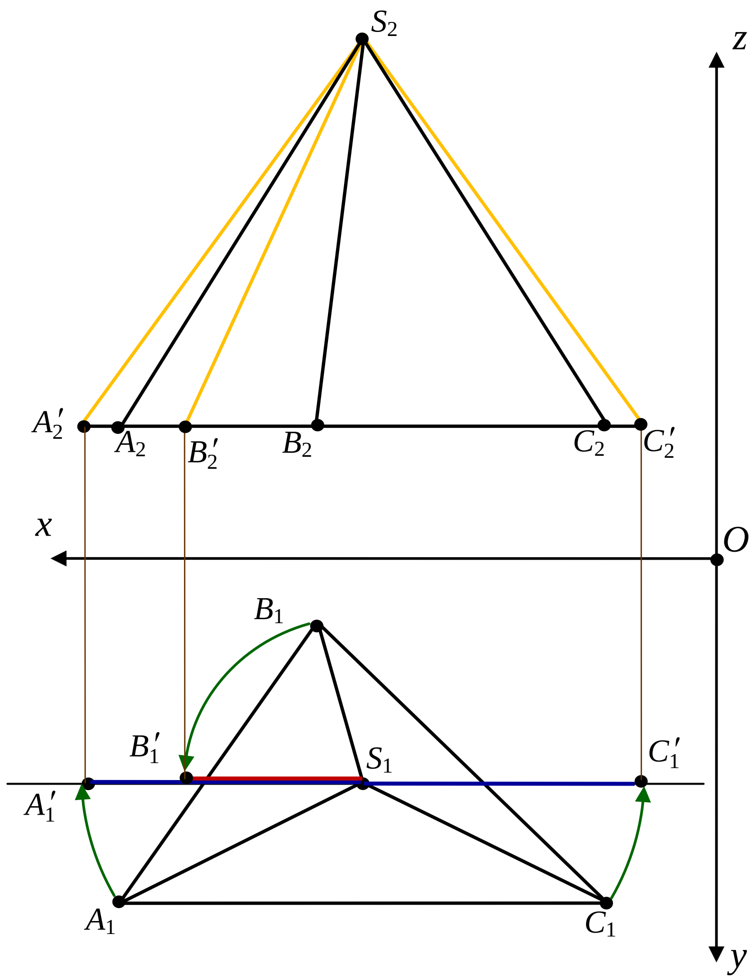

Fig. 3.4

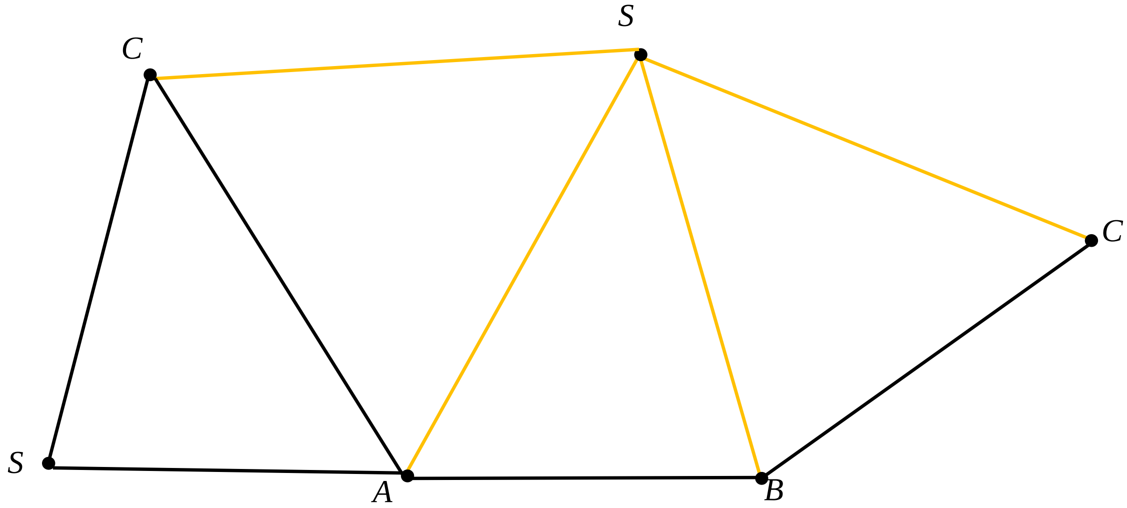

Fig.3.5

Educational publication

PODOKSENOV Mikhail Nikolaevich
KAZHEKINA Lyudmila Viktorovna

DESCRIPTIVE GEOMETRY. HANDS-ON TRAINING

Practical course for self-study activities

Technical editor	*G.V. Razboeva*
Computer design	*A.V. Tabanyukhova*

Signed to print 26.03.2025. Format 60x84 $^{1}/_{16}$. Offset paper.
Conventional printed sheets 2,56. Published sheets 1,53. Circulation 9 copies. Order 34.

Publiser and polygraphic processing – Educational Establishment
“Vitebsk State University named after P.M. Masherov”.

State Registration Certificate as publisher, printer and distributor of editions
№ 1/255 d/d 31.03.2014.

Printed by risograph of Educational Establishment
“Vitebsk State University named after P.M. Masherov”.
210038, Vitebsk, Moskovsky Prospekt, 33.