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***M.V. Linkevich, M.N. Podoksenov*[1]**

Vitebsk State University named after P.M. Masherov, Vitebsk, Belarus

[1]SPIN-код: 5120-2532

SELF-SIMILAR LIE GROUP $\mathcal{E}(1,1)$

Let the Euclidean or Lorentz scalar product be given in Lie algebra $\mathcal{G}$. A linear transformation $f\colon \mathcal{G}\to\mathcal{G}$ is called an *autosimilarity* if it is both an automorphism of the Lie algebra and a similarity with respect to a given scalar product. We will call a Lie algebra self-similar if it admits a one-parameter group of autosimilarities.

To construct a self-similar homogeneous Lorentz manifold of a Lie group equipped with a left-invariant Lorentz metric, we need to construct a one-parameter autosimilarity group of the corresponding Lie algebra equipped with a Lorentz scalar product [1].

All autosimilarities of three-dimensional solvable Lie groups with respect to the Lorentz scalar product were found in [2]. All such Lie algebras contain a two-dimensional commutative ideal $\mathcal{L}$, and the Lie algebra $\mathcal{E}(1,1)$ of the group $E(1,1)$ of motions of the Minkowski plane is one of them.

The basis in the Lie algebra $\mathcal{E}(1,1)$ used in [2] is not convenient for further calculations. In this work we will indicate the matrix representation of the Lie algebra $\mathcal{E}(1,1)$, find the formulas by which the exponential map and its inverse map act in suitable coordinates, find the matrix of the left shift differential in the Lie group $E(1,1)$, and find formulas by which the one-parameter self-similarity group of the Lie algebra $\mathcal{E}(1,1)$ acts in the same coordinates.

In a suitable basis (E_1, E_2, E_3) the bracket operation in the Lie algebra $\mathcal{E}(1,1)$ is given by the formulas $[E_1,E_2]=E_3$, $[E_1,E_3]=E_2$, $[E_2,E_3]=\vec{0}$. Basis vectors and Lie algebra in general have the following matrix representation:

$$E_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\ E_2=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\ E_3=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},$$

$$U=\begin{pmatrix}0&u_1&u_2\\u_1&0&u_3\\0&0&0\end{pmatrix},$$

u_1, u_2, $u_3\in\boldsymbol{R}$. Denote $X=\exp U$. Then

$$X=\begin{pmatrix}\mathrm{ch}x_1&\mathrm{sh}x_1&x_2\\\mathrm{sh}x_1&\mathrm{ch}x_1&x_3\\0&0&1\end{pmatrix},$$

where

$$x_1=u_1,\ x_2=\frac{u_2}{u_1}\mathrm{sh}u_1+\frac{u_3}{u_1}(\mathrm{ch}u_1-1),\ x_3=\frac{u_3}{u_1}\mathrm{sh}u_1+\frac{u_2}{u_1}(\mathrm{ch}u_1-1)$$

These are the formulas for the exponential mapping $\exp:\mathcal{E}(1,1)\to SE(1,1)$ ($SE(1,1)$ is the connected component of the unit element). Formulas of the inverse mapping:

$$u_1=x_1,\ u_2=\frac{x_1x_2\mathrm{sh}\,x_1}{2(\mathrm{ch}\,x_1-1)}-\frac{x_1x_3}{2},\ u_3=\frac{x_1x_3\mathrm{sh}x_1}{2(\mathrm{ch}x_1-1)}-\frac{x_1x_2}{2}.$$

Group operation and the inverse element:

$$(x_1,x_2,x_3)\cdot(y_1,y_2,y_3)=$$
$$=(x_1+y_1,x_2+y_2\mathrm{ch}x_1+y_3\mathrm{sh}x_1,x_3+y_3\mathrm{ch}x_1+y_2\mathrm{sh}x_1),$$
$$(x_1,x_2,x_3)^{-1}=(-x_1,x_3\mathrm{sh}x_1-x_2\mathrm{ch}x_1,x_2\mathrm{sh}x_1-x_3\mathrm{ch}x_1).$$

Thus, the matrix of the differential of the left shift by element $X(x_1,x_2,x_3)$

$$L_X=\begin{pmatrix}1&0&0\\0&\mathrm{ch}x_1&\mathrm{sh}x_1\\0&\mathrm{sh}x_1&\mathrm{ch}x_1\end{pmatrix}$$

is constant, i.e. does not depend on the element $Y(y_1,y_2,y_3)$.

Any basis (V_1, V_2, V_3), in which the bracket operation is given by the equalities $[V_1,V_2]=-V_2$, $[V_1,V_3]=V_3$, $[V_2,V_3]=\vec{0}$ will be called canonical. This could be the basis $V_1 = E_1$, $V_2 = E_2 - E_3$, $V_3 = E_2 + E_3$. The subspaces $I_1 = \boldsymbol{R}V_1$, $I_2 = \boldsymbol{R}V_2$ are one-dimensional ideals. In a suitable canonical basis, the one-parameter autosimilarity group acts by the formulas ([2]):

$$V_1' = V_1,\ V_2' = e^{\mu t}V_2,\ V_3' = e^{2\mu t}V_3,\ \mu > 0,$$

provided that the Gram matrix of this basis has the form:

$$\Gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Unfortunately, in the canonical basis the formulas for the exponential mapping and its inverse mapping look more complicated. That is why we calculated the Gram matrix and the matrix of the one-parameter autosimilarity group in the basis (E_1, E_2, E_3):

$$\Gamma' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix},\ \left[F(t)\right]' = e^{3\nu t}\begin{pmatrix} e^{-3\nu t} & 0 & 0 \\ 0 & \mathrm{ch}\nu t & \mathrm{sh}\nu t \\ 0 & \mathrm{sh}\nu t & \mathrm{ch}\nu t \end{pmatrix},\ \mu = 2\nu.$$

Knowing these matrices and the formulas by which the exponential map and its inverse map operate, makes it possible to construct a self-similar Lorentzian manifold of the Lie group $SE(1,1)$, equipped with a left-invariant Lorentzian metric. This is our immediate goal.

References

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