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SPECIAL THREE-DIMENSIONAL LIE ALGEBRA AND ITS GROUP OF AUTOISOMORPHISMS

Let the Euclidean or Lorentz scalar product be given in Lie algebra $\mathcal{G}$. A linear transformation $f\colon \mathcal{G}\to\mathcal{G}$ is called an *autoisomorphism* (*an autosimilarity*) if it is both an automorphism of the Lie algebra and an isometry (a similarity) with respect to a given scalar product. All autosimilarities of three-dimensional solvable Lie groups with respect to the Lorentz scalar product were found in [1].

According to the classification given in the review by J. Milnor [2], there are, up to isomorphism, 6 unimodular three-dimensional Lie algebras and an infinite family of non-unimodular ones. All non-unimodular three-dimensional Lie algebras are solvable and contain a two-dimensional commutative ideal $\mathcal{L}$, which is the unimodular kernel. Among them, a special case stands out: a Lie algebra in which there exists a vector $X\notin\mathcal{L}$, such that $\operatorname{ad}X$ acts on $\mathcal{L}$, as an identity transformation. We will call such a Lie algebra special and denote it by $\mathcal{S}_3$.

We set the task: find a matrix representation of the Lie algebra $\mathcal{S}_3$ and the corresponding connected simply connected Lie group S_3, introduce coordinates in them and find formulas by which the exponential map and its inverse map act. We will also write down the left shift formulas in the Lie group S_3, and in the Lie algebra $\mathcal{S}_3$ we will write down the formulas by which the one-parameter autoisometry group acts in the presence of a Euclidean scalar product.

In a suitable basis (E_1, E_2, E_3) the bracket operation in the Lie algebra $\mathcal{S}_3$ is given by the formulas $[E_1,E_2]=E_2$, $[E_1,E_3]=E_3$, $[E_2,E_3]=\vec{0}$. We managed to find that these requirements are satisfied by the linearly independent matrices

$$E_1=\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},\ E_2=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix},\ E_3=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Thus, the Lie algebra $\mathcal{S}_3$ can be represented as consisting of matrices of the form

$$U=\begin{pmatrix}0&0&u_1&u_2\\0&u_1&0&u_3\\u_1&0&0&u_2\\0&0&0&0\end{pmatrix},\ u_1,u_2,u_3\in\boldsymbol{R}.$$

In the basis (E_1,E_2,E_3) this matrix has the coordinates (u_1,u_2,u_3). Denote $X=\exp U$. Then

$$X=\begin{pmatrix}\operatorname{ch}x_1&0&\operatorname{sh}x_1&x_2\\0&e^{x_1}&0&x_3\\\operatorname{sh}x_1&0&\operatorname{ch}x_1&x_2\\0&0&0&1\end{pmatrix},$$

where

$$x_1=u_1,\ x_2=\frac{u_2}{u_1}(e^{u_1}-1),\ x_3=\frac{u_3}{u_1}(e^{u_1}-1).$$

These are the formulas for the exponential mapping $\exp:\mathcal{S}_3\to S_3$. Formulas of the inverse mapping:

$$u_1=x_1,\ \ u_2=\frac{x_1x_2}{(e^{x_1}-1)},\ \ u_3=\frac{x_1x_3}{(e^{x_1}-1)}.$$

These formulas show that the Lie group is exponential and the mapping $\exp:\mathcal{S}_3\to S_3$ is a homeomorphism.

Group operation and inverse element:

$$(x_1,x_2,x_3)\cdot(y_1,y_2,y_3)=(x_1+y_1,y_2e^{x_1}+x_2,y_3e^{x_1}+x_3),$$
$$(x_1,x_2,x_3)^{-1}=(-x_1,\ e^{-x_1}x_2,\ e^{-x_1}x_3).$$

Full group of automorphisms of the Lie algebra $\mathcal{S}_3$ in the basis (E_1,E_2,E_3) is defined by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & \alpha_{22} & \alpha_{23} \\ \beta & \alpha_{32} & \alpha_{33} \end{pmatrix}, \alpha, \beta \in \boldsymbol{R}, a_{22}a_{33} - a_{23}a_{32} \neq 0.$$

Thus, if the Euclidean scalar product is introduced in the Lie algebra $\mathcal{S}_3$ then we can always choose an orthonormal basis (E_1', E_2', E_3'), in which the bracket operation is given by the formulas $[E_1', E_2'] = kE_2'$, $[E_1', E_3'] = kE_3'$, $[E_2', E_3'] = \vec{0}$, $k > 0$. In this basis any one-parameter autoisometry group is given by a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos at & -\sin at \\ 0 & \sin at & \cos at \end{pmatrix}, t \in \boldsymbol{R}, a \neq 0.$$

With respect to the Euclidean scalar product, the Lie algebra $\mathcal{S}_3$ does not admit self-similarity at all.

References

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