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M.N. Podoksenov, L.V. Kazhekina

HIGHER MATHEMATICS.
HANDS-ON TRAINING

Practical course for self-study activities

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Authors: Associate Professor of the Department of Mathematics of the Educational Establishment “Vitebsk State University named after P.M. Masherov”, Candidate of Physical and Mathematical Sciences **M.N. Podoksenov**; Senior Lecture of the World Languages Department of the Educational Establishment “Vitebsk State University named after P.M. Masherov”**L.V. Kazhekina**

Reviewer:
Associate Professor of the Department of Mathematics
and Computer Security of the Polotsk State University named after Euphrosyne
of Polotsk, Candidate of Physical and Mathematical Sciences *A.A. Kozlov*

Podoksenov, M.N.

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This study guide is intended for students majoring in general higher education, studying in English as part of an experimental project and mastering the academic disciplines “Higher Mathematics” and “Linear algebra and Analytic Geometry” as an additional program. Theoretical material, examples of problem solutions are presented, individual versions of problems and examples of test assignments are attached.

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CHAPTER 1. HIGHER ALGEBRA

§1. *Matrices and determinants*

Definition. *A matrix* is a rectangular table made up of numbers. A matrix is usually designated by a capital letter of the Latin alphabet, and its elements by the same lowercase letter with two indices, the first (or upper) of which denotes the row number, and the second (or lower) - the number of the column in which the given element is located.

For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \tag{1.1}$$

This is a matrix consisting of 2 rows and 4 columns. We say that it has dimensions 2×4. In it $a_{11}=1$, $a_{12}=2$, and $a_{21}=5$. Instead of numbers, the matrix can contain variables. Two matrices are considered *equal* if they have the same size, and all their elements in the same places are equal.

Definition. A matrix of dimensions $n \times n$ is called a *square* matrix of order n. The elements of a square matrix whose row and column numbers coincide form *the main diagonal*. If all elements outside the diagonal are zero, the matrix is called *diagonal*. A diagonal matrix with ones on its main diagonal is called *a unit matrix* and is denoted by the letter **E**. For example, the unit matrix of order 3 has the form

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If all elements of the matrix below (above) the main diagonal are equal to zero, then the matrix is called *upper triangular* (*lower triangular*). A matrix whose elements are all equal to zero is called zero and is denoted by **O**.

Definition. *The transposition of a matrix* **A** is a permutation of its elements in which each element a_{ij} changes places with the element a_{ji}. The matrix that results from the transposition is denoted by $\mathbf{A}^{\mathrm{T}}$. For example, for the matrix (1.1)

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix},$$

Definition. The concept of determinant is introduced only for square matrices. The determinant of a matrix **A** is denoted by $\det \mathbf{A}$ or $|\mathbf{A}|$. If instead of round brackets there are straight brackets around the matrix elements, then this also means the determinant of the matrix. The determinant of a matrix of order 2 is calculated by the formula:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Let us assume that we already know how to calculate the determinant of a matrix of order $n-1$, and and that **A** is a matrix of order n. If we delete the i-th row and j-th column from the matrix, we get a matrix of order $n-1$. Let M_{ij} be the determinant of this matrix. It is called *the minor complement* to the element a_{ij}. Let us add a minus sign to this minor in the case where $i+j$ is odd. The resulting number is called the *algebraic complement* to the element a_{ij}; we will denote it by A_{ij}. We can write that

$$A_{ij}=(-1)^{i+j}M_{ij}.$$

Now we choose an arbitrary row in the matrix **A** and multiply each element of this row by its algebraic complement; add the resulting numbers. The value that we have calculated in this way is called the determinant of the matrix A. It can be shown that the result of the calculation does not depend on which row of the matrix we choose. For example, if we choose the first row, we get a formula called *the expansion of the determinant by the first row*:

$$\det\mathbf{A} = a_{11}A_{11}+ a_{12}A_{12}+\ldots+a_{1n}A_{1n}.$$

For a matrix of order 3, this formula looks like this:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} =$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

We took the second term with a minus sign because $1+2$ is odd.

Example: $$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\cdot\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\cdot\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\cdot\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} =$$

$$= 1\cdot(5\cdot 9 - 6\cdot 8) - 2\cdot(4\cdot 9 - 6\cdot 7) + 3\cdot(4\cdot 8 - 5\cdot 7) = -3 + 12 - 9 = 0.$$

We formulate the following properties only for rows, but they are also true for columns.

Properties of the determinant

1. If one row of the determinant consists only of zeros, then the determinant is zero.

2. If the determinant contains two identical or proportional rows, then it is zero.

3. When two rows of a matrix are permuted, the determinant changes sign.

4. The common factor of the elements of one row can be taken out of the sign of the determinant.

In the previous example, all elements of the third column are multiples of three. Therefore, we can take the factor 3 outside the determinant sign:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 3 \cdot \begin{vmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \\ 7 & 8 & 3 \end{vmatrix}$$

5. If one row of a determinant is represented as the sum of two rows, then the determinant is equal to the sum of the two corresponding determinants. For example,

$$\begin{vmatrix} 1 & 2 & 3 \\ 4+a & 5+b & 6+c \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ a & b & c \\ 7 & 8 & 9 \end{vmatrix}$$

6. If the elements of one row of a matrix are added to the corresponding elements of another row, multiplied by a certain number, then the determinant of the matrix will not change.

In our example, we add the first row multiplied by –1 to the second and third rows (the first row itself remains in its place without changes):

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix}$$

We got two proportional lines, therefore the determinant is equal to zero.

7. The determinant of a triangular matrix is equal to the product of the diagonal elements:

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 6 \\ 0 & 0 & 9 \end{vmatrix} = 1 \cdot (-3) \cdot 9 = -27.$$

A diagonal matrix is a special case of a triangular matrix. Therefore, its determinant is also equal to the product of the diagonal elements.

§2. *Linear operations on matrices*

Linear operations on matrices are operations of adding two matrices and multiplying a matrix by a number.

Only matrices of the same size can be added. In this case, their elements that are in the same places are added. As a result of addition, a matrix of the same size is obtained.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix}.$$

For example,

$$\mathbf{A}=\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \quad \mathbf{B}=\begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 3 & -3 \end{pmatrix}, \quad \mathbf{A}+\mathbf{B}=\begin{pmatrix} 1+1 & 4-1 \\ 2-2 & 5+2 \\ 3+3 & 6-3 \end{pmatrix}=\begin{pmatrix} 2 & 3 \\ 0 & 7 \\ 6 & 3 \end{pmatrix}.$$

When a matrix is multiplied by a number, each of its elements is multiplied by that number:

$$\lambda\cdot\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2n} \\ \dots & \dots & \dots & \dots \\ \lambda a_{m1} & \lambda a_{m2} & \dots & \lambda a_{mn} \end{pmatrix}.$$

For example, for matrix A from the previous example

$$3\cdot\mathbf{A}=\begin{pmatrix} 3 & 12 \\ 6 & 15 \\ 9 & 18 \end{pmatrix}.$$

We denote the matrix $-1\cdot\mathbf{A}$ as $-\mathbf{A}$.

Properties of linear operations on matrices

1. $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$;

2. $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$;

3. $\mathbf{A}+\mathbf{O}=\mathbf{A}$;

4. $\mathbf{A}+(-\mathbf{A})=\mathbf{O}$;

5. $\lambda(\mathbf{A}+\mathbf{B})=\lambda\mathbf{A}+\lambda\mathbf{B}$;

6. $(\lambda+\mu)\mathbf{A}=\lambda\mathbf{A}+\mu\mathbf{A}$;

7. $(\lambda\mu)\mathbf{A}=\lambda(\mu\mathbf{A})$;

8. $1\cdot\mathbf{A}=\mathbf{A}$.

§3. *Cramer's Rule*

Let a system of linear equations (SLE) be given in which the number of equations coincides with the number of unknowns. We will limit ourselves to the case when this number is equal to 3:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3. \end{cases} \tag{1.2}$$

The numbers a_{ij} are called the coefficients of the system, and the numbers b_1, b_2, b_3 are called *the free terms. A solution to a system of linear equations* is any set of numbers $(\alpha_1, \alpha_2, \alpha_3)$, such that when substituted for the unknowns x_1, x_2, x_3 all equations of the system are transformed into true equalities. The coefficients of the system form the matrix **A**, and the free terms form the column B:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Let us denote $\Delta = \det \mathbf{A}$, and Δ_i is the determinant of the matrix that is obtained from **A** by replacing the i-th column with the column of **B**. For example,

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}.$$

***Theorem* 1.1 (*Cramer's Rule*).** *If* $\Delta \neq 0$, *then the system of linear equations (1.2) has a unique solution. It can be found using the formulas*

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad x_3 = \frac{\Delta_3}{\Delta}.$$

This theorem is also true for systems consisting of an arbitrary number n of equations and unknowns.

Examples of problem solving

1. Find a solution to the system of equations

$$\begin{cases} 5x + 9y = 3, \\ 3x + 5y = 1. \end{cases}$$

Solution.

$$\Delta = \begin{vmatrix} 5 & 9 \\ 3 & 5 \end{vmatrix} = -2, \quad \Delta_1 = \begin{vmatrix} 3 & 9 \\ 1 & 5 \end{vmatrix} = 6, \quad \Delta_2 = \begin{vmatrix} 5 & 3 \\ 3 & 1 \end{vmatrix} = -4.$$

$$x = \frac{\Delta_1}{\Delta} = \frac{6}{-2} = -3, \quad y = \frac{\Delta_2}{\Delta} = \frac{-4}{-2} = 2.$$

Before writing the answer, we perform a check. To do this, we substitute the found values into the system of equations:

$$\begin{cases} 5\cdot(-3)+9\cdot 2 = \ 3, \text{ – true} \\ 3\cdot(-3)+5\cdot 2 = 1 \text{– true.} \end{cases}$$

Answer: (–3, 2).

2. Find a solution to the system of equations

$$\begin{cases} x_1+2x_2+x_3=6, \\ 3x_1+x_2-3x_3=2, \\ 2x_1+x_2-x_3=3. \end{cases}$$

Solution.

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & -3 \\ 2 & 1 & -1 \end{vmatrix} = -3,\ \ \Delta_1 = \begin{vmatrix} 6 & 2 & 1 \\ 2 & 1 & -3 \\ 3 & 1 & -1 \end{vmatrix} = -3,\ \Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 3 & 2 & -3 \\ 2 & 3 & -1 \end{vmatrix} = -6,\ \Delta_3 = \begin{vmatrix} 1 & 2 & 6 \\ 3 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix} = -3,$$

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{-3}{-3} = 1, \quad x_2 = \frac{\Delta_2}{\Delta} = \frac{-6}{-3} = 2, \quad x_3 = \frac{\Delta_3}{\Delta} = \frac{-3}{-3} = 1.$$

Answer: $(1, 2, 1)$.

Excersise. Check this answer independently.

EXAMPLES OF PROBLEM SOLVING

§4. *Matrix multiplication*

Let

$$a=(a_1\ \ a_2\ \dots\ a_k)\quad\text{and}\quad b=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix} -$$

be a row and a column consisting of the same number of elements. Define

$$a\cdot b = a_1b_1+a_2b_2+\dots+a_kb_k.$$

Now let **A** be a matrix of the dimension $m\times k$, and **B** a matrix of size $k\times n$, i.e. the number of columns in matrix **A** is equal to the number of rows in matrix **B**, or, equivalently, the length of a row in matrix **A** is equal to the height of a column in matrix B. Then we can multiply the rows of matrix **A** by the columns of matrix **B**. Let $a_i=(a_{i1}\ \ a_{i2}\ \dots\ a_{ik})$ be the i-th row of matrix **A**, and $b_j=\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{kj} \end{pmatrix}$ be is the j-th column of matrix **B**. Let

$$c_{ij}=a_i\cdot b_j=a_{i1}b_{1j}+a_{i2}b_{2j}+\dots+a_{ik}b_{kj},\quad i=1,\dots,m,\ \ j=1,\dots,n.$$

The numbers c_{ij} form a matrix C of dimensions $m\times n$, which is called the product of matrices **A** and **B**:

$$\mathbf{C}=\mathbf{AB}=\begin{pmatrix} a_1\cdot b_1 & a_1\cdot b_2 & \dots & a_1\cdot b_n \\ a_2\cdot b_1 & a_2\cdot b_2 & \dots & a_2\cdot b_n \\ \dots & \dots & \dots & \dots \\ a_m\cdot b_1 & a_m\cdot b_2 & \dots & a_m\cdot b_n \end{pmatrix}.$$

***Note* 1.** We can rearrange numbers when multiplying. Matrices cannot be rearranged. If the product **AB** is defined, then the product **BA** may not be defined. If both products are defined, then the matrices **AB** and **BA** are necessarily square, but they may have different orders. For example, if **A** has dimensions 2×3 and B has dimensions 3×2, then **AB** has order 2 and **BA** has order 3; it turns out that these matrices cannot be compared at all:

Even if both products **AB** and **BA** are defined and have the same order, we can get $\mathbf{AB}\neq\mathbf{BA}$. For example,

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

It turns out that $\mathbf{AB}=\mathbf{A}$, and $\mathbf{BA}=\mathbf{B}$.

Definition. If $\mathbf{AB}=\mathbf{BA}$, then matrices **A** and **B** are said to commute.

Example.

$$\begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 12 & 5 \\ -5 & 12 \end{pmatrix} = \begin{pmatrix} 33 & 56 \\ -56 & 33 \end{pmatrix} = \begin{pmatrix} 12 & 5 \\ -5 & 12 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}$$

***Note* 2.** From the equality $\mathbf{AB} = \mathbf{O}$ it does not automatically follow that one of the matrices is zero. This is shown by the following example.

$$\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 10 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Properties of the matrix multiplication operation (selective).

1. If $\mathbf{A}$ is a square matrix of order n and $\mathbf{E}$ is the identity matrix of the same order, then $\mathbf{AE} = \mathbf{EA} = \mathbf{A}$. Thus, the identity matrix behaves like the number 1 when multiplied.

2. $\mathbf{AO} = \mathbf{O}$, $\mathbf{OA} = \mathbf{O}$ (if the corresponding products are defined).

3. If the products $\mathbf{AB}$ and $(\mathbf{AB})\mathbf{C}$ are defined, then the products $\mathbf{BC}$ and $\mathbf{A}(\mathbf{BC})$ are defined; moreover, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

4. If $\mathbf{A}(\mathbf{B} + \mathbf{C})$ makes sense, then $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$. If $(\mathbf{A}+\mathbf{B})\mathbf{C}$ makes sense, then, then $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.

5. $\lambda(\mathbf{AB}) = (\lambda\mathbf{A})\mathbf{B} = \mathbf{A}(\lambda\mathbf{B})$.

6. If the product $\mathbf{AB}$ is defined, then $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ is also defined and $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

7. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same order, then $\det\mathbf{AB} =$
$= \det\mathbf{A} \cdot \det\mathbf{B}$.

§5. *Gauss method for solving systems of linear equations*

We will present the Gauss method for systems of 3 linear equations with three unknowns. However, it can be used for systems of arbitrary size.

Let a system of linear equations be given:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3. \end{cases} \tag{1.2}$$

If all coefficients in the first column are equal zero $a_{11}= a_{21}= a_{31}=0$, then the unknown value x_1 does not participate in the equation at all, and it can take any value. Let at least one of the coefficients a_{11}, a_{21}, a_{31} be non-zero. If $a_{11}=0$, then among a_{21}, a_{31} we choose the number that is non-zero and rearrange the corresponding equation to the first place. In the new system we will have $a_{11}\neq 0$. In practice, we put such an equation to the first place so that a_{11} turns out to be a divisor for a_{21} and a_{31}. It is best if after the rearrangement it turns out to be $a_{11}=1$.

Step 1. To the second equation we add the first equation multiplied by $-a_{21}/a_{11}$, and to the third equation we add the first equation multiplied by $-a_{31}/a_{11}$ (the first equation itself remains unchanged). As a result of these actions, we exclude x_1 from the 2nd and 3rd equations, i.e. we obtain a system of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ \qquad\qquad c_{22}x_2 + c_{23}x_3 = d_2, \\ \qquad\qquad c_{32}x_2 + c_{33}x_3 = d_3. \end{cases} \tag{1.3}$$

Step 2. To the third equation we add the second equation multiplied by $-c_{32}/c_{22}$ (the second equation itself remains unchanged). As a result of these actions we exclude x_2 from the third equation, i.e. we obtain a system of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ \qquad\qquad c_{22}x_2 + c_{23}x_3 = d_2, \\ \qquad\qquad\qquad\qquad f_{33}x_3 = g_3. \end{cases} \tag{1.4}$$

This operation cannot be done if $c_{22}=0$. In this case, we will swap equations 2 and 3 and immediately obtain an SLE of the form (1.4).

Step 3. From the third equation we can find x_3 and substitute the found value into the second equation. Then from the second equation we can find the value of x_2. Now we substitute these values into the first equation and find the value of x_1.

In the process of solving, we may get an equation that can be divided by some number. Then it is worth doing.

We may get an equation $0=k$, where k is some number not equal to zero. Then we conclude that the system has no solutions.

We may get an equation $0=0$. Then we can cross out such an equation. How to proceed in this case we will show in example 2.

***Example* 1.** $$\begin{cases} 3x_1 - x_2 + 4x_3 = 1, \\ x_1 + 2x_2 - x_3 = 5, \\ -4x_1 - x_2 + 2x_3 = -6. \end{cases}$$

Let's move the second equation to the first place. Let's ask ourselves the question: by what number should we multiply this equation so that after adding it to the second (in the new order) equation, we get no x_1? The answer is obvious: before adding it, we need to multiply the first equation by –3. In the same way, we determine that before adding it to the third equation, we need to multiply the first by 4. We perform the actions indicated above. In this case, the first equation itself remains unchanged. We denote these actions with arrows:

$$\begin{cases} x_1 + 2x_2 - x_3 = 5, & \Big] {-3} \quad \Big] 4 \\ 3x_1 - x_2 + 4x_3 = 1, & \\ -4x_1 - x_2 + 2x_3 = -6. & \end{cases} \qquad \begin{cases} x_1 + 2x_2 - x_3 = 5, \\ -7x_2 + 7x_3 = -14, \\ 7x_2 - 2x_3 = 14. \end{cases}$$

If you do not have sufficient calculation skills, you can separately, apart from the general solution, perform the above actions in parts. Namely, first multiply the first equation by –3, and then add it to the second equation; first multiply the first equation by 4, and then add it to the third equation. When formalizing the solution, leave these actions on the draft.

Next we see that we can divided the second equation by –7. But it is more convenient to do this after we add the second equation to the third. We also denote these actions.

$$\begin{cases} x_1 + 2x_2 - x_3 = 5, & \\ -7x_2 + 7x_3 = -14, & \Big]+ \quad :(-7) \\ 7x_2 - 2x_3 = 14. & \end{cases} \qquad \begin{cases} x_1 + 2x_2 - x_3 = 5, \\ x_2 - x_3 = 2, \\ 5x_3 = 0. \end{cases} \tag{1.5}$$

Next we draw a curly bracket on three lines and write on the last line $x_3 = 0$. Then on the second line we calculate x_2, and finally on the first line we calculate x_1.

$$\begin{cases} x_1 = 5 - 2x_2 + x_3 = 1, \\ x_2 = x_3 + 2 = 2, \\ x_3 = 0. \end{cases}$$

Checking: $$\begin{cases} 1 + 2\cdot 2 - 0 = 5 - \text{true}, \\ 3 - 2 + 4\cdot 0 = 1 - \text{true}, \\ -4 - 2 + 2\cdot 0 = -6 - \text{true}. \end{cases}$$

Answer: $(1, 2, 0)$.

Transformations can be performed not on the system of equations, but on its extended matrix, reducing its right-hand side to a triangular form. We will demonstrate these actions using the previous example.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 3 & -1 & 4 & -1 \\ -4 & -1 & 2 & 14 \end{array}\right) \begin{matrix} \\ -3 \\ 4 \end{matrix} \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -7 & 7 & -14 \\ 0 & 7 & -2 & 14 \end{array}\right) \begin{matrix} \\ :(-7) \\ + \end{matrix} \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 5 & 0 \end{array}\right)$$

After these transformations we can write out the system of linear equations (1.5) again. But there is another way to continue the solution: using the transformations we bring the matrix of the system of equations (not expanded) to a diagonal form. To do this, we first get one in the last row and use it to zero all the elements above it. In this case, the actions are performed on the entire row, but since the third element in the last column is equal to zero, the elements of the last column will not change.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 5 & 0 \end{array}\right) :5 \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right) \begin{matrix} \\ -4 \\ + \end{matrix} \quad \left(\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The purpose of the next step is to get zeros in the second column above the diagonal. To do this, we first divide the second row by the number in that row on the diagonal in order to get a unit in the place of that number. In our example, we already have a unit in the place we need.

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right) -2 \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

A system of linear equations for this matrix look like this:

$$\begin{cases} x_1 \qquad\qquad = 1, \\ \qquad x_2 \qquad = 2, \\ \qquad\qquad x_3 = 0. \end{cases}$$

In fact, we have obtained a ready-made solution. For systems consisting of two or three equations, the use of matrices does not lead to a reduction in the time spent on the solution. But when solving systems with a large number of equations, the use of matrices is fully justified. We will show this in Example 4.

***Example* 2.** $\begin{cases} x_1+3x_2-11x_3=1, \\ x_1+5x_2-25x_3=7, \\ -5x_1-9x_2+13x_3=13. \end{cases} \begin{matrix} \\ -1 \;\; 5 \\ \\ \end{matrix} \quad \begin{cases} x_1+3x_2-11x_3=1, \\ 2x_2-14x_3=6, \\ 6x_2-42x_3=18. \end{cases} \begin{matrix} :2 \\ \\ -3 \end{matrix}$

$$\begin{cases} x_1+3x_2-11x_3=1, \\ x_2-7x_3=3, \\ 0=0. \end{cases}$$

We can cross out the last equation. The number of equations turned out to be less than the number of unknowns. We can move any of the unknowns x_2 or x_3 to the right side and assign it an arbitrary value, for example, 0 или 1 (as you find more convenient). In our system, it is more convenient to move x_3 to the right side and assign it the value $x_3=0$. Then we will find the corresponding values of x_1 and x_2:

$$\begin{cases} x_1+3x_2=1, \\ x_2=3, \\ x_3=0. \end{cases} \qquad \begin{cases} x_1=1-3x_2=1-9=-8, \\ x_2=3, \\ x_3=0. \end{cases}$$

What we have found is called *a particular solution* of the system. But it is not the only one: we could have given any other value instead of x_3 and obtained a different answer. *The general solution* of the system is sought in the following way. We give x_3 not a numerical value, but the value of an arbitrary parameter, and then find the corresponding values of x_1 and x_2 (which will also depend on this parameter):

$$\begin{cases} x_1+3x_2=1+11a, \\ x_2=3+7a, \\ x_3=a,\ a\in \boldsymbol{R}. \end{cases} \qquad \begin{cases} x_1=1+11a-3x_2=1+11a-3(3+7a)=-8-10a, \\ x_2=3+7a, \\ x_3=a,\ a\in \boldsymbol{R}. \end{cases}$$

Answer: $(-8-10a,\ 3+7a,\ a),\ a\in \boldsymbol{R}$.

***Example* 3.** An SLE is called *inconsistent* if it has no solutions. Now we will give an example of an inconsistent SLE.

$$\begin{cases} 2x_1+3x_2-x_3=-4, \\ 4x_1+5x_2-5x_3=-3, \\ -6x_1-7x_2+9x_3=20. \end{cases} \begin{matrix} \\ -2 \;\; 3 \\ \\ \end{matrix} \quad \begin{cases} 2x_1+3x_2-x_3=-4, \\ -x_2-3x_3=5, \\ 2x_2+6x_3=-2. \end{cases} \begin{matrix} \\ \\ 2 \end{matrix} \quad \begin{cases} 2x_1+3x_2-x_3=-4, \\ -x_2-3x_3=5, \\ 0=8. \end{cases}$$

We got the wrong equality $8=0$.

Answer: The system has no solutions.

§7. *Diagonalization*

Properties 3, 4, 6 and 7 of the determinant (see §1) allow us to use the method of reduction to diagonal form by means of elementary transformations, which we will briefly call the Gauss method, to calculate the determinant. It is very similar to the corresponding method for solving systems of linear equations. But there are differences.

1. In the process of solving systems of linear equations, we could freely rearrange the equations and divide the equation by any number. When calculating the determinant, it should be taken into account that it changes in the process.

2. In the process of solving systems of linear equations, we performed actions only on rows, but when calculating the determinant, we can perform the same actions on columns.

So, we will classify the following actions as elementary transformations of the determinant.

1. A permutation of two rows or columns. In this case, the determinant changes sign.

2. Multiplication of a row or column by a (non-zero number) number different from zero. In this case, the determinant is also multiplied by this number. In other words, we use the property: the common factor of the elements of one row or one column is taken out of the determinant sign.

3. Adding to one row (or column) another row (or column), multiplied by a certain number. In this case, the row (column) that we add does not change itself.

Let us consider this method using the example of calculating a fourth-order determinant. Our task is to obtain zeros below the main diagonal using elementary transformations of the determinant.

Example. Calculate the determinant Δ using the Gauss method if

$$\Delta = \begin{vmatrix} 4 & 13 & -12 & 11 \\ 0 & 2 & 5 & 2 \\ 6 & 15 & -30 & 5 \\ 2 & 6 & -8 & 5 \end{vmatrix}$$

<u>*Step* 1</u>. First, we want to get zeros in the first column in all rows except the first. In the next determinant, it is more convenient for us to put the fourth row in first place. To do this, we need to swap it alternately with rows 3, 2 and 1. This means that we make three permutations of the rows. Each of these permutations changes the sign of the determinant, which means that with a complete permutation, the sign will also change.

$$\Delta = -\begin{vmatrix} 2 & 6 & -8 & 5 \\ 4 & 13 & -12 & 11 \\ 0 & 2 & 5 & 2 \\ 6 & 15 & -30 & 5 \end{vmatrix} \begin{array}{l} \\ -2 \\ \\ -3 \end{array}$$

Here we have indicated further actions with arrows: we add the first line multiplied by –2 to the second line of the determinant, and we add the first line multiplied by –3 to the fourth line (the first line itself, however, remains unchanged). We have obtained zeros in the first column below the main diagonal.

$$\Delta = -\begin{vmatrix} 2 & 6 & -8 & 5 \\ 0 & 1 & 4 & 1 \\ 0 & 2 & 5 & 2 \\ 0 & -3 & -6 & -10 \end{vmatrix}.$$

<u>Step</u> 2. The next step is to get zeros in the second column below the main diagonal. To do this, we add the second row of the matrix multiplied by –2 to the third row, and add the second row multiplied by 3 to the fourth row.

$$\Delta = -\begin{vmatrix} 2 & 6 & -8 & 5 \\ 0 & 1 & 4 & 1 \\ 0 & 2 & 5 & 2 \\ 0 & -3 & -6 & -10 \end{vmatrix} \begin{array}{l} \\ \\ -2 \\ 3 \end{array} = -\begin{vmatrix} 2 & 6 & -8 & 5 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 6 & -7 \end{vmatrix}$$

<u>Step</u> 3. Next we should get zeros in the third column below the main diagonal. As a result, we obtained an upper triangular matrix, the determinant of which we calculate as the product of the diagonal elements.

$$\Delta = -\begin{vmatrix} 2 & 6 & -8 & 5 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 6 & -7 \end{vmatrix} \begin{array}{l} \\ \\ \\ 2 \end{array} = -\begin{vmatrix} 2 & 6 & -8 & 5 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -7 \end{vmatrix} = -2\cdot 1\cdot(-3)\cdot(-7) = -42.$$

Tasks for independent solution

***Problem* 1.** *Matrices* **A** *and* **B** *are given. Calculate the products* **AB**, **BA**, $\mathbf{AB}^{\mathrm{T}}$, $\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}$, *if these products are defined (has sense).*

1. $\mathbf{A}=\begin{pmatrix}1 & -2 & 3\\ 7 & 2 & -3\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & 1\\ 4 & 2\\ 5 & 1\end{pmatrix}$;

2. $\mathbf{A}=\begin{pmatrix}8 & 4 & 1\\ 5 & -3 & 2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & 2\\ -2 & 6\\ 0 & 4\end{pmatrix}$;

3. $\mathbf{A}=\begin{pmatrix}12 & 3 & 2\\ 3 & 5 & 4\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}2 & 1\\ -2 & -4\\ -1 & 0\end{pmatrix}$;

4. $\mathbf{A}=\begin{pmatrix}3 & 6 & 0\\ 8 & 3 & -2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-1 & 4\\ 2 & -2\\ -1 & 1\end{pmatrix}$;

5. $\mathbf{A}=\begin{pmatrix}1 & 2 & 2\\ 3 & 5 & 4\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}2 & 6\\ -2 & -3\\ 1 & 0\end{pmatrix}$;

6. $\mathbf{A}=\begin{pmatrix}1 & 3 & -6\\ -1 & -2 & 4\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}0 & 9\\ 4 & 1\\ 2 & 2\end{pmatrix}$;

7. $\mathbf{A}=\begin{pmatrix}11 & 2 & 5\\ 1 & 9 & 3\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}0 & -1\\ 1 & 3\\ -3 & 1\end{pmatrix}$;

8. $\mathbf{A}=\begin{pmatrix}7 & 2 & -1\\ 2 & 6 & -4\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-5 & 1\\ 3 & -2\\ 2 & 3\end{pmatrix}$;

9. $\mathbf{A}=\begin{pmatrix}9 & 1 & -4\\ 7 & -5 & 2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & 2\\ -5 & 4\\ 1 & 3\end{pmatrix}$;

10. $\mathbf{A}=\begin{pmatrix}2 & 8 & 5\\ -3 & 1 & 2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-7 & 6\\ 3 & 0\\ -2 & 9\end{pmatrix}$;

11. $\mathbf{A}=\begin{pmatrix}6 & -3 & -1\\ -4 & 2 & 1\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}2 & 2\\ 4 & 1\\ 0 & 9\end{pmatrix}$;

12. $\mathbf{A}=\begin{pmatrix}2 & -2 & 1\\ 4 & -5 & 3\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & 0\\ 2 & 3\\ 2 & 6\end{pmatrix}$;

13. $\mathbf{A}=\begin{pmatrix}-3 & 2 & -1\\ 3 & -2 & -7\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}5 & 1\\ 4 & 2\\ 1 & 1\end{pmatrix}$;

14. $\mathbf{A}=\begin{pmatrix}1 & 4 & 8\\ -2 & 3 & -5\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}0 & 4\\ -2 & 6\\ 1 & 2\end{pmatrix}$;

15. $\mathbf{A}=\begin{pmatrix}2 & 3 & 12\\ 4 & 5 & 3\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & 0\\ 2 & -4\\ -2 & 1\end{pmatrix}$;

16. $\mathbf{A}=\begin{pmatrix}1 & 0\\ 2 & 5\\ 3 & -1\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-1 & 3 & 5\\ 2 & -2 & 1\end{pmatrix}$;

17. $\mathbf{A}=\begin{pmatrix}1 & 2\\ 0 & 5\\ 1 & 4\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & 9 & 5\\ 0 & -2 & 1\end{pmatrix}$;

18. $\mathbf{A}=\begin{pmatrix}1 & 2\\ 0 & 5\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-2 & 1\\ 3 & -2\end{pmatrix}$;

19. $\mathbf{A}=\begin{pmatrix}1 & 8\\ 3 & 5\\ 0 & 2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}5 & 4 & 9\\ 1 & -3 & -2\end{pmatrix}$;

20. $\mathbf{A}=\begin{pmatrix}1 & 8\\ 3 & 5\\ 2 & 2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}5 & 4 & 6\\ 1 & -3 & -2\end{pmatrix}$;

***Problem* 2.** *Solve the system of equations using Cramer's rule. Check the answer.*

1. $\begin{cases}6x+11y=14,\\ 8x-5y=-1.\end{cases}$

2. $\begin{cases}6x+7y=9,\\ 9x-5y=-2.\end{cases}$

3. $\begin{cases}9x+12y=3,\\ 6x+10y=3.\end{cases}$

8. $\begin{cases}8x+9y=7,\\ 4x-12y=-2.\end{cases}$

9. $\begin{cases}6x+10y=14,\\ 9x-7y=-1.\end{cases}$

10. $\begin{cases}4x+11y=5,\\ 6x+5y=-4.\end{cases}$

15. $\begin{cases}7x+6y=-2,\\ 5x+8y=-7.\end{cases}$

16. $\begin{cases}12x-5y=1,\\ 4x-7y=-5.\end{cases}$

17. $\begin{cases}9x-8y=11,\\ 12x+5y=-1.\end{cases}$

4. $\begin{cases} 6x-11y=-7, \\ 9x-5y=1. \end{cases}$

5. $\begin{cases} 6x-7y=2, \\ 8x-5y=7. \end{cases}$

6. $\begin{cases} 5x+12y=1, \\ 7x+4y=-5. \end{cases}$

7. $\begin{cases} 8x+9y=11, \\ 5x-12y=1. \end{cases}$

11. $\begin{cases} 11x-6y=14, \\ 5x+8y=1. \end{cases}$

12. $\begin{cases} 7x-6y=9, \\ 5x+9y=2. \end{cases}$

13. $\begin{cases} 12x-9y=3, \\ 10x-6y=3. \end{cases}$

14. $\begin{cases} 11x+6y=7, \\ 5x+9y=-1. \end{cases}$

18. $\begin{cases} 9x-8y=7, \\ 12x+4y=2. \end{cases}$

19. $\begin{cases} 10x-6y=14, \\ 7x+9y=1. \end{cases}$

20. $\begin{cases} 11x-4y=5, \\ 5x-6y=-4. \end{cases}$

Problem 3. *Solve a system of linear equations using the Gauss method.*

1. $\begin{cases} x_1-3x_2+2x_3=4, \\ 2x_1-3x_2+3x_3=3, \\ -3x_1+12x_2-2x_3=-7. \end{cases}$

2. $\begin{cases} x_1-2x_2+3x_3=-3, \\ 3x_1-4x_2+5x_3=-11, \\ 4x_1-4x_2+7x_3=-7. \end{cases}$

3. $\begin{cases} x_1+5x_2+2x_3=-3, \\ -2x_1-7x_2-7x_3=-3, \\ 3x_1+12x_2+5x_3=-4. \end{cases}$

4. $\begin{cases} x_1+3x_2+2x_3=5, \\ -3x_1-5x_2+2x_3=-3, \\ 2x_1+2x_2-5x_3=-4. \end{cases}$

5. $\begin{cases} x_1-4x_2+3x_3=-8, \\ 3x_1-4x_2+5x_3=-8, \\ -2x_1+4x_2-7x_3=2. \end{cases}$

6. $\begin{cases} x_1-3x_2+2x_3=4, \\ 3x_1-12x_2+2x_3=1, \\ 2x_1-3x_2+3x_3=9. \end{cases}$

7. $\begin{cases} x_1-2x_2+3x_3=-2, \\ 4x_1-4x_2+7x_3=-1, \\ 3x_1-4x_2+5x_3=-4. \end{cases}$

8. $\begin{cases} x_1+5x_2-2x_3=1, \\ 3x_1+12x_2-5x_3=3, \\ 2x_1+7x_2-7x_3=-10. \end{cases}$

9. $\begin{cases} x_1+3x_2-2x_3=1, \\ 2x_1+2x_2+5x_3=16, \\ 3x_1+5x_2+2x_3=15. \end{cases}$

10. $\begin{cases} x_1-4x_2+3x_3=-3, \\ 2x_1-4x_2+7x_3=3, \\ 3x_1-4x_2+5x_3=3. \end{cases}$

11. $\begin{cases} x_1+3x_2-2x_3=4, \\ 2x_1+3x_2-3x_3=3, \\ -3x_1-12x_2+2x_3=-7. \end{cases}$

12. $\begin{cases} x_1+2x_2-3x_3=3, \\ 3x_1+4x_2-5x_3=11, \\ 4x_1+4x_2-7x_3=7. \end{cases}$

13. $\begin{cases} x_1-5x_2-2x_3=3, \\ -2x_1+7x_2+7x_3=3, \\ 3x_1-12x_2-5x_3=4. \end{cases}$

14. $\begin{cases} x_1-3x_2-2x_3=5, \\ -3x_1+5x_2-2x_3=-3, \\ 2x_1-2x_2+5x_3=-4. \end{cases}$

15. $\begin{cases} x_1+4x_2-3x_3=8, \\ 3x_1+4x_2-5x_3=8, \\ -2x_1-4x_2+7x_3=-2. \end{cases}$

16. $\begin{cases} x_1+3x_2-2x_3=4, \\ 3x_1+12x_2-2x_3=1, \\ 2x_1+3x_2-3x_3=9. \end{cases}$

17. $\begin{cases} x_1+2x_2-3x_3=2, \\ 4x_1+4x_2-7x_3=1, \\ 3x_1+4x_2-5x_3=4. \end{cases}$

18. $\begin{cases} x_1-5x_2+2x_3=1, \\ 3x_1-12x_2+5x_3=3, \\ 2x_1-7x_2+7x_3=-10. \end{cases}$

19. $\begin{cases} x_1-3x_2+2x_3=1, \\ 2x_1-2x_2-5x_3=16, \\ 3x_1-5x_2-2x_3=15. \end{cases}$

20. $\begin{cases} x_1+4x_2-3x_3=3, \\ 2x_1+4x_2-7x_3=-3, \\ 3x_1+4x_2-5x_3=-3. \end{cases}$

CHAPTER 2. ANALYTICAL GEOMETRY

§1. *Coordinate systems on the plane*

Let us choose a point O on a straight line, which we will call *the origin of coordinates*, and a direction, which we will call positive. Such a straight line is called *an axis*. The direction opposite to positive will be called negative. The positive direction on a drawing is usually depicted to the right of the point and is designated by an arrow. In the positive direction of the axis from point O we will set aside a segment OE, which we will consider to be unitary (it sets the scale). We will sign point E with the number 1. Now we can talk about the distance between two points on the axis.

Let M be an arbitrary point on a line. Let $x=|OM|$ (the distance between points O and M) if M belongs to the positive direction, and $x=-|OM|$ if M belongs to the negative direction. Then x is called the coordinate of point M on the axis (*fig*. 2.1).

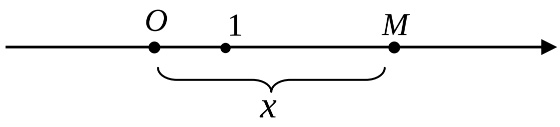


fig.2.1

Let two mutually perpendicular axes be chosen on a plane, which intersect at a point O, called *the origin of coordinates*. We will call these axes *coordinate axes* and denote them Ox and Oy. In this case, we will also require that the rotation by 90°, which combines the positive direction of the Ox axis with the positive direction of the Oy axis, be carried out counterclockwise. We say that these axes together with the point O form a Cartesian coordinate system Oxy on the plane (*fig*. 2.2).

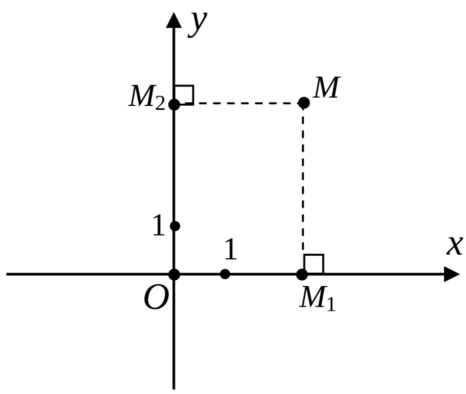


fig.2.2

Let M be an arbitrary point in the plane. Let us drop perpendiculars MM_1 on the Ox axis and MM_2 on the Oy axis. Let point M_1 have coordinate x on axis Ox, and point M_2 have coordinate y on axis Oy Then we say that point M has coordinates (x,y) and write $M(x,y)$. The x coordinate is called the *abscissa* of point M, and the y coordinate is called the *ordinate* of point точки M. The pair of numbers (x,y) is called the Cartesian coordinates of point M.

Let an arbitrary ray OP be given on a plane. We will call it *the polar axis*, and the point O *the origin* or the *pole*. Let M be an arbitrary point on the plane. Denote $r=|OM|$, and φ the angle between the rays OP and OM. In this case, if the rotation from the ray OP to the ray OM is counterclockwise, we consider φ to be positive (*fig*. 2.3), and if clockwise, we consider φ to be negative.

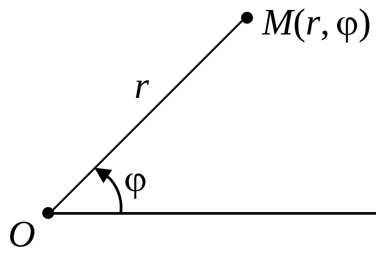


fig.2.3

Definition. The pair (r, φ) is called *the polar coordinates* of the point M, and the set of the point O and the axis OP is called *the polar coordinate system on the plane*.

It is obvious that $0 \le r < +\infty$, and for the angle φ it is usually agreed that $0 \le \varphi < 2\pi$, or that $-\pi < \varphi \le \pi$. If $r = 0$, then φ is considered undefined.

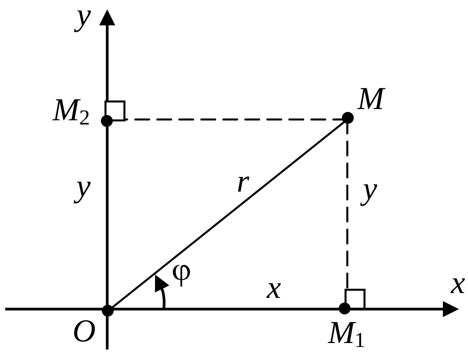


fig.2.4

Now let the Cartesian and polar coordinate systems with a common origin O be simultaneously defined on the plane,and let the positive direction of the Ox axis coincide with the positive direction of the OP axis (*fig*. 2.4). Then from ΔOMM_1 and ΔOMM_2 we get

$$\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi. \end{cases} \qquad (2.1) \qquad \begin{cases} r = \sqrt{x^2 + y^2}, \\ \cos\varphi = x/r, \ \sin\varphi = y/r. \end{cases} \qquad (2.2)$$

Let us emphasize that knowledge of the sine, cosine, or tangent separately does not allow us to uniquely determine the angle φ. That is why both equalities $\cos\varphi = x/r$, $\sin\varphi = y/r$ should be written in formulas (2.2).

§2. *The concept of a vector*

An ordinary segment AB has two ends: A and B (Fig. 2.5 a). If one of the points is called the beginning, and the other is the end, then the segment is called directed. If A is the beginning, and B is the end, then this segment is designated $\overrightarrow{AB}$, and we depict it as in the figure in Fig. 2.5 b).

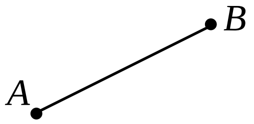


fig.2.5 a)

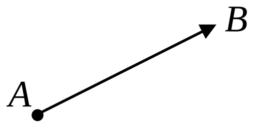


fig.2.5 b)

We denote the length of the directed segment as follows: $|\overrightarrow{AB}|$.

Definition. Directed segments $\overrightarrow{AB}$ and $\overrightarrow{A_1B_1}$ are called *equal* if they have the same length and the same direction (*fig*. 2.6). We write $\overrightarrow{AB} = \overrightarrow{A_1B_1}$.

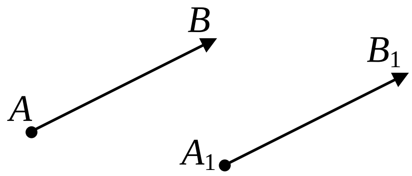


fig.2.6

Definition. *A vector* is a class of equivalent directed segments. This means that *each directed segment defines a vector, and equal directed segments define the same vector*.

Vectors are denoted by lowercase Latin letters with an arrow above: $\vec{a}$, $\vec{b}$, $\vec{c}$… If the vector $\vec{a}$ is defined by the directed segment $\overrightarrow{AB}$, then we write $\vec{a} = \overrightarrow{AB}$, and say that $\overrightarrow{AB}$ is the vector $\vec{a}$, laid off from the point A. On the drawing, the vector is depicted by any of the directed segments defining it.

The length of a vector $\vec{a}$ is denoted by $|\vec{a}|$ and it is equal to the length of the directed segment defining this vector. The direction of a vector is the direction of any of the segments defining this vector.

In other words, a vector has a length, a direction, but no specific beginning or end. To plot a vector from point A means to indicate a directed segment $\overrightarrow{AB} = \vec{a}$.

Example. Let $ABCD$ be a parallelogram (*fig*. 2.7). If $\vec{a} = \overrightarrow{AB}$, then also $\vec{a} = \overrightarrow{DC}$. Similarly, if $\vec{b} = \overrightarrow{BC}$ then also $\vec{b} = \overrightarrow{AD}$.

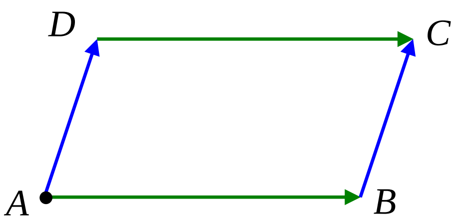


fig.2.7

Definition. A vector whose length is zero is called *a null vector*. We denote it $\vec{o}$; it is defined by a directed segment whose beginning and end coincide. For example, $\vec{o} = \overrightarrow{AA}$. A vector whose length is equal to 1 is called *a unit vector*.

Definition. Vectors that have the same directions are called *co-directed*, and those that have opposite directions are called *oppositely directed*. We write $\vec{a} \uparrow\uparrow \vec{b}$ or $\vec{a} \uparrow\downarrow \vec{b}$. Vectors whose directions coincide or are opposite are called *collinear*. We write $\vec{a} \parallel \vec{b}$. It is considered that $\vec{o}$ has no defined direction and is collinear with any vector. Vectors parallel to one plane are called *coplanar*.

§3. *Linear operations on vectors*

Definition. Let two arbitrary vectors $\vec{a}$ and $\vec{b}$ be given. We choose any point O and plot the vector $\vec{a}$ from it: $\vec{a}=\overrightarrow{OA}$. Then we plot the vector $\vec{b}$ from the point A: $\vec{b}=\overrightarrow{AB}$. Let $\vec{c}$ be the vector that is defined by the directed segment $\overrightarrow{OB}$. Then $\vec{c}$ is called *the sum* of the vectors $\vec{a}$ and $\vec{b}$. We write $\vec{c}=\vec{a}+\vec{b}$.

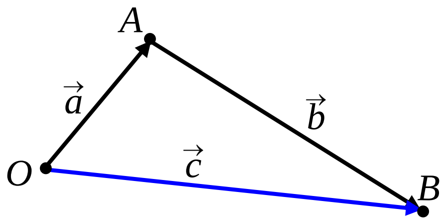


fig.2.8

This method of constructing the sum of two vectors is called *the triangle rule* (*fig*. 2.8). In order to construct the sum of vectors, we must set the second vector aside from the end of the directed segment defining the first vector. We postulate that the result of the construction does not depend on the choice of point O.

Let the vector $\vec{a}$ be defined by the directed segment $\overrightarrow{AB}$, and the vector $\vec{x}$ be defined by the directed segment $\overrightarrow{BA}$. Such a vector $\vec{x}$ is called opposite to the vector $\vec{a}$ and is denoted by $-\vec{a}$. According to the triangle rule, the vector $\vec{a}+\vec{x}$ is defined by the directed segment $\overrightarrow{AA}$, and therefore it is zero vector. Thus, $\vec{a}+(-\vec{a})=\vec{o}$.

Properties of the vector addition operation.

1. $\forall\ \vec{a},\vec{b}$ holds $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (*commutativity*);

2. $\forall\ \vec{a},\vec{b},\vec{c}$ holds $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (*associativity*);

3. $\forall\ \vec{a}$ holds $\vec{a}+\vec{o}=\vec{a}$.

4. $\forall\ \vec{a}\ \exists!\ \vec{x}$ such that $\vec{a}+\vec{x}=\vec{o}$.

The symbol $\forall$ is read as "for any". The symbol $\exists$ is read as "exists". The pair of symbols $\exists!$ is read as "exists the only one". Property 2 allows us to use the notation $\vec{a}+\vec{b}+\vec{c}$ without brackets.

Another way to construct the sum of vectors is the parallelogram rule. Let us set aside $\vec{a}$ and $\vec{b}$ from one point O: $\vec{a}=\overrightarrow{OA}$, $\vec{b}=\overrightarrow{OB}$. Let us complete ΔOAB to a parallelogram $OACB$. Let $\vec{c}$ be the vector defined by the directed segment $\overrightarrow{OC}$ (*fig*. 2.9). Then $\vec{c}=\vec{a}+\vec{b}$.

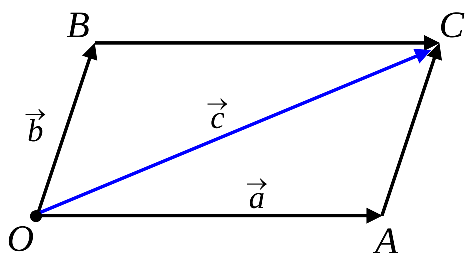


fig.2.9

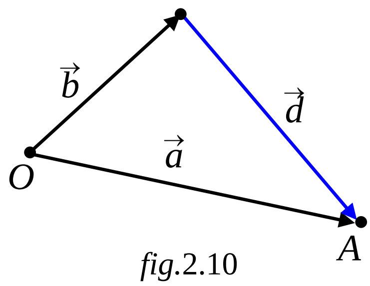


fig.2.10

Definition. We denote the vector $\vec{a}+(-\vec{b})$ as $\vec{a}-\vec{b}$ and call it the difference of the vectors $\vec{a}$ and $\vec{b}$.

Figure 2.10 shows how to construct the vector $\vec{d}=\vec{a}-\vec{b}$. Let us set aside $\vec{a}$ and $\vec{b}$ from one point O: $\vec{a}=\overrightarrow{OA}$, $\vec{b}=\overrightarrow{OB}$. Then $\vec{d}=\overrightarrow{BA}$.

Definition. The product of a vector $\vec{a}$ and a number λ is a vector $\vec{b}$ such that

1. $\vec{a}\uparrow\uparrow\vec{b}$, if $\lambda>0$, and $\vec{a}\uparrow\downarrow\vec{b}$, if $\lambda<0$;

2. $|\vec{b}|=|\lambda|\cdot|\vec{a}|$.

We write $\vec{b}=\lambda\vec{a}$.

In other words, vectors $2\vec{a}$ and $-2\vec{a}$ have lengths twice as large as $\vec{a}$, but $2\vec{a}$ has the same direction as vector $\vec{a}$, and vector $-2\vec{a}$ has the opposite direction.

***Examples* 1.** Let A_1B_1 be the midline in triangle ΔABC, parallel to side AB, $\vec{a}=\overrightarrow{AB}$, $\vec{b}=\overrightarrow{A_1B_1}$ (*fig*. 2.11). Тогда $\vec{b}=\frac{1}{2}\vec{a}$, because these vectors are codirectional, and the length of vector $\vec{b}$ is 2 times less than that of vector $\vec{a}$.

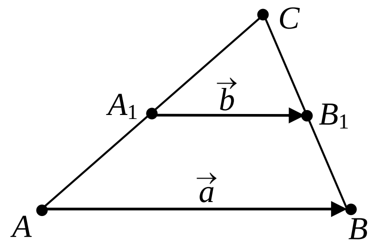


fig.2.11

2. Let AM be the median in the triangle ΔABC, and $\vec{c}=\overrightarrow{AM}$. Let us complete the triangle to a parallelogram $ABCD$ (*fig*. 2.12). Let M be the intersection point of the diagonals, and let $\overrightarrow{AB}=\vec{a}$, $\overrightarrow{AC}=\vec{b}$. Then $\overrightarrow{AD}=\vec{a}+\vec{b}$, and $\overrightarrow{AM}=\frac{1}{2}\overrightarrow{AD}$. This means that the vector $\vec{c}$, which is defined by the median $\overrightarrow{AM}$, is equal to half the sum of the vectors defined by the sides of the triangle $\overrightarrow{AB}$ and $\overrightarrow{AC}$: $\vec{c}=\frac{1}{2}(\vec{a}+\vec{b})$.

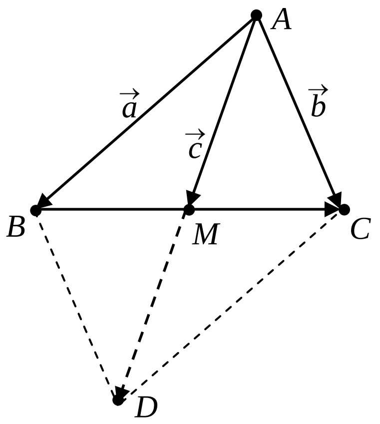


fig.2.12

Properties of the multiplying a vector by a number.

For any vectors $\vec{a},\vec{b}$ and for any $\lambda,\mu\in\boldsymbol{R}$ the following holds:

5. $\lambda(\vec{a}+\vec{b})=\lambda\vec{a}+\lambda\vec{b}$; **6.** $(\lambda+\mu)\vec{a}=\lambda\vec{a}+\mu\vec{a}$;

7. $\lambda(\mu\vec{a})=(\lambda\mu)\vec{a}$; **8.** $1\cdot\vec{a}=\vec{a}$.

§4. *Coordinates of a vector*

Let a vector $\vec{m}$ be given. Let us set it aside from the origin: $\vec{m}=\overrightarrow{OM}$. Let a point M have coordinates $M(m_1,m_2)$ (*fig*. 2.13). Then we will assign the same coordinates to the vector $\vec{m}$: $\vec{m}(m_1,m_2)$. We say, that $\vec{m}$ is the radius-vector of point M. $\vec{m}$

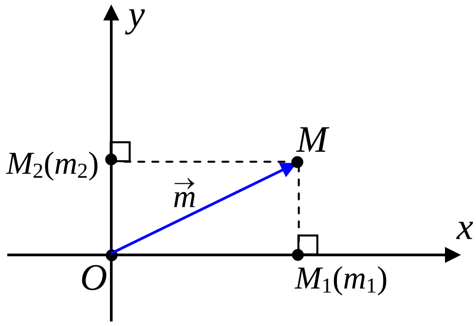


fig.2.13

From ΔOMM_1 by the Pythagorean theorem we find that

$$|OM|=\sqrt{|OM_1|^2+|OM_2|^2}=\sqrt{m_1^2+m_2^2}\,.$$

This means that the length of the vector is calculated using the formula

$$|\vec{m}|=\sqrt{m_1^2+m_2^2}\,. \qquad (2.3)$$

Let us know the coordinates of the beginning and end of the vector $\vec{m}=\overrightarrow{AB}$: $A(x_1,y_1)$, $B(x_2,y_2)$ (*fig*. 2.14). Then the coordinates of the vector $\vec{m}$ are found as follows: $\vec{m}(x_2-x_1,y_2-y_1)$.

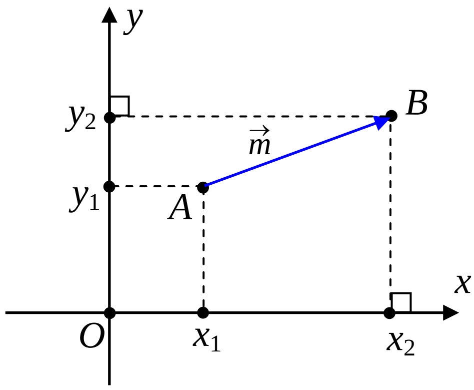


fig.2.14

So, you should learn the rule by heart: *in order to find the coordinates of a vector, you need to subtract the coordinates of its beginning from the coordinates of its end*.

The length of the vector $\vec{m}$ coincides with the length of the segment AB. This value is also called the distance between points A and B. From formula (2.3) follows the formula for calculating the distance between points:

$$|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\,. \qquad (2.4)$$

Let us know the coordinates of the vectors $\vec{m}$ and $\vec{n}$: $\vec{m}(m_1,m_2)$, $\vec{n}(n_1,n_2)$ and $\vec{q}=\vec{m}+\vec{n}$. Then the coordinates of the vector $\vec{q}$ can be found as follows: $\vec{q}(m_1+n_1,m_2+n_2)$. If $\vec{r}=\lambda\vec{m}$, then $\vec{r}(\lambda m_1,\lambda m_2)$. Thus, we should learn the rule by heart: *when adding vectors, their coordinates are added together, and when multiplying a vector by a number, its coordinates are multiplied by this number*.

***Theorem* 2.1** (The sign of collinearity of vectors). *In order for two non-zero vectors to be collinear, it is necessary and sufficient that their coordinates are proportional*:

$$\vec{m}(m_1, m_2) \,||\, \vec{n}(n_1, n_2) \;\Leftrightarrow\; \frac{m_1}{m_1} = \frac{n_2}{n_2}.$$

Unit vectors that are co-directed with the positive directions of the coordinate axes are usually denoted $\vec{e}_1$, $\vec{e}_2$ или $\vec{i}$, $\vec{j}$. Their coordinates are: $\vec{i}(1, 0)$, $\vec{j}(0, 1)$ (*fig*. 2.15). An arbitrary vector $\vec{m}(m_1, m_2)$ can be represented as a combination of these vectors: $\vec{m} = m_1\vec{i} + m_2\vec{j}$.

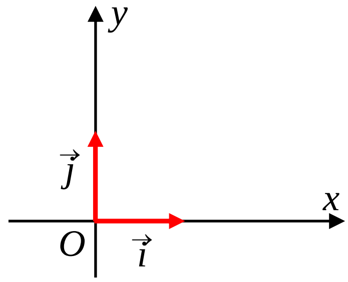


fig. 2.15

§5. *Scalar product of vectors*

Definition. *The scalar product* of two vectors $\vec{a}$ and $\vec{b}$ is a number

$$\vec{a}\cdot\vec{b}=|\vec{a}|\,|\vec{b}|\,\cos\angle(\vec{a},\vec{b}). \qquad (2.5)$$

The number $\vec{a}^2=\vec{a}\cdot\vec{a}$ is called the scalar square of the vector $\vec{a}$.

From the definition we obtain

$$\vec{a}^2=|\vec{a}\,||\vec{a}\,|\cos 0^{\circ}=|\vec{a}\,|^2 \quad\Rightarrow\quad |\vec{a}\,|=\sqrt{\vec{a}\cdot\vec{a}}.$$

It is also obvious from the definition that the equality $\vec{a}\cdot\vec{b}=0$ is possible only in the following cases: 1. $|\vec{a}\,|=0$, 2. $|\vec{b}\,|=0$, 3. $\angle(\vec{a},\vec{b})=\pi/2$.

Thus, we proved the following theorem.

***Theorem* 2.2. 1.** *The scalar square of a vector is equal to the square of its length.*

2. *In order for non-zero vectors $\vec{a}$ and $\vec{b}$ to be perpendicular, it is necessary and sufficient that their scalar product be equal to zero* ($\vec{a}\perp\vec{b} \Leftrightarrow \vec{a}\cdot\vec{b}=0$).

From the definition follows a formula by which one can calculate the angle between vectors:

$$\cos\angle(\vec{a},\vec{b})=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}\,||\vec{b}\,|}. \qquad (2.6)$$

If we know the coordinates of the vectors $\vec{a}(a_1, a_2)$ and $\vec{b}(b_1, b_2)$, then their scalar product is calculated using the formula

$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2. \qquad (2.7)$$

From formulas (2.6) and (2.7) follows the formula for calculating the angle between vectors:

$$\cos\angle(\vec{a},\vec{b})=\frac{a_1b_1+a_2b_2}{\sqrt{a_1^2+a_2^2}\,\sqrt{b_1^2+b_2^2}}, \qquad (2.8)$$

and the condition of perpendicularity of vectors:

$$a_1b_1+a_2b_2=0. \qquad (2.9)$$

In the future, when solving problems, we will allow ourselves a fairly common liberty: we will call the directed segment $\overrightarrow{AB}$ a vector.

Example. The coordinates of the vertices of a quadrilateral are given: $A(-3,-1)$, $B(7,-3)$, $C(8,2)$, $D(-2,4)$. Prove that $ABCD$ is a rectangle.

Proof. We find the coordinates of vectors that are defined by opposite sides. In order to find the coordinates of the vector $\overrightarrow{AB}$, we subtract the coordinates of point A from the coordinates of point B; in order to find the coordinates of the vector $\overrightarrow{DC}$, we subtract the coordinates of point D from the coordinates of point C:

$\overrightarrow{AB}(7-(-3),-3-(-1))$, $\overrightarrow{AB}(7+3,-3+1)$, $\overrightarrow{AB}(10,-2)$

$\overrightarrow{DC}(8-(-2),2-4)$, $\overrightarrow{DC}(8+2,-2)$, $\overrightarrow{DC}(10,-2)$.

We see that $\overrightarrow{AB}=\overrightarrow{DC}$, i.e. the opposite sides of the quadrilateral are parallel and equal. So *ABCD* is a parallelogram. We need to check that the two adjacent sides are perpendicular. We already know the coordinates $\overrightarrow{AB}$. Let's find the coordinates of $\overrightarrow{AD}$: $\overrightarrow{AD}\,(1,5)$. We calculate the scalar product of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$:

$$\overrightarrow{AB}\cdot\overrightarrow{AD}=10\cdot 1+(-2)\cdot 5=10-10=0\,.$$

This means that $\overrightarrow{AB}\perp\overrightarrow{AD}$. Therefore, *ABCD* is a rectangle.

§6. *Midpoint of a segment. Area of a parallelogram and a triangle*

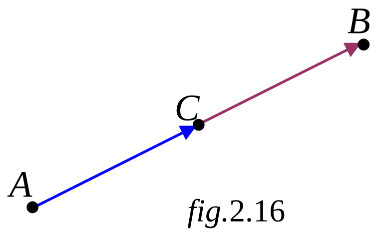


fig.2.16

Let us know the coordinates of the ends of the segment AB: $A(x_1, y_1)$, $B(x_2, y_2)$. Let the point C divide the segment AB in half (*fig*. 2.16). It is required to find its coordinates (x, y).

From the drawing we see that $\overrightarrow{AC} = \overrightarrow{CB}$. We find the coordinates of these vectors: $\overrightarrow{AC}\,(x - x_1, y - y_1)$, $\overrightarrow{CB}(x_2 - x, y_2 - y)$. The vectors are equal, which means their coordinates are also equal:

$$x - x_1 = x_2 - x, \quad y - y_1 = y_2 - y.$$

From here we find that $2x = x_1 + x_2$, $2y = y_1 + y_2$. Finally

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}. \tag{2.10}$$

That is, the coordinates of the middle of a segment are the arithmetic mean of the coordinates of its ends.

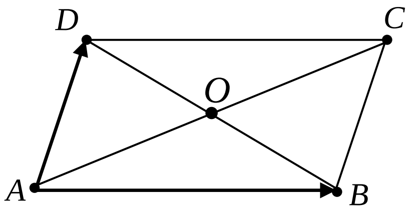


fig.2.17

Theorem 2.3. *Let ABCD be a parallelogram, and let us know the coordinates of the vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{AD}$: $\overrightarrow{AB}(a_1, a_2)$, $\overrightarrow{AD}(b_1, b_2)$ (*fig*. 2.17). *Then the area of the parallelogram is calculated using the formula*

$$S_{ABCD} = \operatorname{mod}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \tag{2.11}$$

(here “mod” means modulus: the determinant can be negative, but the area is necessarily non-negative).

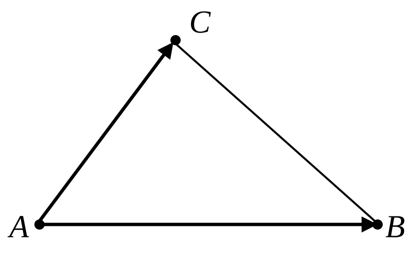


fig.2.18

Corollary. *Let us know the coordinates of the vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$: $\overrightarrow{AB}(a_1, a_2)$, $\overrightarrow{AC}(b_1, b_2)$ (*fig*. 2.18). *Then the area of the triangle ABC is calculated using the formula:*

$$S_{ABC} = \frac{1}{2}\operatorname{mod}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}. \tag{2.12}$$

§7. *Equation of a straight line on a plane*

As part of the school curriculum, you should have learned that the equation

$$y = kx + b \tag{2.13}$$

defines a line on the plane. This means that the line consists of those and only those points whose coordinates (x, y) satisfy equation (2.13).

At $x = 0$ we find from (2.13) that $y = b$. From this follows the geometric meaning of the coefficient b: it is the segment cut off by the line on the Oy axis (it can also be negative).

We choose a direction on the line that corresponds to an increase in y and call it positive. The angle between the positive direction of the Ox axis and the positive direction of the line is called *the slope of the line* (*fig*. 2.20). The geometric meaning of the coefficient k: $k = \operatorname{tg}\alpha$ – the tangent of the slope of the line. Therefore, k is called *the angular coefficient*.

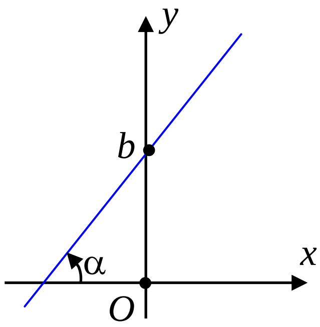


fig.2.20

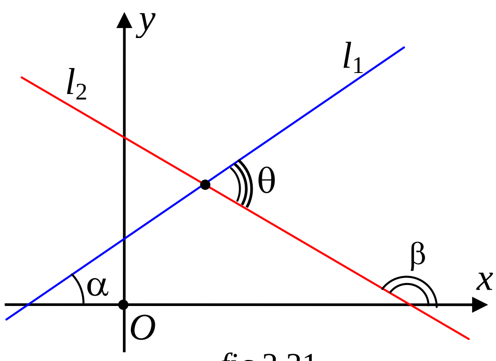


fig.2.21

Definition. The angle between lines on a plane is the smaller of the two angles that the lines form when they intersect. (*fig*. 2.20).

Theorem 2.4. *Let two lines on a plane be given by equations with an angular coefficient*

$$l_1: y = k_1 x + b_1, \qquad l_2: y = k_2 x + b_2.$$

Then

1) *the angle between them is calculated using the formula* $\operatorname{tg}\theta = \dfrac{|k_2 - k_1|}{|1 + k_1 k_2|}$;

2) *the lines coincide if and only if* $k_1 = k_2,\ b_1 = b_2$;

3) *lines are parallel if and only if* $k_1 = k_2,\ b_1 \neq b_2$;

3) *lines are perpendicular if and only if* $k_2 = -1/k_1$.

For example, the lines $l_1: y = 2x + 5$ and $l_2: y = 2x - 3$ are parallel, and the lines $l_1: y = 2x + 5$ и $l_3: y = -0{,}5x - 3$ are perpendicular.

Not every line on the plane can be defined using an equation with an angular coefficient. Lines parallel to the Oy axis are defined by equations of the form $x = c$. Both of these types of equations are combined by an equation of the form

$$ax + by + c = 0, \tag{2.14}$$

which is called *the general equation of a line*. And conversely, any equation of the form (2.14) defines a line on a plane. In order to obtain (2.14) from (2.13), it is sufficient to transfer all the terms of the equation to one part.

Let us recall that the distance from a point to a line is the length of the perpendicular dropped from this point to the line (*fig*. 2.22).

***Theorem* 2.5.** *Let a line l on a plane be given by the general equation* (2.12). *Then the distance from the point* $M(x,y)$ *to this line is calculated by the formula*

$$h=\frac{|ax+by+c|}{\sqrt{a^2+b^2}}. \qquad (2.15)$$

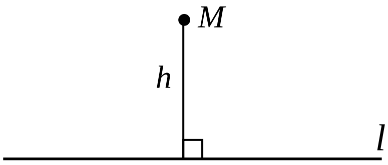


fig.2.22

Let us know the coordinates of two points on the line l: $A(x_1,y_1)$, $B(x_2,y_2)$. The question is: how to write the equation of the line l? First, we find the angular coefficient

$$k=\frac{y_2-y_1}{x_2-x_1}.$$

After this we write the equation

$$y-y_1=k(x-x_1).$$

It remains to open the brackets and move y1 to the right side of the equation. An example of composing an equation is given in paragraph 9.

§8. *Application of determinants to the calculation of areas and volumes in space*

We are not going to explain in detail on how the Cartesian coordinate system is defined in space. As an example, figure 2.22 shows how points $B(5,-1,2)$ and $C(4,5,-3)$ are constructed on the drawing based on their coordinates.

In space the rule also takes place: in order to find the coordinates of a vector, you need to subtract the coordinates of its beginning from the coordinates of its end.

Let a parallelepiped $ABCDA_1B_1C_1D_1$ (*fig*. 2.23) be given in space. Let $\overrightarrow{AB}$, $\overrightarrow{AD}$, $\overrightarrow{AA_1}$ be directed segments lying on the edges of the parallelepiped set aside off one point, $\vec{a}, \vec{b}, \vec{c}$ be the vectors defined by these directed segments. Then we say that the parallelepiped is constructed on vectors $\vec{a}, \vec{b}, \vec{c}$, laid off from one point A.

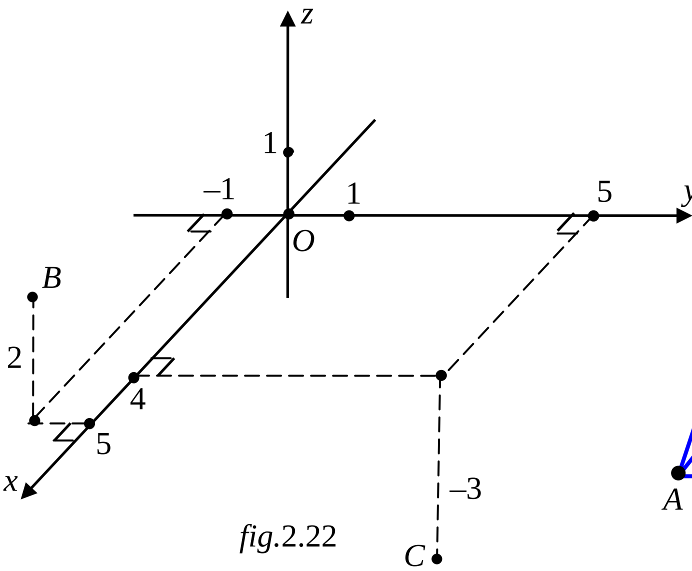


fig.2.22

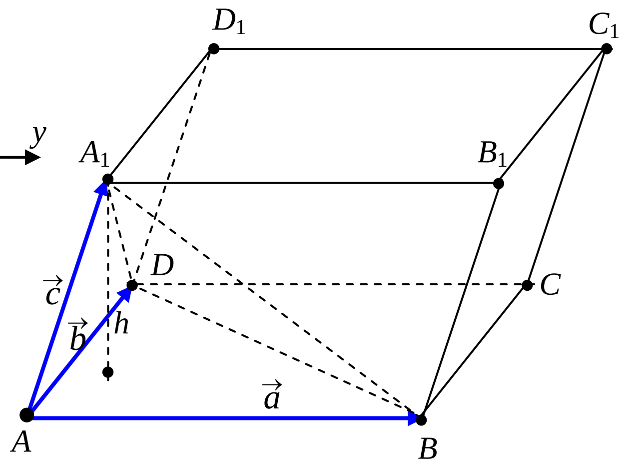


fig.2.23

***Theorem* 2.6.** *Let us know the coordinates of the vectors* $\vec{a}(a_1, a_2, a_3)$, $\vec{b}(b_1, b_2, b_3)$, $\vec{c}(c_1, c_2, c_3)$. *Then the volume of the parallelepiped constructed on these vectors, laid off from one point, is calculated by the formula*:

$$V = \operatorname{mod}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \qquad (2.16)$$

The volume of the triangular pyramid $ABDA_1$ (*fig*. 2.23) is 1/6 of the volume of the parallelepiped. Therefore,

the volume of the triangular pyramid constructed on these vectors, laid off from one point, is calculated using the formula:

$$V = \frac{1}{6} \operatorname{mod} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \qquad (2.17)$$

***Theorem* 2.7.** *Let parallelogram ABCD in space, be constructed on vectors* $\vec{a}(a_1, a_2, a_3)$, $\vec{b}(b_1, b_2, b_3)$, *laid off from one point A. Then its area is calculated by the formula*

$$S = \sqrt{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2}. \qquad (2.18)$$

The area of triangle ΔABD, constructed on vectors $\vec{a}$ and $\vec{b}$, laid off from point A, is half the area of the parallelogram.

§9. *Equations of a line and a plane in space*

The plane π in space can be defined using three points $M_0, M_1, M_2 \in \pi$, which do not lie on the same line. In this case, we can consider two non-collinear vectors $\vec{a} = \overrightarrow{M_0M_1}$ and $\vec{b} = \overrightarrow{M_0M_2}$ parallel to the plane (*fig*. 2.24).

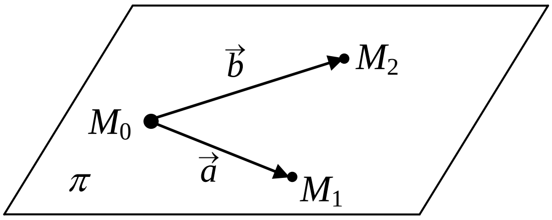


fig.2.24

***Theorem* 2.8.** *The plane π, passing through the point $M_0(x_0, y_0, z_0)$, parallel to two non-collinear vectors $\vec{a}(a_1, a_2, a_3)$, $\vec{b}(b_1, b_2, b_3)$ has the equation*

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0 \qquad (2.19)$$

***Corollary*.** *The plane π passing through three points $M_0(x_0, y_0, z_0)$, $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$, not lying on the same line is given by the equation*

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \\ x_2-x_0 & y_2-y_0 & z_2-z_0 \end{vmatrix} = 0. \qquad (2.20)$$

After expanding the determinant (2.19) or (2.20) and reducing similar terms, we obtain an equation of the form

$$ax + by + cz + d = 0, \qquad (2.21)$$

which is called *the general equation of the plane*. It turns out that the vector $\vec{n}\,(a, b, c)$ is perpendicular to the plane, which is given by equation (2.21), and it is called *the normal vector* to the plane.

***Theorem* 2.9.** *Let the π be defined by the general equation (2.21). Then the distance from an arbitrary point $M(x_0, y_0, z_0)$ t0 the plane is calculated by the formula*

$$h = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \qquad (2.22)$$

***Theorem* 2.10.** *The line l passing through the p0int $M_0(x_0, y_0, z_0)$, parallel to the vector $\vec{a}(a_1, a_2, a_3)$ is given by the canonical equation*

$$\frac{x-x_0}{a_1} = \frac{y-y_0}{a_2} = \frac{z-z_0}{a_3}, \qquad (2.23)$$

or by parametric equati0ns

$$\begin{cases} x = x_0 + a_1 t, \\ y = y_0 + a_2 t, \\ z = z_0 + a_3 t,\ t \in \boldsymbol{R}. \end{cases} \qquad (2.24)$$

The vector $\vec{a} \,||\, l$ is called *the direction vector* of the line l.

***Corollary*.** *A straight line passing through two distinct points $M_0(x_0, y_0, z_0)$ and $M_1(x_1, y_1, z_1)$, is given by the equation*

$$\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}. \qquad (2.25\ )$$

§10. *Equation of a circle*

Definition. *A circle* of radius r with center at point M_o is a set on a plane consisting of all points, the distance from each of which to M_o is equal to r (*fig*. 2.25).

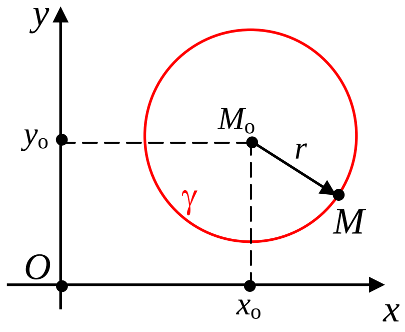


fig.2.25

Theorem **2.11.** *A circle* ω *of radius r, having its center at the point $M_0(x_0, y_0)$, is given by the equation*

$$(x-x_0)^2+(y-y_0)^2=r^2. \qquad (2.23)$$

For example, a circle of radius 5 with center at point $M_o(2,-3$ is given by the equation

$$(x-2)^2+(y+3)^2=25.$$

§11. *Examples of problem solving*

***Task* 1.** *The coordinates of the three vertices of the parallelogram are known*: $A(-5,1)$, $B(1,3)$, $D(-4,5)$ (*fig*. 2.26).

a) *Find the coordinates of the fourth vertex.* C;

b) *calculate the area of a parallelogram*;

c) *find its height drawn from the vertex* D *to the side* AB;

d) *find the coordinates of the point O of intersection of the diagonals*.

Solution. a) Let's find the coordinates of the vector $\vec{AB}$. To do this, we subtract the coordinates of point A from the coordinates of point B:

$\vec{AB}(1-(-5), 3-1) \Leftrightarrow \vec{AB}(6,2)$.

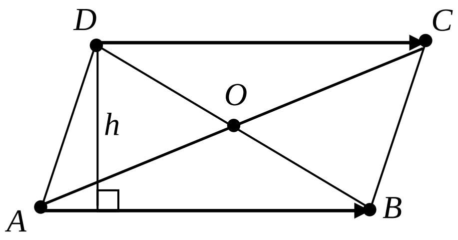


*fig.*2.26

Let $C(x,y)$. Then $\vec{DC}(x+4, y-5)$.

On the other hand $\vec{AB}=\vec{DC}$. Hence $\vec{DC}(6,2)$. We have two equations:

$$x+4=6, y-5=2.$$

Thus $C(2,7)$.

b) We find the coordinates of the vector $\vec{AD}$. To do this, we subtract the coordinates of point A from the coordinates of point D: $\vec{AD}(-4-(-5), 5-1) \Leftrightarrow \vec{AD}(1,4)$. We apply formula (2.11):

$$S_{ABCD} = \mathrm{mod}\begin{vmatrix} 6 & 2 \\ 1 & 4 \end{vmatrix} = |6\cdot 4 - 2\cdot 1| = 22.$$

в) From the school program we know the formula $S_{ABCD} = |AB|\cdot h$. Hence $h=\frac{S_{ABCD}}{|AB|}$. We find the length of the side AB: $|AB|=\sqrt{6^2+2^2}=\sqrt{40}=2\sqrt{10}$. Then

$$h=\frac{22}{2\sqrt{10}}=\frac{11}{\sqrt{10}}=\frac{11\sqrt{10}}{10}.$$

c) The coordinates of point O are calculated as the arithmetic mean of the coordinates of any two opposite vertices: for example, B and D:

$$O\left(\frac{1-4}{2}, \frac{3+5}{2}\right) \Leftrightarrow O(-1{,}5; 4).$$

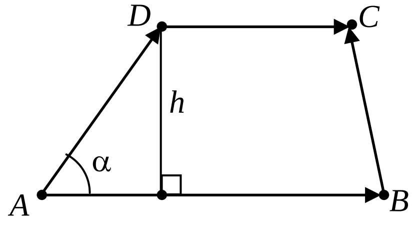
Answer: $C(2,7)$; $S_{ABCD}=22$; $h=\frac{11\sqrt{10}}{10}$, $O(-1{,}5;4)$.

***Task* 2.** *The vertices of the quadrilateral are at the points* $A(1, 2)$, $B(7,-6)$, $C(11,-3)$, $D(8,1)$. *Show that* $ABCD$ *is a trapezoid. Find the lengths of the bases of the trapezoid, its area, and* $\cos\angle DAB$ (рис. 2.27).

Solution. We find the coordinates of the vectors $\vec{AB}(6,-8)$, $\vec{BC}(4, 3)$, $\vec{CD}(-3, 4)$, $\vec{AD}(7,-1)$. We check the vectors determined by the opposite sides of the quadrangle for collinearity:

$$-\frac{6}{3} = -\frac{8}{4}$$ – correct, then $\vec{AB}$ is collinear to $\vec{CD}$.

$$\frac{4}{7} = \frac{3}{-1}$$ – incorrect, then $\vec{BC}$ is not collinear to $\vec{AD}$.



*puc.*2.27

Thus, in a quadrilateral, two opposite sides are collinear, and two are not. So it is a trapezoid, and the bases are *AB* and *CD*. We find the lengths of the sides:

$$|\vec{AB}| = \sqrt{6^2 + 8^2} = 10,$$

and similarly $|\vec{BC}| = 5$; $|\vec{CD}| = 5$; $|\vec{AD}| = 5\sqrt{2}$.

Let us denote $\alpha = \angle BAD$. Then

$$\cos\alpha = \frac{\vec{AB}\cdot\vec{AD}}{|\vec{AB}||\vec{AD}|} = \frac{6\cdot 7 + (-8)\cdot(-1)}{10\cdot 5\sqrt{2}} = \frac{1}{\sqrt{2}},$$

therefore $\angle BAD = 45^{\circ}$. It is not always possible to obtain a tabular angle, so we can do the following: knowing $\cos\alpha$, we find $\sin\alpha$:

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \frac{1}{\sqrt{2}}.$$

Then $h = |\vec{AD}| \cdot \alpha = 5$. Knowing the height and lengths of the bases, we find the area:

$$S = \frac{1}{2}(|AB| + |CD|)\cdot h = \frac{75}{2}.$$

Another way to calculate the height using the equation of a line is given in the following task.

Answer: $|\vec{AB}| = 10$, $|\vec{BC}| = 5$, $\cos\alpha = \frac{1}{\sqrt{2}}$, $S_{ABCD} = \frac{75}{2}$.

Task 3. *The coordinates of the vertices of the triangle are known*: $A(-1, 3)$, $B(11, 0)$, $C(9, 9)$ (*fig*. 2.28).

a) *Make an equation of side AB, height CD and find the coordinates of point D.*

b) *Calculate the height of a triangle using the formula for the distance from a point to a line.*

c) *Calculate the area of* ΔABC.

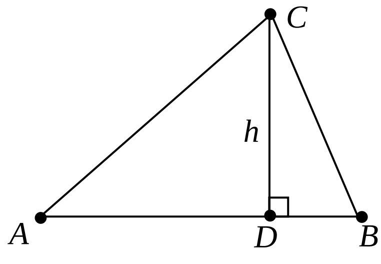


fig.2.27

Solution. a) We find the slope coefficient of the line *AB*:

$$k_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{0-3}{11-(-1)} = -\frac{1}{4}.$$

We make the equation of the straight line *AB*:

$$y-y_A=k_{AB}(x-x_A) \;\Leftrightarrow\; y-3=-\frac{1}{4}(x-(-1)) \;\Leftrightarrow\; y=-\frac{1}{4}x+\frac{11}{4}.$$

Straight lines AB and CD are perpendicular. Therefore $k_{CD}=-\dfrac{1}{k_{AB}}=4$. We compose the equation of straight line CD:

$$y-y_C=k_{CD}(x-x_C) \Leftrightarrow\; y-9=4(x-9) \;\Leftrightarrow\; y=4x-27.$$

Point D is a common point for lines AB and CD. This means that its coordinates must simultaneously satisfy the equations of these two lines. Therefore, to find the coordinates of point D, we combine the equations of lines AB and CD into one system and solve this system.

$$\begin{cases} y=-\frac{1}{4}x+\frac{11}{4}, \\ y=4x-27. \end{cases} \Leftrightarrow \begin{cases} 4x-27=-\frac{1}{4}x+\frac{11}{4}, \\ y=4x-27. \end{cases} \Leftrightarrow \begin{cases} x=7, \\ y=1. \end{cases} \Rightarrow D(7,1).$$

b) We rewrite the equation of the line AB in the form of a general equation:

$$y=-\frac{1}{4}x+\frac{11}{4} \;\Leftrightarrow\; 4y=-x+11 \;\Leftrightarrow\; x+4y-11=0.$$

Let us apply formula (2.15) to this equation and to the point C:

$$h=\frac{|9+4\cdot 9-11|}{\sqrt{4^2+1^2}}=\frac{|34|}{\sqrt{17}}=2\sqrt{17}.$$

c) $$S_{ABC}=\frac{1}{2}\operatorname{mod}\begin{vmatrix} 12 & -3 \\ 10 & 6 \end{vmatrix}=\frac{1}{2}|12\cdot 6-10\cdot(-3)|=\frac{1}{2}|72+30|=\frac{1}{2}\cdot 102=51.$$

This is consistent with the previously obtained result.

Answer: a) AB: $y=-\frac{1}{4}x+\frac{11}{4}$, CD: $y=4x-27$; $D(7,1)$;

b) $h=2\sqrt{17}$; c) $|S=51$.

***Task* 5.** *The coordinates of the vertices of a triangular pyramid are given:* $SABC$: $A(4,0,1)$, $B(5,-1,1)$, $C(4,7,-5)$, $S(7,5,2)$ (*fig*. 2.28).

a) *Find the volume of the pyramid, the area of the base ABC and the height*.

b) *Make an equation of the plane of the base and calculate the height using the formula for the distance from a point to the plane. Compare with the previously obtained result.*

Solution. a) From one vertex A the vectors $\vec{a}=\overrightarrow{AB}$, $\vec{b}=\overrightarrow{AC}$, $\vec{c}=\overrightarrow{AS}$ emanate. We find their coordinates:

$$\vec{a}(1,-1,0),\quad \vec{b}(0,7,-6),\quad \vec{c}(3,5,1).$$

We find the volume of the pyramid according to formula (2.17):

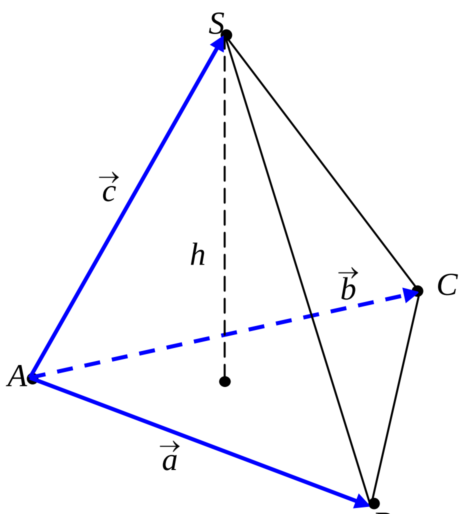


fig.2.28

$$V=\frac{1}{6}\operatorname{mod}\begin{vmatrix}1 & -1 & 0\\ 0 & 7 & -6\\ 3 & 5 & 1\end{vmatrix}=$$

$$=\frac{1}{6}\operatorname{mod}\left(1\cdot\begin{vmatrix}7 & -6\\ 5 & 1\end{vmatrix}-(-1)\cdot\begin{vmatrix}0 & -6\\ 3 & 1\end{vmatrix}+0\right)=\frac{1}{6}|37+18|=\frac{55}{6}.$$

The area of a triangle is is equal to half the area of the parallelogram. Therefore

$$S_{\text{base}}=\frac{1}{2}\sqrt{\begin{vmatrix}-1 & 0\\ 7 & -6\end{vmatrix}^2+\begin{vmatrix}1 & 0\\ 0 & -6\end{vmatrix}^2+\begin{vmatrix}1 & -1\\ 0 & 7\end{vmatrix}^2}=$$

$$=\frac{1}{2}\sqrt{6^2+6^2+7^2}=\frac{1}{2}\sqrt{36+36+49}=\frac{11}{2}.$$

It should be known from school mathematics that $V=\frac{1}{3}S_{\Delta ABC}\cdot h$. From here

$$h=\frac{3V}{S_{\Delta ABC}}=\frac{55/2}{11/2}=5.$$

b) The plane passing through the point $M(x_0, y_0, z_0)$, parallel to two given vectors $\vec{a}(a_1, a_2, a_3)$, $\vec{b}(b_1, b_2, b_3)$ is given by the equation

$$\begin{vmatrix}x-x_0 & y-y_0 & z-z_0\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=0$$

We substitute into this equation the coordinates of point A and vectors $\vec{a}=\overrightarrow{AB}$, $\vec{b}=\overrightarrow{AC}$:

$$\begin{vmatrix}x-4 & y+0 & z-1\\ 1 & -1 & 0\\ 0 & 7 & -6\end{vmatrix}=0.$$

We expand the determinant by the first line:

$$6(x-4)+6y+7(z-1)=0 \iff 6x+6y+7z-31=0.$$

It is recommended to check by substituting the coordinates of the points A, B, C into this equation.

The height of the pyramid is equal to the distance from the point S to the plane $\pi=ABC$. We can calculate it using formula (2.22):

$$h=\frac{|6\cdot 7+6\cdot 5+7\cdot 2-31|}{\sqrt{6^2+6^2+7^2}}=\frac{|42+30+14-31|}{\sqrt{36+36+49}}=\frac{55}{11}=5.$$

This is consistent with the previously obtained result.

Answer: a) $V=\frac{55}{6}$, $S_{\text{осн.}}=\frac{11}{2}$, $h=5$; b) ABC: $6x+6y+7z-31=0$, $h=5$.

***Task* 6.** *Given the equation of the plane*

$$\pi\colon 11x - 10y - 2z + 105 = 0$$

and the coordinates of the point $S(6,–5,–2)$. *Write the equation of the perpendicular SH to the plane and find the coordinates of the point* H (*fig*. 2.29).

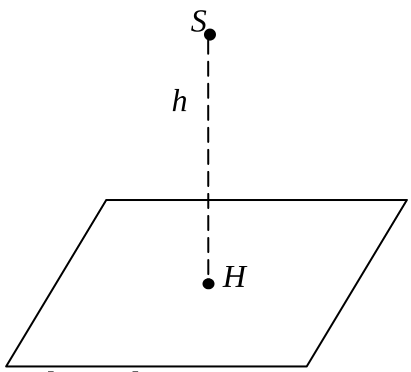


*рис.*2.29

Solution. From the equation of the plane we find that the vector $\vec{n}(11,–10,–2)$ is the normal vector to the plane. This same vector will be the directrix for the line $h = SH$.

The parametric equation of a straight line passing through a given point $M(x_0, y_0, z_0)$ with a direction vector $\vec{a}(a_1, a_2, a_3)$ has the form

$$\begin{cases} x = x_0 + a_1 t, \\ y = y_0 + a_2 t, \\ z = z_0 + a_3 t. \end{cases}$$

In our case we get the equation:

$$SH\colon \begin{cases} x = \ 6 + 11t, \\ y = –5 - 10t, \\ z = –2 - 2t. \end{cases} \qquad (*)$$

Let's find the base of the perpendicular. This is the point of intersection of the line SH with the plane π. To do this, we must solve the equations SH and π. We substitute x, y, z from the equation SH into the equation π:

$$11(6 + 11t) - 10(–5 - 10t) - 2(–2 - 2t) + 105 = 0,$$

$$66 + 121t + 50 + 100t + 4 + 4t + 105 = 0,$$

$$225y = –225, \quad t = –1.$$

Substitute the found t into the equation SH and find the coordinates of the point H:

$$\begin{cases} x = \ 6 + 11\cdot(–1) = –5, \\ y = –5 - 10\cdot(–1) = 5, \\ z = –2 - 2\cdot(–1) = 0. \end{cases}$$

$H(–5, 5, 0)$.

Answer: Equation of a perpendicular $SH\colon \begin{cases} x = \ 6 + 11t, \\ y = –5 - 10t, \\ z = –2 - 2t. \end{cases}$ $H(–5, 5, 0)$.

Tasks for independent solution

***Task* 1.** *The coordinates of the vertices of the triangle* ΔABC *are known* (*fig*. 2.27).

a) *Write an equation of side AB, height CD and find the coordinates of point D.*

b) *Calculate the height of a triangle using the formula for the distance from a point to a line.*

c) *Calculate the area of* ΔABC *using formula* (2.12).

1. $A(-3,1)$, $B(9,5)$, $C(-2,8)$.
2. $A(-2,-5)$, $B(1,7)$, $C(8,1)$.
3. $A(-5,-4)$, $B(10,-1)$, $C(-1,2)$.
4. $A(0,1)$, $B(12,-3)$, $C(5,6)$.
5. $A(0,2)$, $B(9,-4)$, $C(7,6)$.
6. $A(-1,2)$, $B(5,-2)$, $C(6,6)$.
7. $A(0,4)$, $B(3,-2)$, $C(-4,-3)$.
8. $A(0,-4)$, $B(8,0)$, $C(4,-7)$.
9. $A(0,-3)$, $B(9,0)$, $C(5,-8)$.
10. $A(0,-3)$, $B(-12,0)$, $C(-2,6)$.
11. $A(-6,0)$, $B(6,4)$, $C(-5,7)$.
12. $A(-1,-5)$, $B(2,7)$, $C(8,-3)$.
13. $A(-5,1)$, $B(10,-5)$, $C(2,4)$.
14. $A(-6,1)$, $B(6,-3)$, $C(-1,6)$.
15. $A(-6,2)$, $B(3,-4)$, $C(1,6)$.
16. $A(1,-2)$, $B(7,2)$, $C(8,-6)$.
17. $A(2,5)$, $B(5,-1)$, $C(-2,-2)$.
18. $A(4,0)$, $B(0,-8)$, $C(6,-6)$.
19. $A(0,3)$, $B(9,0)$, $C(5,8)$.
20. $A(-2,0)$, $B(1,-12)$, $C(8,-6)$.

***Task* 2.** *The coordinates of the vertices of the triangular pyramid SABC are given* (*fig*. 2.28).

a) *Find the volume of the pyramid, the area of the base ABC and the height.*

b) *Make an equation of the plane of the base and calculate the height using the formula for the distance from a point to a plane. Compare with the previously obtained result.*

1. $A(-1,1,2)$, $B(-5,4,-2)$, $C(-1,2,3)$, $S(-8,-5,4)$.
2. $A(0,2,2)$, $B(0,4,3)$, $C(1,4,2)$, $D(7,-1,7)$.
3. $A(1,1,2)$, $B(1,2,4)$, $C(4,1,4)$, $S(2,-7,3)$.
4. $A(-1,1,-2)$, $B(-1,-2,-1)$, $C(1,-2,0)$, $S(5,-2,-12)$.
5. $A(-5,1,2)$, $B(-5,-2,6)$, $C(-4,4,-2)$, $S(2,12,4)$.
6. $A(-5,1,2)$, $B(-5,-2,6)$, $C(-4,4,-2)$, $S(2,12,4)$.
7. $A(-6,0,1)$, $B(-6,-3,5)$, $C(-5,3,-3)$, $S(1,9,-1)$.
8. $A(1,0,-1)$, $B(2,0,4)$, $C(4,2,3)$, $S(10,-11,-8)$.
9. $A(-1,3,0)$, $B(-1,-1,2)$, $C(0,5,-2)$, $S(7,2,6)$.
10. $A(1,4,2)$, $B(7,6,3)$, $C(3,4,3)$, $S(6,-7,-7)$.
11. $A(2,1,4)$, $B(3,-1,2)$, $C(3,7,6)$, $S(-7,6,-7)$.

12. $A(1,-1,0)$, $B(2,1,2)$, $C(1,1,1)$, $S(3,-2,7)$.

13. $A(-1,1,1)$, $B(-1,3,2)$, $C(0,3,1)$, $S(6,-2,6)$.

14. $A(-1,-1,0)$, $B(-1,0,2)$, $C(2,-1,2)$, $S(0,-9,1)$.

15. $A(2,-1,1)$, $B(3,-1,2)$, $C(-2,-5,4)$, $S(4,-8,-5)$.

16. $A(-2,0,-3)$, $B(-2,-3,-2)$, $C(0,-3,-1)$, $S(4,-3,-13)$.

17. $A(6,1,1)$, $B(9,2,1)$, $C(6,2,3)$, $S(4,-11,11)$.

18. $A(1,0,-2)$, $B(2,-3,-2)$, $C(0,2,4)$, $S(2,4,-6)$.

19. $A(2,3,1)$, $B(6,3,0)$, $C(2,0,2)$, $S(-1,3,5)$.

20. $A(2,-2,4)$, $B(8,7,12)$, $C(2,-1,3)$, $S(5,1,0)$.

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