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On quasinormal Fitting classes

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ABSTRACT

Let Z_p be a group of order p and $G \wr Z_p$ the regular wreath product of the group G with Z_p. A Fitting class $\mathfrak{F}$ is said to be $\mathfrak{X}$-quasinormal if there exists a natural number m such that $G^m \wr Z_p \in \mathfrak{F}$ whenever $\mathfrak{F} \subseteq \mathfrak{X}$, p is a prime, $G \in \mathfrak{F}$ and $G \wr Z_p \in \mathfrak{X}$. In this paper, we generalize the well known theorem of Blessenohl and Gaschütz and prove that the intersection of any set of nonidentity $\mathfrak{X}$-quasinormal Fitting classes is a nonidentity $\mathfrak{X}$-quasinormal Fitting class. For a Fitting class $\mathfrak{F}$, we also discuss the relation between its $\mathfrak{X}$-normality and its $\mathfrak{X}$-quasinormality, and give the condition of $\mathfrak{X}_* = \mathfrak{X}_{\circledast}$, where "$*$" is Lockett star operation and $\mathfrak{X}_{\circledast}$ is the smallest nonidentity $\mathfrak{X}$-quasinormal Fitting class.

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1. Introduction

Throughout this paper all groups are finite and G denotes a finite group. A class of groups [4, Chapter II, Definition (1.1)] is a collection $\mathfrak{X}$ of groups with the property that if $G \in \mathfrak{X}$ and $G \cong H$, then $H \in \mathfrak{X}$. Let π be a set of some primes. We denote by $\mathfrak{S}$, $\mathfrak{N}$, $\mathfrak{E}_\pi$, $\mathfrak{N}_p$, $\mathfrak{E}_{p'}$ $\mathfrak{S}_\pi$ the class of all soluble groups, the class of all nilpotent groups, the class of all π-groups, the class of all p-groups, the class of all p'-groups, the class of all soluble π-groups, respectively.

A class $\mathfrak{F}$ of groups is called a Fitting class if $\mathfrak{F}$ is closed under taking normal subgroups and products of normal $\mathfrak{F}$-subgroups. It is clear that for any non-empty Fitting class $\mathfrak{F}$, every group G has the largest normal $\mathfrak{F}$-subgroup, which is called the $\mathfrak{F}$-*radical* of G and denoted by $G_\mathfrak{F}$. In particular, $G_\mathfrak{N} = F(G)$.

In the construction and development of the structural theory of Fitting classes many well-known results are related to the research of so-called normal Fitting classes (see [4, Chapters IX–XI]).

Let $\mathfrak{X}$ be a class of groups. A Fitting class $\mathfrak{F} \neq (1)$ is called *normal in the class* $\mathfrak{X}$ (or $\mathfrak{X}$-*normal*) [11, definition 1.1] if $\mathfrak{F} \subseteq \mathfrak{X}$ and the $\mathfrak{F}$-radical $G_\mathfrak{F}$ is an $\mathfrak{F}$-maximal subgroup of G for any $G \in \mathfrak{X}$. In particular, if $\mathfrak{X} = \mathfrak{S}$, then a $\mathfrak{X}$-normal Fitting $\mathfrak{F}$ is said to be normal.

Let G and H be groups. Then $G \wr H$ denotes the regular wreath product of G with H, and $G^\natural$ denotes the base group of $G \wr H$.

Blessenohl and Gaschütz [2, Theorem 6.2] proved that the intersection of any sets of nonidentity normal Fitting classes is a nonidentity normal Fitting class. Hence there exists the smallest nonidentity normal Fitting class. Makan [13] gave the characterization of normality of Fitting classes in terms of regular wreath products of groups and proved that $\mathfrak{F}$ is normal if and only if for each prime p and $G \in \mathfrak{S}$, there exists a natural number n such that $G^n \wr Z_p \in \mathfrak{F}$. By using the characterization, Hauck [10, Definition 5.1] generalized the concept of normal Fitting class and gave the following notion of quasinormal Fitting classes:

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A Fitting class $\mathfrak{F}$ is called *quasinormal* in a class of groups $\mathfrak{X}$ (or $\mathfrak{X}$-*quasinormal* for short) if the following condition holds: if $\mathfrak{F} \subseteq \mathfrak{X}$, $G \in \mathfrak{F}$, p is a prime and $G \wr Z_p \in \mathfrak{X}$, then there exists a natural number m such that $G^m \wr Z_p \in \mathfrak{F}$.

By the above Makan theorem, if $\mathfrak{X} = \mathfrak{S}$, then a $\mathfrak{X}$-quasinormal Fitting class $\mathfrak{F}$ is a normal Fitting class.

In connection with the above, the following problems naturally arise.

Question 1. For a Fitting class, whether its $\mathfrak{X}$-normality implies its $\mathfrak{X}$-quasinormality and its $\mathfrak{X}$-quasinormality implies its $\mathfrak{X}$-normality. arise.

Question 2. Is it true that the intersection of any sets of nonidentity $\mathfrak{X}$-quasinormal Fitting classes is a nonidentity $\mathfrak{X}$-quasinormal Fitting class? arise.

The following Theorems 1 and 2 give the answers to Questions 1 and 2, respectively.

Theorem 1. *For a Fitting class, its $\mathfrak{X}$-normality does not imply $\mathfrak{X}$-quasinormality and its $\mathfrak{X}$-quasinormality does not imply $\mathfrak{X}$-normality, in general. arise.*

Theorem 2. *The intersection of any sets of nonidentity $\mathfrak{X}$-quasinormal Fitting classes is a nonidentity $\mathfrak{X}$-quasinormal Fitting class. In particular, there exists the smallest nonidentity $\mathfrak{X}$-quasinormal Fitting class.*

We denote by $\mathfrak{X}_{\circledast}$ the smallest nonidentity $\mathfrak{X}$-quasinormal Fitting class.

The theorem of Blessenohl and Gaschütz [2, Theorem 6.2] is now a corollary of Theorem 2 in the case when $\mathfrak{X} = \mathfrak{S}$,

In [12], Lockett introduced the star operations "*" and "$_*$", which are called Lockett star operations (see also [4, Chapter X]): If $\mathfrak{F}$ is a nonempty Fitting class, then $\mathfrak{F}^*$ is the smallest Fitting class including $\mathfrak{F}$ such that $(G \times H)_{\mathfrak{F}^*} = G_{\mathfrak{F}^*} \times H_{\mathfrak{F}^*}$ for all groups G and H. The class $\mathfrak{F}_*$ is the intersection of all Fitting classes $\mathfrak{X}$ with $\mathfrak{X}^* = \mathfrak{F}^*$. If $\mathfrak{F}^* = \mathfrak{F}$, then the Fitting class $\mathfrak{F}$ is called a *Lockett class* [4, Chapter X, Definition (1.12)]. Clearly, $\mathfrak{N}$ is a Lockett class.

In this connection, the following question naturally arises.

Question 3. Under what conditions do we have $\mathfrak{X}_* = \mathfrak{X}_{\circledast}$?

In order to solve Question 3, we need the following concepts, which were proposed by Hauck [10].

Let $p \in \mathbb{P}$. A Fitting class $\mathfrak{F}$ is called *a class with limited p-property of wreath product* if there exists a natural number n such that $G^n \wr Z_p \in \mathfrak{F}$ whenever $1 \neq G \in \mathfrak{F}$ and $O^{p'}(G) = G$. Let $\pi \subseteq \mathbb{P}$. A Fitting class $\mathfrak{F}$ is called *a class with limited π-property of wreath product* if there exists a natural number n such that $G^n \wr Z_p \in \mathfrak{F}$ whenever $1 \neq G \in \mathfrak{F}$ and $O^{p'}(G) = G$ for each $p \in \pi$. If $\pi = \mathbb{P}$, then a class $\mathfrak{F}$ with limited π-property of wreath product is called *a class with limited property of wreath product* (see [10, Definition 2.7]).

Recall also that if $\mathfrak{F}$ and $\mathfrak{H}$ are two Fitting classes, then the product $\mathfrak{F}\mathfrak{H}$ of $\mathfrak{F}$ and $\mathfrak{H}$ is the class $(G : G/G_{\mathfrak{F}} \in \mathfrak{H})$. It is easy to prove that $\mathfrak{F}\mathfrak{H}$ is a Fitting class and the multiplication of Fitting classes satisfies the associative law (see [4, Chapter IX, Theorem (1.12)]).

Let f be a function of the form: $\mathbb{P} \to \{\text{Fitting classes}\}$. Then $Supp(f) = \{p \in \mathbb{P} : f(p) \neq \varnothing\}$ is called the support of f. Let $LR(f) = \mathfrak{E}_\pi \cap (\cap_{p \in \pi} f(p)\mathfrak{N}_p\mathfrak{E}_{p'})$, where $\pi = Supp(f)$. A Fitting class $\mathfrak{F}$ is said to be *local* if there exists a function f such that $\mathfrak{F} = LR(f)$. In this case, the function f is called an H-function of the local Fitting class $\mathfrak{F}$ (see [8, 9, 15]).

The following theorem gave an answer to Question 3.

Theorem 3. *1) If $\mathfrak{F}$ is a local Fitting class, then $\mathfrak{F}$ is a Fitting class with limited π-property of wreath product.*

2) Suppose that $\mathfrak{X}$ and $\mathfrak{F}$ are Fitting classes and $\mathfrak{F} \subseteq \mathfrak{X} \subseteq \mathfrak{S}_\pi$, where $\varnothing \neq \pi \subseteq \mathbb{P}$.
i) If $\mathfrak{X}^*$ is a Fitting class with limited π-property of wreath product, then $\mathfrak{X}_* = \mathfrak{X}_{\circledast}$;
ii) If $\mathfrak{X}^*$ is a local Fitting class, then $\mathfrak{X}_* = \mathfrak{X}_{\circledast}$.

All unexplained notations and terminology are standard. The reader is referred to [1, 4] or [7] if necessary.

2. Preliminaries

In this section, we cite some known results, which are often used in our proofs.

Lemma 2.1. [4, Chapter X, Theorem (2.9)] *Let $\mathfrak{F}$ be a Fitting class and $G \in \mathfrak{F}$. Let P_0 be a nontrivial p-group for some $p \in \mathbb{P}$, and assume that $G \wr P_0 \in \mathfrak{F}^*$. Then for all p-groups P we have $G^2 \wr P \in \mathfrak{F}$.*

Lemma 2.2. *1)* [4, Chapter X, Lemma (2.3)]. *Let $\mathfrak{F}$ be a Fitting class, G a group and H a nilpotent group. If $G \wr H \in \mathfrak{F}$, then $G^n \wr H \in \mathfrak{F}$ for all $n \in \mathbb{N}$.*

2) [3, Corollary 2.2]. *Let N be a normal subgroup of G and C a supplement of N in G such that $C/C \cap N$ is nilpotent and $N \cap C$ is central in N. If $\mathfrak{F}$ is a Fitting class containing G which is closed under subdirect product, then $C \in \mathfrak{F}$.*

Lemma 2.3. *1)* [4, Chapter X, Proposition (2.1)(a)] Let $\mathfrak{F}$ be a Lockett class and G be a group. If $G \notin \mathfrak{F}$, then $(G \wr H)_{\mathfrak{F}} = (G_{\mathfrak{F}})^{\natural}$ for all groups H.

2) [4, Chapter A, Lemma (18.2)(d)] Let $W = G \wr H$. If $K \trianglelefteq G$, then $W/K^{\natural} \cong (G/K) \wr H$.

3) [4, Chapter A, Lemma (18.2)(c)] Let $W = G \wr H$. If $K \leq G$, then $K^{\natural}H \cong K \wr H \leq W$.

Lemma 2.4. *Let $\mathfrak{F}$ be a Fitting class, G a group, and H a nilpotent group.*

1) [4, Chapter X, Lemma (2.4)] *If there exists a natural number m such that $G^m \wr H \in \mathfrak{F}$, then $G^n \wr H \in \mathfrak{F}^*$ for all $n \in \mathbb{N}$.*

2) [4, Chapter X, Lemma (2.11)(a)] *If there exists a natural number n such that $G^n \wr Z_p \in \mathfrak{F}^*$, then $G^2 \wr Z_p \in \mathfrak{F}$.*

Lemma 2.5. [14, Theorem 1] *Let $\mathfrak{F} = LR(f)$ for some H-function f and $\pi = Supp(f)$. Then $LR(f) = LR(f^*)$, where f^* is an H-function such that $f^*(p) = (f(p))^*$ for all $p \in \pi$.*

Recall that the socle of G, denoted by $\mathrm{Soc}(G)$, is the subgroup generated by the identity and its minimal normal subgroups.

Lemma 2.6. [4, Chapter IX, Examples (2.9)(a)] *Let $\mathfrak{X} = (G : \mathrm{Soc}(G) \leq Z(G))$. Then $\mathfrak{X}$ is a Fitting class and $\mathfrak{X}$ is closed under taking subdirect products.*

Lemma 2.7. [5, Lemma 3] *Let A be a soluble group of nilpotent length u and B a soluble permutation group of nilpotent length v. Let G be the permutational wreath product of A by B. Then the nilpotent length of the group G is $u + v - 1$ if and only if A has a nilpotent chain of length u whose first section is a p-group and B has a nilpotent chain of length v whose last section is a p-group for the same prime p.*

Lemma 2.8. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be Fitting classes. Then:*

1) [4, Chapter X, Theorems (1.15) and (1.8)(a)] $\mathfrak{F}_* = (\mathfrak{F}_*)_* = (\mathfrak{F}^*)_* \subseteq \mathfrak{F} \subseteq \mathfrak{F}^* = (\mathfrak{F}_*)^* = (\mathfrak{F}^*)^*$.

2) [4, Chapter X, Theorem (1.8)(b)] If $\mathfrak{F} \subseteq \mathfrak{H}$, then $\mathfrak{F}^* \subseteq \mathfrak{H}^*$.

3. The proof of Theorem 1

Proof of Theorem 1. We first prove that there are $\mathfrak{X}$-quasinormal Fitting classes which are not $\mathfrak{X}$-normal.

Let $\mathfrak{N}$ be the class of all nilpotent groups and $\mathfrak{X} = (G : \mathrm{Soc}(G) \leq Z(G))$. Clearly, $\mathfrak{N} \subseteq \mathfrak{X}$ and $\mathfrak{N} \subseteq \mathfrak{X} \cap \mathfrak{N}^2$, where $\mathfrak{N}^2 = \mathfrak{N}\mathfrak{N}$ is the class of all metanilpotent groups.

We show that $\mathfrak{N}$ is $\mathfrak{X}$-quasinormal. The only thing to show is that $\mathfrak{N}$ is $(\mathfrak{X} \cap \mathfrak{N}^2)$-quasinormal. By Lemma 2.6, it is easy to see that $\mathfrak{X} \cap \mathfrak{N}^2$ is a Fitting class and it is closed under subdirect product. Let $G \in \mathfrak{N}$ and $G \wr Z_p \in \mathfrak{X} \cap \mathfrak{N}^2$ for some $p \in \mathbb{P}$. We first prove that G is a p-group.

If $G \wr Z_p \in \mathfrak{N}$, then by Lemma 2.7, G is a p-group. Note that $\mathfrak{N}$ is a Lockett class. Assume that $G \wr Z_p \in (\mathfrak{X} \cap \mathfrak{N}^2) \setminus \mathfrak{N}$. Then by Lemma 2.3(1), $(G \wr Z_p)_{\mathfrak{N}} = \mathrm{F}(G \wr Z_p) = \mathrm{F}(G)^{\natural}$. By Lemma 2.3(2), $(G \wr Z_p)/\mathrm{F}(G \wr Z_p) = (G \wr Z_p)/F(G)^{\natural} \simeq (G/\mathrm{F}(G)) \wr Z_p$ and it is a p-group by Lemma 2.7.

Assume that $\mathrm{O}_{p'}(\mathrm{F}(G \wr Z_p)) \neq 1$. Then $\mathrm{O}_{p'}(\mathrm{F}(G)) \neq 1$, and so for a prime $q \neq p$, G has a minimal normal q-subgroup N. Since $G \in \mathfrak{X}$, $N \leq Z(G)$. Clearly, $N^{\natural}Z_p$ is a supplement of $G^{\natural}$ in $G \wr Z_p$, $N^{\natural}Z_p/(G^{\natural} \cap N^{\natural}Z_p) = N^{\natural}Z_p/N^{\natural} \simeq Z_p \in \mathfrak{N}$ and $N^{\natural}Z_p \cap G^{\natural} \subseteq Z(G^{\natural})$. By Lemma 2.2(2), $N^{\natural}Z_p \simeq N \wr Z_p \in \mathfrak{X} \cap \mathfrak{N}^2$. Because $q \neq p$, $N^{\natural} \leq \mathrm{Soc}(N^{\natural}Z_p)$. But as $N^{\natural}Z_p \in \mathfrak{X}$, we have that $N^{\natural} \leq Z(N^{\natural}Z_p)$. Hence $N^{\natural}Z_p \simeq N \wr Z_p$ is an abelian group. This is impossible because $N \neq 1$. This contradiction shows that $\mathrm{F}(G \wr Z_p)$ is a p-group.

It follows that G is a p-group. Consequently, $G \wr Z_p$ is a p-group and therefore $G \wr Z_p \in \mathfrak{N}$. Therefore, $\mathfrak{N}$ is $\mathfrak{X} \cap \mathfrak{N}^2$-quasinormal and so $\mathfrak{N}$ is quasinormal in $\mathfrak{X}$.

We now show that $\mathfrak{N}$ is not $\mathfrak{X}$-normal. First note that for any class $\mathfrak{F}$ of groups, an $\mathfrak{F}$-*injector* of a group G is a subgroup V of G with the property that $V \cap K$ is an $\mathfrak{F}$-maximal subgroup of K for all subnormal subgroups K of G. Let $G = GL(2, 3)$. Since $Z(G)$ is the unique minimal normal subgroup of G, we have that $G \in \mathfrak{X}$, and clearly a Sylow 2-subgroup of G is an $\mathfrak{N}$-injector of G. But the Sylow 2-subgroup of G is not normal in G. If $\mathfrak{N}$ is $\mathfrak{X}$-normal, then $G_{\mathfrak{N}} = F(G)$ is $\mathfrak{N}$-maximal in G, and so $G_{\mathfrak{N}}$ is an $\mathfrak{N}$-injector of G. It follows that the Sylow 2-subgroup of G is normal in G, a contradiction. Thus $\mathfrak{N}$ is not $\mathfrak{X}$-normal.

Next we prove that there exists an $\mathfrak{X}$-normal Fitting class which is not $\mathfrak{X}$-quasinormal.

Let $\varnothing \neq \pi \subseteq \mathbb{P}$, $\mathfrak{S}_\pi$ be the class of all soluble π-groups and $\mathfrak{X} = \mathfrak{S}_\pi \mathfrak{S}_{\pi'}$.

Let $G \in \mathfrak{X}$ and let V be an $\mathfrak{S}_\pi$-injector of G. Then V is a Hall π-subgroup of G and $G/G_{\mathfrak{S}_\pi} \in \mathfrak{S}_{\pi'}$. By the Chunihin theorem (see [6, Theorem 1.7.6]), $V \leq G_{\mathfrak{S}_\pi} \leq V$, so $V = G_{\mathfrak{S}_\pi}$ is normal in G. This shows that $\mathfrak{S}_\pi$ is an $\mathfrak{X}$-normal Fitting class.

We now show that $\mathfrak{S}_\pi$ is not $\mathfrak{X}$-quasinormal. Let $G \in \mathfrak{S}_\pi$ and $W = G \wr Z_p \in \mathfrak{X}$ for some $p \in \pi'$. Then $W/(G)^{\natural} \simeq Z_p \in \mathfrak{S}_{\pi'}$. Since $G^{\natural} \in \mathfrak{S}_\pi$, we have that $W \in \mathfrak{S}_\pi \mathfrak{S}_{\pi'} = \mathfrak{X}$. But since $p \notin \pi$, clearly, for any natural number m, $G^m \wr Z_p \notin \mathfrak{S}\pi$. Hence $\mathfrak{S}_\pi$ is not quasinormal in $\mathfrak{X}$. The theorem is proved. □

4. The proof of Theorem 2

Following [1, p. 263], for a class $\mathfrak{F}$ of groups, the set $\mathrm{Char}(\mathfrak{F}) = \{p : p \in \mathbb{P} \text{ and } Z_p \in \mathfrak{F}\}$ is called the characteristic of $\mathfrak{F}$.

The proof of Theorem 2. Let $\{\mathfrak{F}_i\}_{i \in I}$ be a set of some nonidentity $\mathfrak{X}$-quasinormal Fitting classes, and let $\mathfrak{F} = \bigcap_{i \in I} \mathfrak{F}_i$.

We first prove that $\mathfrak{F}$ is $\mathfrak{X}$-quasinormal. It is clear that $\mathfrak{F} \subseteq \mathfrak{X}$. Let $G \in \mathfrak{F}$ and $G \wr Z_p \in \mathfrak{X}$ for some $p \in \mathbb{P}$. Since $\mathfrak{F} \subseteq \mathfrak{F}_i$ for all $i \in I$, we have that $G \in \mathfrak{F}_i$ for any $i \in I$. As $\mathfrak{F}_i$ is $\mathfrak{X}$-quasinormal for any $i \in I$, there exist natural number m_i such that $G^{m_i} \wr Z_p \in \mathfrak{F}_i$ for any $i \in I$. Then by Lemma 2.4, $G^2 \wr Z_p \in \mathfrak{F}_i$ for $i \in I$. It follows that $G^2 \wr Z_p \in \mathfrak{F}$. Hence $\mathfrak{F}$ is $\mathfrak{X}$-quasinormal.

In order to prove $\mathfrak{F} \neq (1)$, where (1) is the class of all identity groups, we need to prove that $\mathrm{Char}(\mathfrak{F}_i) = \mathrm{Char}(\mathfrak{X})$ for any $i \in I$. Assume that there exists a prime $p \in \mathrm{Char}(\mathfrak{X}) \setminus \mathrm{Char}(\mathfrak{F}_i)$ for some $i \in I$. Then $1 \wr Z_p \simeq Z_p \in \mathfrak{X} \setminus \mathfrak{F}_i$, and clearly for any natural number m, $1^m \wr Z_p \simeq Z_p \notin \mathfrak{F}$. This shows that $\mathfrak{F}_i$ is not $\mathfrak{X}$-quasinormal. The contradiction obtained implies $\mathrm{Char}(\mathfrak{X}) \subseteq \mathrm{Char}(\mathfrak{F}_i)$ for all $i \in I$. On the other hand, $\mathrm{Char}(\mathfrak{F}_i) \subseteq \mathrm{Char}(\mathfrak{X})$ for all $i \in I$ since $\mathfrak{F}_i \subseteq \mathfrak{X}$. Hence $\mathrm{Char}(\mathfrak{F}_i) = \mathrm{Char}(\mathfrak{X})$ for every $i \in I$.

From the above it follows that $\mathrm{Char}(\mathfrak{F}_i) = \mathrm{Char}(\mathfrak{X}) = \mathrm{Char}(\mathfrak{F})$ for every $i \in I$. Since $\mathfrak{F}_i \neq (1)$ for every $i \in I$, there exists a prime p such that $Z_p \in \mathfrak{F}_i$. Hence $Z_p \in \mathfrak{F}$, and so $\mathfrak{F} \neq (1)$.

Consequently, there exists the smallest nonidentity $\mathfrak{X}$-quasinormal Fitting class. The theorem is proved. □

5. The proof of Theorem 3

In order to prove Theorem 3, we first need to prove the following lemmas.

Lemma 5.1. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be Fitting classes. If $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ and $\mathfrak{F}_1^* = \mathfrak{F}_2^*$, then $\mathfrak{F}_1$ is quasinormal in $\mathfrak{F}_2$.*

Proof. Let $G \in \mathfrak{F}_1$ and $G \wr Z_p \in \mathfrak{F}_2$ for some $p \in \mathbb{P}$. We need to prove that there exists a natural number m such that $G^m \wr Z_p \in \mathfrak{F}_1$. In fact, since $\mathfrak{F}_1^* = \mathfrak{F}_2^*$, we have that $G \wr Z_p \in \mathfrak{F}_2 \subseteq \mathfrak{F}_2^* = \mathfrak{F}_1^*$. Hence by Lemma 2.1, $G^2 \wr Z_p \in \mathfrak{F}_1$. Lemma is proved. □

Lemma 5.2. *The relation of quasinormality is transitive.*

Proof. Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{F}_3$ be Fitting classes. We prove that if $\mathfrak{F}_1 \subseteq \mathfrak{F}_2 \subseteq \mathfrak{F}_3$, $\mathfrak{F}_1$ is quasinormal in $\mathfrak{F}_2$ and $\mathfrak{F}_2$ is quasinormal in $\mathfrak{F}_3$, then $\mathfrak{F}_1$ is quasinormal in $\mathfrak{F}_3$.

By Lemma 2.4, we can see that a Fitting class $\mathfrak{F}$ is $\mathfrak{X}$-quasinormal if and only if $G^2 \wr Z_p \in \mathfrak{F}$ whenever $\mathfrak{F} \subseteq \mathfrak{X}$, $G \in \mathfrak{F}$ and $G \wr Z_p \in \mathfrak{X}$.

Let $G \in \mathfrak{F}_1$ and $G \wr Z_p \in \mathfrak{F}_3$ for some $p \in \mathbb{P}$. We show that $G^2 \wr Z_p \in \mathfrak{F}_1$.

Since $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$, $G \in \mathfrak{F}_2$. But as $\mathfrak{F}_2$ is quasinormal in $\mathfrak{F}_3$, we have that $G^2 \wr Z_p \in \mathfrak{F}_2$. Then since $G \wr Z_p = G^{\natural} Z_p$ is subnormal embedded into $(G^2)^{\natural} Z_p = G^2 \wr Z_p$, we have that $G \wr Z_p \in \mathfrak{F}_2$. Now since $\mathfrak{F}_1$ is quasinormal in $\mathfrak{F}_2$, we obtain that $G^2 \wr Z_p \in \mathfrak{F}_1$. Hence $\mathfrak{F}_1$ is quasinormal in $\mathfrak{F}_3$. □

Lemma 5.3. *Let $\mathfrak{F}$, $\mathfrak{X}$ and $\mathfrak{Y}$ be Fitting classes. If $\mathfrak{Y} \subseteq \mathfrak{X}$ and $\mathfrak{F}$ is quasinormal in $\mathfrak{X}$, then $\mathfrak{Y} \bigcap \mathfrak{F}$ is quasinormal in $\mathfrak{Y}$.*

Proof. Let $G \in \mathfrak{Y} \bigcap \mathfrak{F}$ and $G \wr Z_p \in \mathfrak{Y}$ for some $p \in \mathbb{P}$. We show that there exists a natural number m such that $G^m \wr Z_p \in \mathfrak{Y} \bigcap \mathfrak{F}$.

Since $\mathfrak{F}$ is $\mathfrak{X}$-quasinormal and $G \wr Z_p \in \mathfrak{Y} \subseteq \mathfrak{X}$, there exists a natural number m such that $G^m \wr Z_p \in \mathfrak{F}$. On the other hand, by Lemma 2.2(1), $G^m \wr Z_p \in \mathfrak{Y}$. This implies that $G^m \wr Z_p \in \mathfrak{Y} \bigcap \mathfrak{F}$. Therefore $\mathfrak{Y} \bigcap \mathfrak{F}$ is quasinormal in $\mathfrak{Y}$. □

Lemma 5.4. *Let $\mathfrak{F}$ and $\mathfrak{X}$ be Fitting classes. If $\mathfrak{F}$ is quasinormal in $\mathfrak{X}$, then $\mathfrak{F}$ is quasinormal in $\mathfrak{X}^*$.*

Proof. By Lemma 2.8(1), $\mathfrak{X} \subseteq \mathfrak{X}^*$ and $\mathfrak{X}^* = (\mathfrak{X}^*)^*$. Then by Lemma 5.1, $\mathfrak{X}$ is quasinormal in $\mathfrak{X}^*$. Hence by Lemma 5.2, $\mathfrak{F}$ is quasinormal in $\mathfrak{X}^*$. □

The following proposition describes the properties of the operator "$_{\circledast}$".

Proposition 5.5. *Let $\mathfrak{F}$ and $\mathfrak{X}$ be Fitting classes.*

1) If $\mathfrak{F} \subseteq \mathfrak{X}$, then $\mathfrak{F}_{\circledast} \subseteq \mathfrak{X}_{\circledast}$.

2) $\mathfrak{X}_{\circledast} = (\mathfrak{X}_{\circledast})_{\circledast} = (\mathfrak{X}_*)_{\circledast} = (\mathfrak{X}_{\circledast})_*$.

3) If $\mathfrak{X}_{\circledast} \subseteq \mathfrak{F} \subseteq \mathfrak{X}$, then $\mathfrak{F}_{\circledast} = \mathfrak{X}_{\circledast}$.

Proof. 1) Since $\mathfrak{X}_{\circledast}$ is $\mathfrak{X}$-quasinormal, $\mathfrak{F} \cap \mathfrak{X}_{\circledast}$ is quasinormal in $\mathfrak{F}$ by Lemma 5.3. Since $\mathfrak{F}_{\circledast}$ is the smallest $\mathfrak{F}$-quasinormal Fitting class, $\mathfrak{F}_{\circledast} \subseteq \mathfrak{F} \cap \mathfrak{X}_{\circledast} \subseteq \mathfrak{X}_{\circledast}$. Hence $\mathfrak{F}_{\circledast} \subseteq \mathfrak{X}_{\circledast}$.

2) Since $(\mathfrak{X}_{\circledast})_{\circledast}$ is quasinormal in $\mathfrak{X}_{\circledast}$ and $\mathfrak{X}_{\circledast}$ is quasinormal in $\mathfrak{X}$, $(\mathfrak{X}_{\circledast})_{\circledast}$ is $\mathfrak{X}$-quasinormal by Lemma 5.2. But $\mathfrak{X}_{\circledast}$ is the smallest $\mathfrak{X}$-quasinormal Fitting class, so $\mathfrak{X}_{\circledast} \subseteq (\mathfrak{X}_{\circledast})_{\circledast}$. On the other hand, by the definition of "$\circledast$", $(\mathfrak{X}_{\circledast})_{\circledast} \subseteq \mathfrak{X}_{\circledast}$. Therefore $\mathfrak{X}_{\circledast} = (\mathfrak{X}_{\circledast})_{\circledast}$.

By Lemma 2.8(1), $(\mathfrak{X}_{\circledast})_* \subseteq \mathfrak{X}_{\circledast}$ and $((\mathfrak{X}_{\circledast})_*)^* = (\mathfrak{X}_{\circledast})^*$. Then by Lemma 5.1, $(\mathfrak{X}_{\circledast})_*$ is quasinormal in $\mathfrak{X}_{\circledast}$. It follows from Lemma 5.2 that $(\mathfrak{X}_{\circledast})_*$ is quasinormal in $\mathfrak{X}$. But as $\mathfrak{X}_{\circledast}$ is the smallest $\mathfrak{X}$-quasinormal Fitting class, we have that $\mathfrak{X}_{\circledast} \subseteq (\mathfrak{X}_{\circledast})_*$. Hence $(\mathfrak{X}_{\circledast})_* = \mathfrak{X}_{\circledast}$.

Finally we prove that $\mathfrak{X}_{\circledast} = (\mathfrak{X}_*)_{\circledast}$. In fact, since $(\mathfrak{X}_*)^* = \mathfrak{X}^*$ and $\mathfrak{X}_* \subseteq \mathfrak{X}$ by Lemma 2.8(1), $\mathfrak{X}_*$ is $\mathfrak{X}$-quasinormal by Lemma 5.1. But as $\mathfrak{X}_{\circledast}$ is the smallest $\mathfrak{X}$-quasinormal Fitting class, we have that $\mathfrak{X}_{\circledast} \subseteq \mathfrak{X}_*$. Hence $(\mathfrak{X}_{\circledast})_{\circledast} \subseteq (\mathfrak{X}_*)_{\circledast} \subseteq \mathfrak{X}_{\circledast}$ by the statement 1) of the Proposition. Thus $\mathfrak{X}_{\circledast} = (\mathfrak{X}_*)_{\circledast} = (\mathfrak{X}_{\circledast})_{\circledast}$.

3) Let $\mathfrak{X}_{\circledast} \subseteq \mathfrak{F} \subseteq \mathfrak{X}$. Then by the statements 1) and 2), $\mathfrak{X}_{\circledast} = (\mathfrak{X}_{\circledast})_{\circledast} \subseteq \mathfrak{F}_{\circledast} \subseteq \mathfrak{X}_{\circledast}$. Thus $\mathfrak{F}_{\circledast} = \mathfrak{X}_{\circledast}$.

The proposition is proved. □

Proof of Theorem 3. 1) Let $\mathfrak{F}$ is a local Fitting class. Then $\mathfrak{F} = LR(f) \doteq \mathfrak{E}_\pi \cap (\cap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{E}_{p'})$ for some H-function f with the support π.

Suppose that $1 \neq G \in \mathfrak{F}$ and $O^{p'}(G) = G$ for any $p \in \pi$. We need to prove that there exists a natural number n such that $G^n \wr Z_p \in \mathfrak{F}$.

Let $W = G \wr Z_p$. We first prove that $W \in \mathfrak{F}$.

Since $G \in \mathfrak{F} = \mathfrak{E}_\pi \cap (\cap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{E}_{p'})$, we have that $G \in \mathfrak{E}_\pi$, and so $W \in \mathfrak{E}_\pi$. We now prove that $W \in \cap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{E}_{p'}$ for all $p \in \pi$. Clearly, $G \in f(p)\mathfrak{N}_p\mathfrak{E}_{p'}$ for every $p \in \pi$. By Lemmas 2.5 and 2.8(1), $f(p)\mathfrak{N}_p\mathfrak{E}_{p'} = f^*(p)\mathfrak{N}_p\mathfrak{E}_{p'}$ and $(f^*(p))^* = f^*(p)$ for all $p \in \pi$. Hence without loss of generality, we may assume that f is an H-function of $\mathfrak{F}$ such that $f(p)$ is a Lockett class for every $p \in \pi$.

If $G \in f(p)$, then $G \in f(p)\mathfrak{N}_p$ and so $W \in f(p)\mathfrak{N}_p \subseteq f(p)\mathfrak{N}_p\mathfrak{E}_{p'}$. Suppose that $G \in f(p)\mathfrak{N}_p \setminus f(p)$. By Lemma 2.3(1), $W_{f(p)} = (G_{f(p)})^\natural$, so $W/W_{f(p)} = W/(G_{f(p)})^\natural$. Hence by Lemma 2.3(2), $W/W_{f(p)} \simeq (G/G_{f(p)}) \wr Z_p$. Since $G \in f(p)\mathfrak{N}_p$, $G/G_{f(p)} \in \mathfrak{N}_p$. Hence $W \in f(p)\mathfrak{N}_p \subseteq f(p)\mathfrak{N}_p\mathfrak{E}_{p'}$.

If $G \in f(p)\mathfrak{N}_p\mathfrak{E}_{p'} \setminus f(p)\mathfrak{N}_p$, then, with a similar discussion in the above paragraph, $W/W_{f(p)\mathfrak{N}_p} \simeq (G/G_{f(p)\mathfrak{N}_p}) \wr Z_p$. Since $G \in f(p)\mathfrak{N}_p\mathfrak{E}_{p'}$ for every $p \in \pi$, $G/G_{f(p)\mathfrak{N}_p} \in \mathfrak{E}_{p'}$. But $G = O^{p'}(G)$, so we have that $G = G_{f(p)\mathfrak{N}_p}$. Hence $G \in f(p)\mathfrak{N}_p$. It follows that $W \in f(p)\mathfrak{N}_p \subseteq f(p)\mathfrak{N}_p\mathfrak{E}_{p'}$ for all $p \in \pi$.

The above shows that $W = G \wr Z_p \in \mathfrak{F}$. Hence by Lemma 2.2(1), $G^n \wr Z_p \in \mathfrak{F}$ for any natural n. This shows that $\mathfrak{F}$ is a class with limited π-property of wreath product.

2) In order to prove the statement i), we first prove that $\mathfrak{X}^* = \mathfrak{F}^*$ for every $\mathfrak{X}$-quasinormal Fitting class $\mathfrak{F}$.

Since $\mathfrak{F} \subseteq \mathfrak{X}$, we have that $\mathfrak{F}^* \subseteq \mathfrak{X}^*$ by Lemma 2.8(2). Assume that $\mathfrak{X}^* \neq \mathfrak{F}^*$ and G is a group in $\mathfrak{X}^* \setminus \mathfrak{F}^*$ of minimal order. Then, clearly, $G_{\mathfrak{F}^*}$ is the unique maximal normal subgroup of G (in fact, we only need to show that $G_{\mathfrak{F}^*}$ is a maximal normal subgroup of G. If not, then there exists a maximal normal subgroup M of G such that $G_{\mathfrak{F}^*} < M \lhd G \in \mathfrak{X}^*$. Then $M \in \mathfrak{X}^*$ since $\mathfrak{X}^*$ is a Fitting class. But by the choice of G, we have that $M \in \mathfrak{F}^*$, so $M \subseteq G_{\mathfrak{F}^*}$, a contradiction).

We claim that $(\mathfrak{S}_\pi)^* = \mathfrak{S}_\pi$. Firstly, by Lemma 2.8(1), $\mathfrak{S}_\pi \subseteq (\mathfrak{S}_\pi)^*$. On the other hand, by [4, Chapter X, Proposition (1.20)], $(\mathfrak{S}_\pi^*) \subseteq Q\mathfrak{S}_\pi$, where $Q\mathfrak{F}$ is the set of epimorphisms of all $\mathfrak{F}$-groups (see [4, p. 264]) and clearly, $Q\mathfrak{S}_\pi \subseteq \mathfrak{S}_\pi$. It follows that $(\mathfrak{S}_\pi)^* \subseteq \mathfrak{S}_\pi$. Hence $(\mathfrak{S}_\pi)^* = \mathfrak{S}_\pi$.

Since $\mathfrak{X} \subseteq \mathfrak{S}_\pi$, $\mathfrak{X}^* \subseteq (\mathfrak{S}_\pi)^* = \mathfrak{S}_\pi$ by Lemma 2.8(1) again. Hence $G \in \mathfrak{S}_\pi$. This implies that $G/G_{\mathfrak{F}^*} \in \mathfrak{S}_\pi$, and so $G/G_{\mathfrak{F}^*}$ is a cyclic p-group for some prime $p \in \pi$.

By Lemma 2.3(3), $G_{\mathfrak{F}} \wr Z_p \simeq (G_{\mathfrak{F}})^\natural Z_p \trianglelefteq \trianglelefteq G^\natural Z_p \simeq G \wr Z_p$. Note also that $O^{p'}(G) = G$ since $G \in \mathfrak{X} \subseteq \mathfrak{S}_\pi$. As $\mathfrak{X}^*$ is a Fitting class with limited π-property of wreath product by the hypothesis, there exists a natural number n such that $G^n \wr Z_p \in \mathfrak{X}^*$. Hence by Lemmas 2.4(1) and 2.8(1), $G \wr Z_p \in (\mathfrak{X}^*)^* = \mathfrak{X}^*$. This implies that $G_{\mathfrak{F}} \wr Z_p \in \mathfrak{X}^*$.

Since $\mathfrak{F}$ is $\mathfrak{X}$-quasinormal, $\mathfrak{F}$ is $\mathfrak{X}^*$-quasinormal by Lemma 5.4. Hence there exists a natural number m such that $(G_{\mathfrak{F}})^m \wr Z_p \in \mathfrak{F}$. Then by Lemmas 2.4(1) and 2.3(3), $G_{\mathfrak{F}} \wr Z_p \cong (G_{\mathfrak{F}})^\natural Z_p \in \mathfrak{F}^*$. Since $(G_{\mathfrak{F}})^\natural Z_p \trianglelefteq \trianglelefteq G^\natural Z_p \cong G \wr Z_p$, we have that $(G_{\mathfrak{F}})^\natural Z_p \leq (G \wr Z_p)_{\mathfrak{F}^*}$. But since $G \notin \mathfrak{F}^*$ and $(\mathfrak{F}^*)^* = \mathfrak{F}^*$ (that

is, $\mathfrak{F}^*$ is a Lockett class), $(G \wr Z_p)_{\mathfrak{F}^*} = (G_{\mathfrak{F}^*})^\natural$ by Lemma 2.3(1), which is impossible since $Z_p \not\subseteq (G_{\mathfrak{F}^*})^\natural$. This contradiction shows that $\mathfrak{X}^* = \mathfrak{F}^*$.

Now since $\mathfrak{X}_\circledast$ is $\mathfrak{X}$-quasinormal, $(\mathfrak{X}_\circledast)^* = \mathfrak{X}^*$ by above. But as $\mathfrak{X}_*$ is the intersection of all Fitting classes $\mathfrak{F}$ with $\mathfrak{X}^* = \mathfrak{F}^*$, we have that $\mathfrak{X}_* \subseteq \mathfrak{X}_\circledast$. On the other hand, $\mathfrak{X}_* \subseteq \mathfrak{X}$ by Lemma 2.8(1) and $(\mathfrak{X}_*)^* = \mathfrak{X}^*$ by Lemma 2.8(2). Hence by Lemma 5.1, $\mathfrak{X}_*$ is also $\mathfrak{X}$-quasinormal, so $\mathfrak{X}_\circledast \subseteq \mathfrak{X}_*$. Therefore $\mathfrak{X}_* = \mathfrak{X}_\circledast$.

ii) This directly follows from statements 1) and i) of the theorem.

The theorem is proved. □

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References

[1] Ballester-Bolinches, A., Ezquerro, L. M. (2006). *Classes of Finite Groups*. Dordrecht: Springer.
[2] Blessenohl, D., Gaschütz, W. (1970). Über normale Schunk- und Fittingklassen. *Math. Z.* 118:1–8.
[3] Bryce, R. A., John, C. (1973). Subdirect product closed Fitting classes. In: Proceedings of the Second International Conference on the Theory of Groups, Canberra, pp. 158–164.
[4] Doerk, K., Hawkes, T. (1992). *Finite Soluble Groups*. Berlin, New York: Walter de Gruyter.
[5] Frick, M., Newman, M. F. (1972). Soluble linear groups. *Bull. Australian Math. Soc.* 6:31–44.
[6] Guo, W. (2000). *The Theory of Classes of Groups*. Beijing-New York-Dordrecht-Boston-London: Science Press-Kluwer Academic Publishers.
[7] Guo, W. (2015). *Structure Theory for Canonical Classes of Finite Groups*. Berlin: Springer.
[8] Guo, W., Vorob'ev, N. T. (2008). On injectors of finite groups. *Commun. Algebra* 36:3200–3208.
[9] Guo, W., Zhang, L., Vorob'ev, N. T. (2020). On σ-local Fitting classes. *J. Algebra* 542:116–129.
[10] Hauck, P. (1977). Zur Theorie der Fittingklassen endlicher auflösbarer Gruppen. Dis. Doctor der Naturwissenschaft. Mainz.
[11] Laue, H. (1977). Über nichtauflösbare normale Fittingklassen. *J. Algebra* 45:274–283.
[12] Lockett, P. (1974). The fitting class $\mathfrak{F}^*$. *Math. Z.* 137:131–136.
[13] Makan, A. R. (1974). Fitting classes with the wreath product property are normal. *J. London Math. Soc.* 8:245–246.
[14] Vorob'ev, N. T. (1992). On maximal and minimal group functions of local Fitting classes. *Problems Algebra* 7:60–69 (in Russian).
[15] Vorob'ev, N. T. (1996). On Hawkes's conjecture for radical classes. *Siberian Math. J.* 37:1137–1142.