FULL GROUP OF ISOMETRIES OF SPECIAL THREE-DIMENSIONAL LIE GROUP

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The purpose of this work is to construct a complete group of isometries of one special connected simply connected three-dimensional Lie group S_3, the Lie algebra of which belongs to Bianchi type V.

Material and methods. We consider the Lie group S_3 equipped with a left-invariant Lorentz metric. Methods from the theory of groups and Lie algebras are used.

Findings and their discussion. Let G be a Lie algebra equipped with an inner product. Let us call a linear transformation $f\colon G\to G$ *an autoisometry* if it is both an isometry and an automorphism of the Lie algebra.

Let g be a left-invariant metric on a connected Lie group G. Any isometry $h: G\to G$ of a homogeneous space (G, g), can be decomposed into a composition $L_a\circ f$, where f is a similarity leaving the unit element $e\in G$ in place, and L_a is left shift, $a\in G$. Therefore, in order to find the complete group of isometries of the manifold (G, g), it is necessary first to find isometries that leave the unit element of the Lie group fixed.

In a suitable basis (E_1, E_2, E_3), the bracket operation in the considered Lie algebra of V Bianchi type is given by the formulas $[E_1,E_2]=E_2, [E_1,E_3]=E_3, [E_2,E_3]=\vec{0}$. We will call this Lie algebra and the corresponding connected simply connected Lie group *special* and denote them S_3 and S_3, respectively, and the above basis will be called *canonical*. It is shown in [1], that the group and Lie algebra under consideration can be represented in the form of matrices

$$U=\begin{pmatrix}0&0&u_1&u_2\\0&u_1&0&u_3\\u_1&0&0&u_2\\0&0&0&0\end{pmatrix},\ u_1,u_2,u_3\in\boldsymbol{R},\quad X=\begin{pmatrix}ch\,x_1&0&sh\,x_1&x_2\\0&e^{x_1}&0&x_3\\sh\,x_1&0&ch\,x_1&x_2\\0&0&0&1\end{pmatrix},\ x_1,x_2,x_3\in\boldsymbol{R}.$$

We assign coordinates (x_1, x_2, x_3) and (u_1, u_2, u_3) to these matrices, respectively. We will call these coordinates *canonical*. In canonical coordinates, the mappings $\exp:\ \mathrm{S}_3\to S_3$ and $\exp^{-1}:\ S_3\to \mathrm{S}_3$ are given respectively by the formulas

$$x_1=u_1,\ x_2=\frac{u_2}{u_1}(e^{u_1}-1),\ x_3=\frac{u_3}{u_1}(e^{u_1}-1);\quad u_1=x_1,\ u_2=\frac{x_1x_2}{(e^{x_1}-1)},\ u_3=\frac{x_1x_3}{(e^{x_1}-1)}.$$

The left shift by element $X(x_1, x_2, x_3)$ and the inverse element are determined by the formulas:

$$L_X(y_1,y_2,y_3)=(x_1+y_1, y_2e^{x1}+x_2, y_3e^{x1}+x_3);\quad (x_1,x_2,x_3)^{-1}=(-x_1,\ e^{-x1}x_2,\ e^{-x1}x_3).$$

Let a Eucledean scalar product be introduced in the Lie algebra S_3. Then the Gram matrix of a given scalar product in some canonical basis (E_1', E_2', E_3') can be drawn to the form

$$\Gamma = \begin{pmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ k > 0. \qquad (1)$$

If we want to make our basis orthonormal (i.e. make the Gram matrix unit), we must additionally multiply E_1' by the number $m = 1/\sqrt{k}$. In the new basis, composed of vectors $V_1 = mE_1'$, $V_2 = E_2'$, $V_3 = E_3'$, the bracket operation is given by the equalities

$$[V_1, V_2] = mV_2,\ [V_1, V_3] = mV_3,\ [V_2, V_3] = \vec{0},\ m > 0.$$

In any of the two bases (V_1, V_2, V_3) or (E_1', E_2', E_3') an arbitrary one-parameter group of autoisometries is specified by a matrix of the form

$$[F_t] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos at & -\sin at \\ 0 & \sin at & \cos at \end{pmatrix},\ t \in \boldsymbol{R},\ a \neq 0. \qquad (1)$$

Using the Gram matrix (1) and the left shift differential, we can construct a metric left-invariant tensor of the Lie group S_3; using matrix (2) and the exponential map, we found the formulas by which the one-parameter group of isometries acts, leaving the unit element fixed.

***Theorem* 1.** *Any left-invariant Riemannian metric tensor on the Lie group S_3 can be drawn to the following form with respect to natural coordinates:*

$$[g(x)] = \begin{pmatrix} k & 0 & 0 \\ 0 & e^{-2x_1} & 0 \\ 0 & 0 & e^{-2x_1} \end{pmatrix}.$$

The one-parameter group of isometries, leaving the unit element of the Lie group S_3 invariant, acts according to the formulas:

$$f_t(x_1, x_2, x_3) = (x_1, x_2\cos at - x_3\sin at, x_2\sin at + x_3\cos at)),\ a \neq 0,\ t \in \boldsymbol{R}.$$

2. *The full group of isometries of the resulting manifold is connected, four-dimensional, and consists of transformations acting according to the formulas*

$$\begin{cases} x_1' = x_1 + h_1, \\ x_2' = (x_2\cos at - x_3\sin at)e^{h_1} + h_2, \\ x_3' = (x_2\sin at + x_3\cos at)e^{h_1} + h_3, \end{cases} \qquad \begin{cases} x_1' = x_1 + h_1, \\ x_2' = (x_2\sin at + x_3\cos at)e^{h_1} + h_2, \\ x_3' = (x_2\cos at - x_3\sin at)e^{h_1} + h_3, \end{cases} \qquad (13)$$

where $H(h_1, h_2, h_3)$ *is an arbitrary element of the Lie group* S_3, $a \neq 0$, $t \in \boldsymbol{R}$.

Remark. The homogeneous Riemannian manifold obtained in this paper is a manifold of constant positive Ricci curvature. Using the Excel workbook, the development of which is described in [2], we found out that in the orthonormal basis (V_1, V_2, V_3) the matrices of operators $R(V_1, V_2)$, $R(V_1, V_3)$, $R(V_2, V_3)$ have the form, respectively:

$$\begin{pmatrix} 0 & -m^2 & 0 \\ m^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 0 & -m^2 \\ 0 & 0 & 0 \\ m^2 & 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -m^2 \\ 0 & m^2 & 0 \end{pmatrix}.$$

The nonzero components of the curvature tensor of type (4.0) are: $R_{1221} = R_{1331} = R_{2332} = -m^2$, and those obtained from them as a result of permutations. The Ricci curvature in the direction of basis vectors: $r(V_1) = r(V_2) = r(V_3) = 2m^2$. This means that for any unit vector holds $r(V_1) = 2m^2$.

Conclusion. We constructed a homogeneous manifold of the special Lie group S_3 equipped with a left-invariant Riemannian metric and found its complete isometry group. The constructed manifold turned out to be a manifold of constant Ricci curvature.

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2. Подоксёнов, М.Н. Тензор кривизны самоподобных лоренцевых многообразий некоторых четырёхмерных групп Ли / М.Н. Подоксёнов, Ю.А. Шпакова // Математические структуры и моделирование. 2023.–№ 2 (67). – С. 16–22.