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ON HAWKES'S CONJECTURE FOR RADICAL CLASSES[†)]

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Solving a series of classification problems of the theory of finite soluble groups is connected with studying totally local or primitive saturated formations (see, for instance, [1, Chapter 5; 2, Chapter 11]). Recall that, in accord with the definition of a multiply-local formation which is proposed by A. N. Skiba [3], every formation is assumed 0-local. A formation $\mathfrak{F}$ is *n-local* (with n a natural number) if all nonempty values of its local screen are $(n-1)$-local formations. A formation $\mathfrak{F}$ is *totally local* (or, which is the same, primitive saturated in the sense of Hawkes [4]) if $\mathfrak{F}$ is an n-local formation for every natural n.

For soluble formations of finite groups, Hawkes [4] established in 1971 that every totally local formation is hereditary and radical and conjectured that the totally local formations are exactly the hereditary radical saturated formations (see [4, p. 586]).

The conjecture was corroborated by Bryce and Cossey who demonstrated [5, Theorems A and 4] that every soluble hereditary radical saturated formation is totally local. Recall that a formation $\mathfrak{F}$ is a *radical* formation [2] if $\mathfrak{F}$ is simultaneously a radical class (a Fitting class), i.e., $\mathfrak{F}$ is closed under the taking of normal subgroups and products of normal subgroups.

In the present article we validate Hawkes's conjecture for soluble radical classes: we prove that a soluble radical class is totally local if and only if it is hereditary. To this end, we use the principally new local method for constructing radicals and radical classes of finite soluble groups which was proposed by Hartley [6]. In line with [6], let f be a mapping from the set $\mathbb{P}$ of all prime numbers into the set of soluble radical classes F and let $\pi = \operatorname{Supp} f$ stand for the support of f; i.e., $\pi = \{p \in \mathbb{P} \mid f(p) \neq \varnothing\}$. We call f (at L. A. Shemetkov's suggestion) a *local Hartley function* or a *local H-function.* Denote by $\mathrm{LR}(f)$ the class of those π-groups whose quotient groups by $f(p)$-radicals are extensions of p-groups by p'-groups for all prime $p \in \pi$. A radical class $\mathfrak{F}$ is called *local* [7] if $\mathfrak{F} = \mathrm{LR}(f)$ for some local H-function f. Involving the notion of product of radical classes, observe that

$$\mathfrak{F} = \mathfrak{S}_\pi \cap \Big(\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'}\Big).$$

Recall that the product of radical classes $\mathfrak{F}$ and $\mathfrak{H}$ is the class of all groups G such that $G/G_{\mathfrak{F}} \in \mathfrak{H}$ where $G_{\mathfrak{F}}$ is the product of all normal $\mathfrak{F}$-subgroups of G. In line with [3], introduce the notion of a multiply-local radical class: every radical class is assumed 0-local. Call a radical class $\mathfrak{F}$ *n-local* (with n a natural number) if all nonempty values of its local H-function are $(n-1)$-local radical classes. Call $\mathfrak{F}$ *totally local* if $\mathfrak{F}$ is n-local for every natural n.

In the article we deal with only finite soluble groups. The unrevealed definitions and notations can be found in [2, 8].

A radical class is called *hereditary* or *S-closed* if it is closed under the taking of subgroups. In Lemma 1 we present simplest properties of hereditary radical classes which are of service in the article.

Lemma 1. *The following assertions hold*:

(1) *if* $\{\mathfrak{F}_i \mid i \in I\}$ *is a set of hereditary radical classes then the intersection* $\mathfrak{F} = \bigcap_{i\in I}\mathfrak{F}_i$ *is a hereditary radical class*;

(2) *each finite product* $\mathfrak{F} = \prod_{i=1}^{n}\mathfrak{F}_i$ *of hereditary radical classes* $\mathfrak{F}_i$ *is a hereditary radical class.*

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PROOF. The first assertion of the lemma is obvious. To prove item (2), by associativity of the product of radical classes (see [2, IX.1.12, c]), it suffices to validate the claim for $n = 2$. Let G be a group in the class $\mathfrak{F}_1\mathfrak{F}_2$ and let H be an arbitrary subgroup of G. Then $G/G_{\mathfrak{F}_1} \in \mathfrak{F}_2$ and since the radical class $\mathfrak{F}_2$ is hereditary, we have

$$H/H \cap G_{\mathfrak{F}_1} \cong HG_{\mathfrak{F}_1}/G_{\mathfrak{F}_1} \in \mathfrak{F}_2.$$

Since $H \cap G_{\mathfrak{F}_1} \trianglelefteq H$ and $\mathfrak{F}_1$ is a hereditary class, it follows that $H \cap G_{\mathfrak{F}_1} \subseteq H_{\mathfrak{F}_1}$. Since the radical class $\mathfrak{F}_2$ is hereditary, the Bryce–Cossey theorem (see [2, XI.1.1]) implies that $\mathfrak{F}_2$ is a formation. Therefore, $H/H \cap G_{\mathfrak{F}_1}/H_{\mathfrak{F}_1}/H \cap G_{\mathfrak{F}_1} \cong H/H_{\mathfrak{F}_1} \in \mathfrak{F}_2$. Hence, $H \in \mathfrak{F}_1\mathfrak{F}_2$, which completes the proof of the lemma.

Call a local H-function f *hereditary* if $f(p)$ is a hereditary radical class for all prime p.

Lemma 2. *Each finite product $\mathfrak{F} = \prod_{i=1}^{n} \mathfrak{F}_i$ $(n \geq 2)$ of radical classes, where $\mathfrak{F}_i = \mathfrak{S}_{\pi_i}$ for some set π_i of prime numbers, is a radical class determined by a hereditary local H-function f such that the following equality holds for every prime p:*

$$f(p) = \begin{cases} \mathfrak{F}_1 & \text{if } p \in \pi_1 \setminus (\pi_2 \cup \pi_3 \cup \dots \cup \pi_n), \\ \mathfrak{F}_1\mathfrak{F}_2 & \text{if } p \in \pi_2 \setminus (\pi_3 \cup \pi_4 \cup \dots \cup \pi_n), \\ \dots\dots\dots\dots\dots\dots & \\ \mathfrak{F} & \text{if } p \in \pi_n, \\ \varnothing & \text{if } p \in (\pi_1 \cup \pi_2 \cup \dots \cup \pi_n)'. \end{cases}$$

PROOF. Let $\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2 \dots \mathfrak{F}_n$ be a product of radical classes; moreover, the following equalities hold for some sets $\pi_1, \pi_2, \dots, \pi_n$ of prime numbers: $\mathfrak{F}_1 = \mathfrak{S}_{\pi_1}, \mathfrak{F}_2 = \mathfrak{S}_{\pi_2}, \dots, \mathfrak{F}_n = \mathfrak{S}_{\pi_n}$ and $\sigma_1 = \pi_1 \setminus (\pi_2 \cup \pi_3 \cup \dots \cup \pi_n)$, $\sigma_2 = \pi_2 \setminus (\pi_3 \cup \dots \cup \pi_n)$, ..., $\sigma_n = \pi_n$, $\sigma = \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_n$.

Define a local H-function f on $\mathbb{P}$ as follows:

$$f(p) = \begin{cases} \mathfrak{F}_1 & \text{if } p \in \sigma_1, \\ \mathfrak{F}_1\mathfrak{F}_2 & \text{if } p \in \sigma_2, \\ \dots\dots\dots & \\ \mathfrak{F} & \text{if } p \in \sigma_n, \\ \varnothing & \text{if } p \in \sigma'. \end{cases}$$

Since the radical class $\mathfrak{F}_i$ is hereditary for every $i \in \{1, 2, \dots, n\}$, item (2) of Lemma 1 implies that f is a hereditary local H-function.

We demonstrate that f determines the radical class $\mathfrak{F}$. Put $\mathfrak{M} = \mathrm{LR}(f)$. By Lemma 5 of [9], $\pi(\mathfrak{F}_1\mathfrak{F}_2 \dots \mathfrak{F}_n) = \sigma$ and $\sigma = \operatorname{Supp} f$; therefore,

$$\mathfrak{M} = \mathfrak{S}_\sigma \cap \Big(\bigcap_{p\in\sigma} f(p)\mathfrak{N}_p\mathfrak{S}_{p'}\Big).$$

Leaning on the definition of f and using Lemma 1 of [10], we obtain

$$\mathfrak{M} = \mathfrak{S}_\sigma \cap \Big(\mathfrak{F}_1\Big(\bigcap_{p\in\sigma_1} \mathfrak{S}_{p'}\Big) \cap \Big(\mathfrak{F}_1\mathfrak{F}_2\Big(\bigcap_{p\in\sigma_2} \mathfrak{S}_{p'}\Big)\Big) \cap \dots \cap \Big(\mathfrak{F}\Big(\bigcap_{p\in\sigma_n} \mathfrak{S}_{p'}\Big).$$

It is easy to see that $\bigcap_{p\in\sigma_i} \mathfrak{S}_{p'} = \mathfrak{S}_{\sigma_i'}$ for every $i \in \{1, 2, \dots, n\}$. In consequence,

$$\mathfrak{M} = \mathfrak{S}_\sigma \cap \mathfrak{F}_1\mathfrak{S}_{\sigma_1'} \cap (\mathfrak{F}_1\mathfrak{F}_2)\mathfrak{S}_{\sigma_2'} \cap \dots \cap \mathfrak{F}\mathfrak{S}_{\sigma_n'}.$$

Since $\mathfrak{F}_2 \dots \mathfrak{F}_n \subseteq \mathfrak{S}_{\sigma_1'}$, $\mathfrak{F}_3 \dots \mathfrak{F}_n \subseteq \mathfrak{S}_{\sigma_2'}$, ..., $\mathfrak{F}_n \subseteq \mathfrak{S}_{\sigma_{n-1}'}$, we have $\mathfrak{F}_1\mathfrak{S}_{\sigma_1'} = \mathfrak{F}\mathfrak{S}_{\sigma_1'}$, $\mathfrak{F}_1\mathfrak{F}_2\mathfrak{S}_{\sigma_2'} = \mathfrak{F}\mathfrak{S}_{\sigma_2'}$, ..., $\mathfrak{F}_1\mathfrak{F}_2 \dots \mathfrak{F}_{n-1}\mathfrak{S}_{\sigma_{n-1}'} = \mathfrak{F}\mathfrak{S}_{\sigma_{n-1}'}$. Using Lemma 1 of [10] once again, we hence infer that

$$\mathfrak{M} = \mathfrak{S}_\sigma \cap \mathfrak{F}(\mathfrak{S}_{\sigma_1'} \cap \mathfrak{S}_{\sigma_2'} \cap \dots \cap \mathfrak{S}_{\sigma_n'}) = \mathfrak{S}_\sigma \cap \mathfrak{F}\mathfrak{S}_{\sigma'} = \mathfrak{F}.$$

In the case of $\sigma = \varnothing$ we have $\mathfrak{F} = \mathfrak{G}$, with $\mathfrak{G}$ the identity local radical class, and the lemma is obvious. This completes the proof of the lemma.

In line with [11], call a local H-function f of a radical class $\mathfrak{F}$

(1) *inner* if $f(p) \subseteq \mathfrak{F}$ for all prime numbers p;

(2) *full* if $f(p)\mathfrak{N}_p = f(p)$ for every prime p;

(3) *full inner* if f is simultaneously a full and inner local H-function.

Lemma 3. *If $\mathfrak{F} = \mathrm{LR}(f)$ for some hereditary local H-function f then $\mathfrak{F}$ is hereditary and $\mathfrak{F} = \mathrm{LR}(\varphi) = \mathrm{LR}(\psi) = \mathrm{LR}(h)$, where φ, ψ, and h are respectively full, inner, and full inner local H-functions.*

PROOF. By hypothesis, $\mathfrak{F} = \mathfrak{S}_\pi \cap \left(\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'}\right)$ and $f(p)$ is hereditary and local for all prime $p \in \pi$ ($\pi = \mathrm{Supp}\, f$).

Therefore, by Lemma 1 the radical class $\mathfrak{F}$ is hereditary. Now, construct local H-functions φ, ψ, and h as follows: $\varphi(p) = f(p)\mathfrak{N}_p$, $\psi(p) = f(p) \cap \mathfrak{F}$, and $h(p) = \psi(p)\mathfrak{N}_p$ for every prime p. Since the product of radical classes is associative, it follows that $\varphi(p)\mathfrak{N}_p = f(p)\mathfrak{N}_p$ for all prime p. Hence, $\mathfrak{F} = \mathrm{LR}(\varphi)$. Since $\psi(p) \subseteq f(p)$ for all prime p, we have $\mathrm{LR}(\psi) \subseteq \mathfrak{F}$. Let G be a group in $\mathfrak{F}$. Then $G^{\mathfrak{N}_p\mathfrak{S}_{p'}} \in f(p) \cap \mathfrak{F} = \psi(p)$ for all prime $p \in \pi$. Moreover, it is easily seen that $G \in \mathfrak{S}_\pi$, $\pi = \mathrm{Supp}\,\psi$. Hence, $G \in \mathfrak{S}_\pi \cap \left(\bigcap_{p\in\pi} \psi(p)\mathfrak{N}_p\mathfrak{S}_{p'}\right) = \mathrm{LR}(\psi)$. Thus, $\mathfrak{F} = \mathrm{LR}(\psi)$. But then since by Lemma 1 of [11] (also see [12, Lemma 6]) every local radical class is determined by a full inner local H-function, we conclude that $\mathfrak{F} = \mathrm{LR}(h)$. The fact that each of the local H-functions φ, ψ, and h is hereditary follows from Lemma 1. The lemma is proven.

Suppose that $\mathfrak{F} = \mathrm{LR}(f)$ for some hereditary local H-function f and let Ω be the set of all full hereditary local H-functions of the class $\mathfrak{F}$. In line with [8], define some order $\leq$ on Ω as follows: if $f, \varphi \in \Omega$ then $f \leq \varphi$ if and only if $f(p) \subseteq \varphi(p)$ for all prime p. Call a minimal element of Ω a *minimal full hereditary local H-function of the class $\mathfrak{F}$.*

If $\mathfrak{X}$ is a set of groups then denote by $\mathrm{SFit}\,\mathfrak{X}$ the hereditary radical class that is generated by $\mathfrak{X}$.

Lemma 4. *Assume that $\mathfrak{F} = \mathrm{LR}(h)$ for some hereditary local H-function h. Then the following assertions hold:*

(1) *$\mathfrak{F}$ is determined by a unique minimal full hereditary local H-function f such that the following equality is valid for every prime p:*

$$f(p) = \begin{cases} (\mathrm{SFit}\{G \in \mathfrak{F} \mid G = 0^{p'}(G)\})\mathfrak{N}_p & \text{if } p \in \pi(\mathfrak{F}), \\ \varnothing & \text{if } p \in \pi'(\mathfrak{F}); \end{cases} \tag{1}$$

(2) *if $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are radical classes determined by minimal full local H-functions f_1 and f_2 then $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ if and only if $f_1 \leq f_2$.*

PROOF. By hypothesis, the radical class $\mathfrak{F}$ is determined by a hereditary local H-function h; therefore, by Lemma 3, $\mathfrak{F}$ is also determined by some full hereditary local H-function. In consequence, the set Ω of all full hereditary local H-functions of the class $\mathfrak{F}$ is nonempty. Let φ be an arbitrary element in Ω. Then $\mathfrak{F} = \mathrm{LR}(\varphi)$; moreover, $\varphi(p) = \varphi(p)\mathfrak{N}_p$ and $\varphi(p)$ is a hereditary radical class for all prime p. In view of Hartley's result (see [6, 3.1.1]) claiming that the condition $p \in \pi(\mathfrak{F})$ is equivalent to $\mathfrak{N}_p \subseteq \mathfrak{F}$, we easily see that $\pi(\mathfrak{F}) = \mathrm{Supp}\,\varphi$.

Take $p \in \pi(\mathfrak{F})$. Define the set

$$\psi(p) = \{G \in \mathfrak{F} \mid G = 0^{p'}(G)\}.$$

Let $(\mathrm{SFit}\,\psi(p))\mathfrak{N}_p = f(p)$. If $X \in \psi(p)$ then $X = 0^p(X)$ and $X \in \mathfrak{F}$. Therefore, $X/X_{\varphi(p)} \in \mathfrak{S}_{p'}$ and $X \in \varphi(p)$. Thus, $\psi(p) \subseteq \varphi(p)$. But then $f(p) \subseteq \mathrm{SFit}\,\varphi(p) = \varphi(p)$ for all prime $p \in \pi(\mathfrak{F})$. Hence, $f \leq \varphi$, and therefore

$$\mathfrak{M} = \mathfrak{S}_{\pi(\mathfrak{F})} \cap \left(\bigcap_{p\in\pi(\mathfrak{F})} f(p)\mathfrak{N}_p\mathfrak{S}_{p'}\right) \subseteq \mathfrak{F}.$$

We now establish the reverse inclusion $\mathfrak{F} \subseteq \mathfrak{M}$. Let Y be an arbitrary group in $\mathfrak{F}$. Since $O^{p'}(O^{p'}(Y)) = O^{p'}(Y)$ and $O^{p'}(Y) \in \mathfrak{F}$ for all prime $p \in \pi(\mathfrak{F})$, it follows that Y is an extension of a $\psi(p)$-group by a p'-group for some $p \in \pi(\mathfrak{F})$. However, $\psi \leq f$. Therefore, $Y \in \bigcap_{p \in \pi(\mathfrak{F})} f(p)\mathfrak{N}_p\mathfrak{S}_{p'}$. Furthermore, $Y \in \mathfrak{F}$. In consequence, $Y \in \mathfrak{S}_{\pi(\mathfrak{F})}$. Hence, $Y \in \mathfrak{M}$ and $\mathfrak{M} = \mathfrak{F}$, completing the proof of the first assertion of the lemma.

If $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ then, by item (1) and the definition of the operator SFit, we have $f_1 \leq f_2$. The converse assertion is obvious. The lemma is proven.

Lemma 5. *The union of an arbitrary chain of radical classes each of which is determined by a hereditary local H-function is a radical class determined by a hereditary local H-function.*

PROOF. Let $\{\mathfrak{F}_i \mid i \in I\}$ be an arbitrary nonempty chain of radical classes each of which is determined by a hereditary local H-function. By Lemma 4, each radical class $\mathfrak{F}_i$ is determined by a unique minimal full hereditary local H-function f_i ($i \in I$). Let $\mathfrak{F} = \bigcup_{i \in I} \mathfrak{F}_i$ and $f = \bigcup_{i \in I} f_i$. We first establish that f is a local H-function. Take $G \in f(p)$ (p is a prime) and let K be a normal subgroup of G. Then $G \in f_{i_0}(p)$ for some $i_0 \in I$. Since f_{i_0} is a local H-function, it follows that $K \in f_{i_0}(p)$, and hence $K \in f(p)$. Therefore, the group class $f(p)$ is normally hereditary. Now, if $L = K_1K_2$, with K_1 and K_2 normal $f(p)$-subgroups of L, then $K_1 \in f_{i_k}(p)$ and $K_2 \in f_{i_l}(p)$ for some $i_k, i_l \in I$. Since either $f_{i_k} \leq f_{i_l}$ or $f_{i_l} \leq f_{i_k}$ and f_{i_k} and f_{i_l} are local H-functions, we see that L is either an $f_{i_k}(p)$-group or an $f_{i_l}(p)$-group. However, in each of these cases it is obvious that $L \in f(p)$. Thus, f is a local H-function. The fact that $\mathfrak{F}$ is a radical class is proved by analogy.

We verify that f locally determines $\mathfrak{F}$. Let $\mathfrak{M} = \mathrm{LR}(f)$. Since $f_i \leq f$ and $\mathfrak{F}_i = \mathrm{LR}(f_i)$ for all $i \in I$, it is obvious that $\mathfrak{F} \subseteq \mathfrak{M}$. Now, take $M \in \mathfrak{M}$. Then $M^{\mathfrak{N}_p\mathfrak{S}_{p'}} \in f(p)$ for all prime $p \in \operatorname{Supp} f$. It is easy to see that $\operatorname{Supp} f = \pi(\mathfrak{F})$. Therefore, $M^{\mathfrak{N}_p\mathfrak{S}_{p'}} \in f_{i_r}(p)$ for $i_r \in I$ and all prime $p \in \pi(\mathfrak{F})$. Thereby $M \in \bigcap_{p \in \pi(\mathfrak{F}_{i_r})} f_{i_r}(p)\mathfrak{N}_p\mathfrak{S}_{p'}$. Since $M \in \mathfrak{S}_{\pi(\mathfrak{F})}$, it follows that $M \in \mathfrak{S}_{\pi(\mathfrak{F}_{i_m})}$ for some $i_m \in I$. However, since $\mathfrak{F}_{i_r}$ and $\mathfrak{F}_{i_m}$ are elements of a chain, either $\mathfrak{F}_{i_r} \subseteq \mathfrak{F}_{i_m}$ or $\mathfrak{F}_{i_m} \subseteq \mathfrak{F}_{i_r}$. Consequently, either $\mathfrak{S}_{\pi(\mathfrak{F}_{i_r})} \subseteq \mathfrak{S}_{\pi(\mathfrak{F}_{i_m})}$ or $\mathfrak{S}_{\pi(\mathfrak{F}_{i_m})} \subseteq \mathfrak{S}_{\pi(\mathfrak{F}_{i_r})}$, and by item (2) of Lemma 4 either $f_{i_r} \leq f_{i_m}$ or $f_{i_m} \leq f_{i_r}$. In each of these cases we obtain $M \in \mathfrak{F}$. Thus, $\mathfrak{M} = \mathfrak{F}$.

It remains to demonstrate that the local H-function f is hereditary. Indeed, if $R \in f(p)$ and U is an arbitrary subgroup in R then $R \in f_{i_s}(p)$ for some $i_s \in I$. However, the local H-function f_{i_s} is hereditary. Hence, $U \in f_{i_s}(p) \subseteq f(p)$, which completes the proof of the lemma.

The next lemma is a consequence of the familiar results by Bryce and Cossey [5, 13, 14] and Hawkes [4].

Lemma 6. *If $\mathfrak{F}$ is a hereditary radical class such that $\mathfrak{F} \subseteq \mathfrak{N}^k$ for some natural k then $\mathfrak{F} = \bigcap_{i=1}^{\infty} \mathfrak{F}_i$, where $\mathfrak{F}_i$ is the finite product of radical classes $\mathfrak{S}_{\pi_1}, \mathfrak{S}_{\pi_2}, \dots, \mathfrak{S}_{\pi_n}$ for some sets $\pi_1, \pi_2, \dots, \pi_n$ of prime numbers.*

PROOF. Since the radical class $\mathfrak{F}$ is hereditary, Theorem 1.1 of [13] implies that $\mathfrak{F}$ is a formation. But then, by Theorem 4 of [5], the formation $\mathfrak{F}$ is totally local. Now, the claim is immediate from the condition $\mathfrak{F} \subseteq \mathfrak{N}^k$ in view of Lemma 2.3 of [14].

The following theorem classifies hereditary radical classes in terms of local Hartley classes and corroborates Hawkes's conjecture which is mentioned in the introduction for radical classes.

Theorem. *Let $\mathfrak{F}$ be a nonempty radical class. The following assertions hold:*

(1) *$\mathfrak{F}$ is hereditary if and only if $\mathfrak{F} = \mathrm{LR}(f)$ for the local H-function f whose values are defied by* (1);

(2) *$\mathfrak{F}$ is totally local if and only if $\mathfrak{F}$ is hereditary.*

PROOF. Let $\mathfrak{F}$ be the hereditary class determined by the local H-function f whose values are given by the formula (1). Then we have the equality $\mathfrak{F} = \mathfrak{S}_\sigma \cap \left(\bigcap_{p \in \sigma} f(p)\mathfrak{N}_p\mathfrak{S}_{p'}\right)$, where $\sigma = \operatorname{Supp} f = \pi(\mathfrak{F})$, and by Lemma 1 $\mathfrak{F}$ is a hereditary radical class.

Suppose that a nonempty radical class $\mathfrak{F}$ is hereditary. If $\mathfrak{F}$ has bounded nilpotent length, i.e., $\mathfrak{F} \subseteq \mathfrak{N}^k$ for some natural k, then in this event Lemma 6 yields $\mathfrak{F} = \bigcap_{i=1}^{\infty} \mathfrak{F}_i$, where $\mathfrak{F}_i$ is some product of radical classes $\mathfrak{S}_{\pi_1}, \mathfrak{S}_{\pi_2}, \dots, \mathfrak{S}_{\pi_n}$ for some sets $\pi_1, \pi_2, \dots, \pi_n$ of natural numbers and some natural number n. By Lemma 2, each of the products $\mathfrak{F}_i$ is determined by a hereditary local H-function f_i.

Then by Lemma 3 of [10] $\mathfrak{F}$ is local and radical and its local H-function is $\varphi = \bigcap_{i \in I} f_i$ (see the proof of Lemma 3 in [10]). Thus, $\mathfrak{F} = \mathrm{LR}(\varphi)$ and the local H-function φ is hereditary by Lemma 1. Hence, by item (1) of Lemma 4 $\mathfrak{F} = \mathrm{LR}(f)$ for the local H-function f whose values are given by the formula (1).

To prove the first assertion of the theorem, we are left with settling the case in which the nilpotent length of the class $\mathfrak{F}$ is unbounded. Let $\mathfrak{F}_i = \mathfrak{F} \cap \mathfrak{N}^i$, $i \geq 1$. Then, for each natural i, the class $\mathfrak{F}_i$ has bounded nilpotent length and is hereditary by Lemma 1. It follows that, as before, $\mathfrak{F}_i = \mathrm{LR}(f_i)$ for some hereditary H-function f_i. But then $\mathfrak{F} = \bigcup_{i=1}^{\infty} \mathfrak{F}_i$ is the union of a chain of radical classes each of which is determined by a local H-function. Hence, by Lemma 5 $\mathfrak{F} = \mathrm{LR}(\psi)$ for some hereditary local H-function ψ. But then by item (1) of Lemma 4 $\mathfrak{F} = \mathrm{LR}(f)$ and the values of the local H-function f are defined by the formula (1). The first assertion of the theorem is proven.

If $\mathfrak{F}$ is hereditary then $\mathfrak{F}$ is totally local by item (1) of the theorem. Prove the converse. To this end, in line with [4] (also see [2, VII.3]) describe the family $\mathcal{T}$ of all totally local radical classes as follows: Let $\mathcal{F}_0 = \{\varnothing, \mathfrak{G}, \mathfrak{S}\}$. For every natural $i > 0$, define the family $\mathcal{F}$ of radical classes by induction: $\mathfrak{X} \in \mathcal{F}$ if and only if $\mathfrak{X} \in \mathcal{F}_{i-1}$ or $\mathfrak{X} = \mathrm{LR}(x)$ for a local H-function x such that $x(p) \in \mathcal{F}_{i-1}$ for all prime $p \in \mathrm{Supp}\, x$. Then $\mathcal{T}$ is the family of all radical classes $\mathfrak{X}$ such that $\mathfrak{X} = \bigcup_j \mathfrak{X}_j$, where $\mathfrak{X}_j \in \bigcup_j \mathcal{F}_i$ and $\mathfrak{X}_j \subseteq \mathfrak{X}_{j+1}$.

Suppose that $\mathfrak{F} \neq \varnothing$ and $\mathfrak{F} \in \mathcal{T}$. Then $\mathfrak{F} = \bigcup_{i=1}^{\infty} \mathfrak{F}_i$, where $\mathfrak{F}_1 \subseteq \mathfrak{F}_2 \subseteq \dots$ and $\mathfrak{F}_i \in \mathcal{F}_{\infty} = \bigcup_{i=0}^{\infty} \mathcal{F}_i$. Clearly, $\mathcal{F}_0$ consists of sets of hereditary radical classes. Using induction on i and item (1) of the theorem, we easily infer that all radical classes in $\mathcal{F}_i$ are hereditary for all natural i. But then the radical class $\mathcal{F}$ too is hereditary as the union of a chain of hereditary radical classes.

The above-proven theorem shows that the family of local radical classes is rather wide: it contains all nonempty hereditary radical classes. However, not every local radical class is hereditary, as we see from the following

EXAMPLE. Suppose that $\varnothing \subset \pi \subset \mathbb{P}$ and let $\mathfrak{X}^{\pi}$ be the class of all groups G whose π-radicals are hypercentral in G. Then by [2, IX.2.5,a] $\mathfrak{X}^{\pi}$ is a nonhereditary radical class. Let $\mathfrak{F} = \mathfrak{X}^{\pi}\mathfrak{N}$. It is easy that $\mathfrak{F} = \mathrm{LR}(f)$ for a local H-function f such that $f(p) = \mathfrak{X}^{\pi}$ for all prime p. Demonstrate that the class $\mathfrak{F}$ is not hereditary. Since the radical class $\mathfrak{X}^{\pi}$ is not hereditary, there is a group $G \in \mathfrak{X}^{\pi}$ such that some subgroup H of G does not belong to $\mathfrak{X}^{\pi}$. Let p be a prime number not dividing $|G|$ and let $W = G \wr C_p$ be the regular wreath product of G with a cyclic group of order p. It is obvious that $W \in \mathfrak{F}$. If we suppose that $\mathfrak{F}$ is hereditary then $\Gamma = H \wr C_p \in \mathfrak{F}$. By Claim X.1.25 of [2] the radical class $\mathfrak{X}^{\pi}$ is a Lockett class; i.e., $(X \times X)_{\mathfrak{X}^{\pi}} = X_{\mathfrak{X}^{\pi}} \times X_{\mathfrak{X}^{\pi}}$ for every group X. Therefore, by Lemma 2.2 of [15] (also see [2, X.2.1,a]) $\Gamma_{\mathfrak{X}^{\pi}} = (H_{\mathfrak{X}^{\pi}})^*$, where $(H_{\mathfrak{X}^{\pi}})^*$ is the base of the wreath product $H_{\mathfrak{X}^{\pi}} \wr C_p$. Now, from $\Gamma \in \mathfrak{F}$ and $H_{\mathfrak{X}^{\pi}} \trianglelefteq H$, it follows that [2, A.18.2,d] $(H/H_{\mathfrak{X}^{\pi}}) \wr C_p \cong \Gamma/(H_{\mathfrak{X}^{\pi}})^* \in \mathfrak{N}$. But then the fact that p does not divide $|G|$ implies that $H = H_{\mathfrak{X}^{\pi}}$, which contradicts the choice of H. Hence, $\mathfrak{F}$ is nonhereditary local radical class.

Observe that the condition of solubility of the radical class $\mathfrak{F}$ in the theorem is essential, since by L. A. Shemetkov and A. F. Vasil′ev's results [16] there exist nonsoluble hereditary radical classes which are nonlocal.

The proven theorem implies in particular that the main result of [10] claiming that every nonempty soluble hereditary radical class is local fails in general for soluble formations.

References

1. L. A. Shemetkov and A. N. Skiba, Formations of Algebraic Systems [in Russian], Nauka, Moscow (1989).
2. K. Doerk and T. Hawkes, Finite Soluble Groups, Walter de Gruyter, Berlin and New York (1992).
3. A. N. Skiba, "Multiply local formations," in: Abstracts: The Tenth All-Union Symposium on Group Theory, Minsk, 1986, p. 211.
4. T. O. Hawkes, "Sceletal classes of soluble groups," Arch. Math., **22**, No. 6, 577–589 (1971).
5. R. A. Bryce and J. Cossey, "Fitting formations of finite soluble groups," Math. Z., **127**, No. 3, 217–223 (1972).
6. B. Hartley, "On Fisher's dualization of formation theory," Proc. London Math. Soc., **3**, No. 9, 103–115 (1969).

7. P. D'Arcy, "Locally defined Fitting classes," J. Austral. Math. Soc., **20**, No. 1, 25–32 (1975).
8. L. A. Shemetkov, Formations of Finite Groups [in Russian], Nauka, Moscow (1978).
9. N. T. Vorob'ëv, "Local products of Fitting classes," Vestsi Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk, No. 6, 22–26 (1991).
10. N. T. Vorob'ëv, "Localization of soluble hereditary Fitting classes," Mat. Zametki, **51**, No. 3, 3–8 (1992).
11. N. T. Vorob'ëv, "On radical classes of finite groups with Lockett's condition," Mat. Zametki, **43**, No. 2, 161–168 (1988).
12. A. Grytczuk and N. T. Vorob'ëv, "On Lockett's conjecture for finite groups," Tsukuba J. Math., **18**, No. 2, 63–67 (1994).
13. R. A. Bryce and J. Cossey, "Subgroup closed Fitting classes are formations," Math. Proc. Cambridge Philos. Soc., **91**, No. 2, 225–258 (1982).
14. R. A. Bryce and J. Cossey, "A problem in the theory of normal Fitting classes," Math. Z., **141**, No. 2, 99–110 (1975).
15. J. Cossey, "Products of Fitting classes," Math. Z., **141**, No. 3, 289–295 (1975).
16. L. A. Shemetkov and A. F. Vasil'ev, "On finite groups with prescribed Hall's subgroups," in: Abstracts: The Sixth Conference of the Mathematicians of Belarus, Grodno, 1992, Part I, p. 56.

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