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ON THE MODULARITY OF A LATTICE OF τ-CLOSED n-MULTIPLY ω-COMPOSITE FORMATIONS

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Let $n \geq 0$, let ω be a nonempty set of prime numbers and let τ be a subgroup functor (in Skiba's sense) such that all subgroups of any finite group G contained in $\tau(G)$ are subnormal in G. It is shown that the lattice of all τ-closed n-multiply ω-composite formations is algebraic and modular.

Introduction

In [1], it is shown that the lattice of all (saturated) formations is modular. Later, this result was developed in various directions. Thus, in [2], Shemetkov and Skiba proved the modularity of the lattice of all n-multiply saturated formations. In [3], Ballester-Bolinches and Shemetkov proved modularity of the lattice of all ω-saturated formations. The modularity of the lattice of all τ-closed n-multiply saturated formations was established by Skiba in [4]. Later, in [5, 6], Skiba and Shemetkov proved the modularity of the lattices of n-multiply ω-saturated formations and n-multiply $\mathfrak{L}$-composite formations, respectively. In [7], Shabalina established the modularity of the lattice of all τ-closed n-multiply ω-saturated formations. In [8], Zadorozhnyuk proved the modularity of the lattice of all τ-closed ω-composite formations. The modularity of the lattice of all totally saturated formations and the lattice of all τ-closed totally saturated formations was proved by Safonov in [9, 10].

In the present paper, the functor approach is applied to develop the methods of the theory of modular lattices of partially composite formations. It is shown that the lattice of all τ-closed n-multiply ω-composite formations is algebraic and modular (Theorem 3.1). Moreover, the inductance of the indicated lattice is established (Theorem 2.1). Note that all results established for the modular lattices of formations and mentioned above are special cases of Theorem 3.1 (see Corollaries 3.1–3.4).

We use the standard terminology accepted in [2, 4–6, 11, 12]. All groups studied in the present paper are finite.

1. Preliminary Data

Recall that a formation is defined as a class of groups closed relative to the homomorphic images and finite subdirect products.

In what follows, ω denotes a nonempty set of prime numbers and $\omega' = \mathbb{P} \setminus \omega$. By $\pi(G)$ we denote the set of all different prime divisors of the order of the group G. Thus, $\pi(\mathfrak{X})$ is the union of the sets $\pi(G)$ for all groups G from $\mathfrak{X}$, $[K]H$ is the semidirect product of the group K by a certain group of its operators H, and $A \wr B$ is the regular interlacing of the group A with the group B. For any class of groups $\mathfrak{F} \supseteq (1)$, by $G^{\mathfrak{F}}$

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we denote the intersection of all normal subgroups N such that $G/N \in \mathfrak{F}$ and by $G_{\mathfrak{F}}$ we denote the product of all normal $\mathfrak{F}$-subgroups of the group G. In particular, $O_p(G) = G_{\mathfrak{N}_p}$ and $F_p(G) = G_{\mathfrak{G}_{p'}\mathfrak{N}_p}$. By $\mathfrak{N}_p$, $\mathfrak{G}$, $\mathfrak{G}_{p'}$, and $\mathfrak{G}_{cp}$ we denote the class of all p-groups, the class of all groups, the class of all p'-groups, and the class of all groups for which all chief p-factors are central, respectively.

Any function of the form

$$f\colon \omega \cup \{\omega'\} \to \{\text{formations of groups}\}$$

is called an ω-composite satellite [6]. As in [11], by $C^p(G)$ we denote the intersection of centralizers of all chief factors of the group G whose composition factors have a prime order p ($C^p(G) = G$ if G does not contain chief factors with this property). By $R_\omega(G)$ we denote the maximum normal soluble ω-subgroup of the group G and $\mathrm{Com}(G)$ stands for the class of all simple Abelian groups A such that $A \cong H/K$ for a certain composition factor H/K of the group G. According to [6], any ω-composite satellite f is associated with a class of groups

$$CF_\omega(f) = (G \mid G/R_\omega(G) \in f(\omega') \quad \text{and} \quad G/C^p(G) \in f(p) \quad \text{for all} \quad p \in \omega \cap \pi(\mathrm{Com}(G))).$$

If a formation $\mathfrak{F}$ is such that $\mathfrak{F} = CF_\omega(f)$ for some ω-composite satellite f, then we say that it is ω-composite and f is an ω-composite satellite of this formation [6]. Moreover, if all values of f lie in $\mathfrak{F}$, then the satellite f is called *internal*.

According to the concept of multiple localization proposed by Skiba (see [13, 6]), any formation is regarded as 0-multiply ω-composite. Moreover, for $n > 0$, the formation $\mathfrak{F}$ is called n-multiply ω-composite if $\mathfrak{F} = CF_\omega(f)$, where all nonempty values of the function f are $(n-1)$-multiply ω-composite formations.

Let Θ be a complete lattice of formations. By $\Theta\mathrm{form}\,\mathfrak{X}$ we denote the intersection of all formations from Θ containing a collection of groups $\mathfrak{X}$. Thus, in particular, we write $\Theta\mathrm{form}\,G$ in the case where $\mathfrak{X} = \{G\}$. Any formation of this type is called a one-generated formation from Θ. The sign Θ is omitted if Θ is the collection of all formations. Recall that a satellite f is called Θ-*valued* if all its values belong to Θ. Following [6], we denote the complete lattice of formations with Θ-valued ω-composite satellites by Θ^{ω_c}.

For any collection of groups $\mathfrak{X}$, we set (see [6]):

$$\mathfrak{X}(C^p) = \begin{cases} \mathrm{form}\,(G/C^p(G) \mid G \in \mathfrak{X}) & \text{for} \quad p \in \pi(\mathrm{Com}(\mathfrak{X})), \\ \varnothing & \text{for} \quad p \notin \pi(\mathrm{Com}(\mathfrak{X})). \end{cases}$$

Let $\mathfrak{F} = CF_\omega(F)$, where $F(\omega') = \mathfrak{F}$ and $F(p) = \mathfrak{N}_p\mathfrak{F}(C^p)$ for all $p \in \omega$. Then the satellite F is called a canonical ω-*composite satellite* [6].

Recall several well-known assertions required for the proof of the principal result of the present paper.

Lemma 1.1 ([6], Lemma 8). *Let Θ be a complete lattice of formations such that $\Theta^{\omega_c} \subseteq \Theta$ and let the formation $\mathfrak{N}_p\mathfrak{H} \in \Theta$ for any formation $\mathfrak{H} \in \Theta$ and any $p \in \omega$.*

In this case, if $\mathfrak{F} = CF_\omega(F) \in \Theta^{\omega_c}$, then the satellite F is Θ-valued.

Lemma 1.2 ([6], Lemma 4). *If* $\mathfrak{F} = CF_{\omega}(f)$ *and* $G/O_p(G) \in f(p) \cap \mathfrak{F}$ *for some* $p \in \omega$, *then* $G \in \mathfrak{F}$.

Lemma 1.3 ([6], Remark 1). *Any* ω-*composite formation possesses a canonical* ω-*composite satellite.*

In an arbitrary group G, we select a system of subgroups $\tau(G)$. We say that τ is a subgroup functor (in Skiba's sense [4]) if the following conditions are satisfied:

(i) $G \in \tau(G)$;

(ii) the inclusions $H^{\varphi} \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$ hold for any epimorphism $\varphi : A \mapsto B$ and all groups $H \in \tau(A)$ and $T \in \tau(B)$.

If $\tau(G) = \{G\}$, then the functor τ is called trivial. A formation $\mathfrak{F}$ is called τ-closed [4] if $\tau(G) \subseteq \mathfrak{F}$ for any group G from $\mathfrak{F}$. In what follows, we consider only subgroup functors τ such that, for any group G, all its subgroups contained in $\tau(G)$ are subnormal in G.

The collection of all τ-closed n-multiply ω-composite formations $c^{\tau}_{\omega_n}$ is a complete lattice relative to the inclusion $\subseteq$. By $c^{\tau}_{\omega_0}$ and c^{ω}_{n} we denote the lattice of all τ-closed formations and the lattice of all n-multiply ω-composite formations, respectively. Note that if τ is a trivial subgroup functor, then $c^{\tau}_{\omega_n} = c^{\omega}_{n}$.

Let $\{f_i \mid i \in I\}$ be the set of all ω-composite $c^{\tau}_{\omega_{n-1}}$-valued satellites of the formation $\mathfrak{F}$. By virtue of Lemma 2 in [6],

$$f = \bigcap_{i \in I} f_i$$

is an ω-composite $c^{\tau}_{\omega_{n-1}}$-valued satellite of the formation $\mathfrak{F}$. It is called the *minimum* satellite of the formation.

The next statement gives a method for the construction of the minimum $c^{\tau}_{\omega_{n-1}}$-valued satellite of the formation $\mathfrak{F} = c^{\tau}_{\omega_n} \operatorname{form} \mathfrak{X}$.

Lemma 1.4 ([6], Lemma 5). *Let* $\mathfrak{F} = c^{\tau}_{\omega_n} \operatorname{form} \mathfrak{X}$ *and let* $\pi = \omega \cap \pi(\mathrm{Com}(\mathfrak{X}))$. *Then the minimum* ω-*composite* $c^{\tau}_{\omega_{n-1}}$-*valued satellite* f *of the formation* $\mathfrak{F}$ *is such that*

(i) $f(\omega') = c^{\tau}_{\omega_{n-1}} \operatorname{form}\, (G/R_{\omega}(G) \mid G \in \mathfrak{X})$;

(ii) $f(p) = c^{\tau}_{\omega_{n-1}} \operatorname{form}\, (G/C^{p}(G) \mid G \in \mathfrak{X})$ *for all* $p \in \pi$;

(iii) $f(p) = \varnothing$ *for all* $p \in \omega \setminus \pi$;

(iv) if $\mathfrak{F} = CF_\omega(h)$ *and the satellite* h *is* $c^\tau_{\omega_{n-1}}$*-valued, then, for all* $p \in \pi$,

$$f(p) = c^\tau_{\omega_{n-1}} \operatorname{form}\,(A \mid A \in h(p) \cap \mathfrak{F},\ O_p(A) = 1)$$

and, in addition,

$$f(\omega') = c^\tau_{\omega_{n-1}} \operatorname{form}\,(A \mid A \in h(\omega') \cap \mathfrak{F},\ R_\omega(A) = 1).$$

Lemma 1.5 ([4], Corollary 1.2.24)**.** *For any collection of* τ*-closed formations* $\{\mathfrak{M}_i \mid i \in I\}$,

$$\tau\operatorname{form}\left(\bigcup_{i\in I}\mathfrak{M}_i\right) = \operatorname{form}\left(\bigcup_{i\in I}\mathfrak{M}_i\right).$$

Lemma 1.6 ([12], Theorem 2.2)**.** *The following equality is true for any class* $\mathfrak{X}$:

$$\operatorname{form}\mathfrak{X} = \mathrm{QR}_0\,\mathfrak{X}$$

We set $f \le h$ if $f(a) \subseteq h(a)$ for all $a \in \omega \cup \{\omega'\}$.

Lemma 1.7 ([6], Lemma 6)**.** *Let* f_1 *and* f_2 *be the minimum* ω*-composite* $c^\tau_{\omega_{n-1}}$*-valued satellites of the formations* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$*, respectively. Then* $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ *if and only if* $f_1 \le f_2$.

Lemma 1.8 ([6], Lemma 10)**.** *The formation* $\mathfrak{F}$ *is n-multiply* ω*-composite if and only if it has a satellite* f *all values of which* $f(a)$ *are* $(n-1)$*-multiply* ω*-composite for all* $a \in \omega$.

Lemma 1.9 ([6], Lemma 2)**.** *Let* $\mathfrak{F} = \bigcap_{i\in I}\mathfrak{F}_i$*, where* $\mathfrak{F}_i = CF_\omega(f_i)$*. Then* $\mathfrak{F} = CF_\omega(f)$*, where* $f = \bigcap_{i\in I} f_i$.

2. Inductance of the Lattice $c^\tau_{\omega_n}$

We now recall the definition of inductive lattice of formations. Let Θ be a complete lattice of formations. For any collection of formations $\{\mathfrak{F}_i \mid i \in I\}$ from Θ, we set

$$\vee_\Theta(\mathfrak{F}_i \mid i \in I) = \Theta\operatorname{form}\left(\bigcup_{i\in I}\mathfrak{F}_i\right).$$

Let $\{f_i \mid i \in I\}$ be a collection of Θ-valued satellites. Then $\vee_\Theta(f_i \mid i \in I)$ denotes a satellite f such that

$$f(a) = \Theta\text{form}\left(\bigcup_{i\in I} f_i(a)\right)$$

for any $a \in \omega \cup \{\omega'\}$.

As in [4], we say that the complete lattice of formations Θ is *inductive* if the following equality holds for any collection $\{\mathfrak{F}_i \mid i \in I\}$ of formations $\mathfrak{F}_i \in \Theta^{\omega_c}$ and any collection $\{f_i \mid i \in I\}$ of internal Θ-valued ω-composite satellites f_i, where f_i is an ω-composite satellite of the formation $\mathfrak{F}_i$:

$$\vee_{\Theta^{\omega_c}}(\mathfrak{F}_i \mid i \in I) = CF_\omega(\vee_\Theta(f_i \mid i \in I)).$$

In the present section, we prove the property of inductance of the lattice of all τ-closed n-multiply ω-composite formations. The proof of the property of modularity of this lattice is based on the property of inductance.

Lemma 2.1. *Let $\mathfrak{F} = CF_\omega(F)$ be a τ-closed n-multiply ω-composite formation and let $n \geq 1$. Then the satellite F is $c^\tau_{\omega_{n-1}}$-valued.*

Proof. According to Lemma 1.1, it suffices to check the fact that, for any $p \in \mathbb{P}$ and any τ-closed n-multiply ω-composite formation $\mathfrak{H}$ $(n \geq 0)$, $\mathfrak{M} = \mathfrak{N}_p\mathfrak{H}$ is a τ-closed n-multiply ω-composite formation.

Note that the formation $\mathfrak{N}_p\mathfrak{H}$, where $\mathfrak{H}$ is a τ-closed formation, is τ-closed for any $p \in \mathbb{P}$. Thus, the required assertion is true for $n = 0$.

Now let $n > 0$. Assume that the assertion of the lemma holds for $n - 1$. First, we show that $\mathfrak{M}$ is an n-multiply ω-composite formation. Let $\mathfrak{H} = CF_\omega(h)$, where h is an internal $c^\tau_{\omega_{n-1}}$-valued ω-composite satellite. The formation $\mathfrak{N}_p$ possesses an internal ω-composite satellite f such that $f(p) = (1)$, $f(\omega') = (1)$, and $f(q) = \varnothing$ for all $q \in \omega\setminus\{p\}$. It is easy to see that the formation $\mathfrak{M}$ possesses a satellite m such that $m(p) = \mathfrak{H}$, $m(\omega') = \mathfrak{M}$, and $m(q) = \varnothing$ for all $q \in \omega\setminus\{p\}$ (see [6], Theorem 6). However, according to the assumption, $\mathfrak{M} = \mathfrak{N}_p\mathfrak{H}$ is an $(n-1)$-multiply ω-composite formation. Hence, $\mathfrak{M}$ is an n-multiply ω-composite formation.

We now prove that the formation $\mathfrak{M}$ is τ-closed. Assume the contrary. Then there exist a group $G \in \mathfrak{M}$ and a subgroup $H \in \tau(G)$ such that $H \notin \mathfrak{M}$. Since $G \in \mathfrak{N}_p\mathfrak{H} = \mathfrak{M}$, we have $G^{\mathfrak{H}} \in \mathfrak{N}_p$ and $G/G^{\mathfrak{H}} \in \mathfrak{H}$. By the assumption, the formation $\mathfrak{H}$ is τ-closed. Thus, for any group $\bar{H} \in \tau(G/G^{\mathfrak{H}})$, we conclude that $\bar{H} \in \mathfrak{H}$. However, $HG^{\mathfrak{H}}/G^{\mathfrak{H}} \in \tau(G/G^{\mathfrak{H}})$ and, therefore,

$$HG^{\mathfrak{H}}/G^{\mathfrak{H}} \cong H/H \cap G^{\mathfrak{H}} \in \mathfrak{H}.$$

At the same time, $H \cap G^{\mathfrak{H}} \lhd H$ and $H \cap G^{\mathfrak{H}}$ is a p-group. Therefore, $H \cap G^{\mathfrak{H}} \subseteq O_p(H)$ and, thus, $H^{\mathfrak{H}} \subseteq O_p(H)$, i.e., $H^{\mathfrak{H}} \in \mathfrak{N}_p$. This yields $H \in \mathfrak{N}_p\mathfrak{H} = \mathfrak{M}$, which is a contradiction. Hence, the formation $\mathfrak{M}$ is τ-closed.

Let F be a canonical c_{n-1}^{ω}-valued ω-composite satellite of a τ-closed n-multiply ω-composite formation $\mathfrak{F}$. We show that the formation $F(a)$ is τ-closed. If $a = \{\omega'\}$, then the formation $F(\omega') = \mathfrak{F}$ is τ-closed by the assumption.

Assume that $a = p \in \omega$. We consider a group $G \in F(p)$ and $H \in \tau(G)$. Let P be a nonidentity group and let $D = P \wr G = [K]G$, where K is a basis of the regular interlacing of D. Then $HK \in \tau(D)$. Indeed, let $\varphi : D \to D/K$ be a canonical epimorphism of the group D on D/K. Then $HK/K = H^{\varphi}$. Therefore, $HK/K \in \tau(D/K)$. Since $HK = (HK/K)^{\varphi^{-1}}$ is the complete preimage of the subgroups HK/K under the epimorphism φ, we conclude that $HK \in \tau(D)$.

Since the satellite F is internal and $G \cong D/K \cong D/O_p(D) \in \in F(p)$, by virtue of Lemma 1.2, we conclude that $D \in \mathfrak{F}$. Since the formation $\mathfrak{F}$ is τ-closed, we have $HK \in \mathfrak{F}$. Let $M = HK$. Then $M/C^p(M) \in F(p)$, where $p \in \pi(\mathrm{Com}(M))$. Since K is a normal p-subgroup of the group M, we find $K \cap O_{p'}(M) = 1$. Hence, $O_{p'}(M) \subseteq C_M(K)$. According to the property of regular interlacings, $C_G(K) \subseteq K$. Therefore, $O_{p'}(M) = 1$ and, thus, $O_p(M) = F_p(M) = C^p(M)$.

In view of the facts that

$$O_p(M) = O_p(M) \cap M = O_p(M) \cap KH = K(O_p(M) \cap H)$$

and $O_p(M) \cap H \subseteq O_p(H)$, we get

$$O_p(M) = K(O_p(M) \cap H) \subseteq KO_p(H) \subseteq O_p(M).$$

Therefore, $KO_p(H) = O_p(M)$. This yields

$$M/C^p(M) = KH/O_p(M) = KH/KO_p(H) \cong H/O_p(H)(K \cap H)$$

$$= H/O_p(H) \in F(p) = \mathfrak{N}_p F(p),$$

i.e., $H \in \mathfrak{N}_p(\mathfrak{N}_p F(p)) = (\mathfrak{N}_p \mathfrak{N}_p) F(p) = \mathfrak{N}_p F(p) = F(p)$. Hence, the formation $F(p)$ is τ-closed.

The lemma is proved.

In the case where $\Theta = c_{\omega_n}^{\tau}$, we write $\vee_{\omega_n}^{\tau}$ instead of $\vee_{c_{\omega_n}^{\tau}}$.

The following lemma gives the description of a satellite of the lattice union of two τ-closed n-multiply ω-composite formations:

Lemma 2.2. *Let $\mathfrak{F}_i = CF_{\omega}(f_i)$, where f_i is the internal $c_{\omega_{n-1}}^{\tau}$-valued ω-composite satellite of the formation $\mathfrak{F}_i$. Moreover, let $f_i(\omega') = \mathfrak{F}_i$, $i = 1, 2$. In this case, if $\mathfrak{F} = \mathfrak{F}_1 \vee_{\omega_n}^{\tau} \mathfrak{F}_2$, then $\mathfrak{F} = CF_{\omega}(f)$, where $f = f_1 \vee_{\omega_{n-1}}^{\tau} f_2$.*

Proof. Let h_i be the minimum $c^{\tau}_{\omega_{n-1}}$-valued ω-composite satellite of the formation $\mathfrak{F}_i = CF_{\omega}(f_i)$, $i = 1, 2$, and let $p \in \omega$. By virtue of Lemmas 2.1 and 3.1, the following inclusion is true:

$$h_i(p) \subseteq f_i(p) \subseteq \mathfrak{N}_p h_i(p) = F_i(p) \in c^{\tau}_{\omega_{n-1}},$$

where F_i is the canonical $c^{\tau}_{\omega_{n-1}}$-valued ω-composite satellite of the formation $\mathfrak{F}_i$, $i = 1, 2$.

Let $\mathfrak{F} = CF_{\omega}(F)$, where F is the canonical $c^{\tau}_{\omega_{n-1}}$-valued ω-composite satellite of the formation $\mathfrak{F}$ and let h be the minimum $c^{\tau}_{\omega_{n-1}}$-composite satellite of the formation $\mathfrak{F}$. By Lemma 1.4, we have

$$h(p) = c^{\tau}_{\omega_{n-1}} \text{form}\,((\mathfrak{F}_1 \cup \mathfrak{F}_2)(C^p)) = c^{\tau}_{\omega_{n-1}} \text{form}\,(\mathfrak{F}_1(C^p) \cup \mathfrak{F}_2(C^p))$$

$$= c^{\tau}_{\omega_{n-1}} \text{form}\,(h_1(p) \cup h_2(p)) \subseteq f(p) \subseteq \mathfrak{N}_p c^{\tau}_{\omega_{n-1}} \text{form}\,(h_1(p) \cup h_2(p)) \subseteq \mathfrak{N}_p h(p) = F(p).$$

Thus, $h(p) \subseteq f(p) \subseteq F(p)$ for all $p \in \omega$. It is clear that $h(\omega') \subseteq f(\omega') \subseteq F(\omega')$ and, hence, $h(a) \subseteq f(a) \subseteq F(a)$ for all $a \in \omega \cup \{\omega'\}$. Therefore, $h \le f \le F$. This means that $\mathfrak{F} = CF_{\omega}(f)$.

The lemma is proved.

Theorem 2.1. *The lattice of all* τ-*closed* n-*multiply* ω-*composite formations is inductive.*

Proof. Let $\{\mathfrak{F}_i \mid i \in I\}$ be an arbitrary collection of τ-closed n-multiply ω-composite formations and let f_i be an internal $c^{\tau}_{\omega_{n-1}}$-valued ω-composite satellite of the formation $\mathfrak{F}_i$. By induction on i, we now prove the equality

$$\vee^{\tau}_{\omega_n} (\mathfrak{F}_i \mid i \in I) = CF_{\omega}(\vee^{\tau}_{\omega_{n-1}} (f_i \mid i \in I)).$$

If $i = 2$, then the theorem is true by virtue of Lemma 2.2.

Assume that $i > 2$ and that the theorem is true for $i = r - 1$. Then

$$\mathfrak{F}_1 \vee^{\tau}_{\omega_n} \ldots \vee^{\tau}_{\omega_n} \mathfrak{F}_{r-1} = CF_{\omega}(f_1 \vee^{\tau}_{\omega_{n-1}} \ldots \vee^{\tau}_{\omega_{n-1}} f_{r-1}).$$

At the same time, by Lemma 2.2,

$$\mathfrak{F} = \mathfrak{F}_1 \vee^{\tau}_{\omega_n} \ldots \vee^{\tau}_{\omega_n} \mathfrak{F}_r = c^{\tau}_{\omega_n} \text{form}\,((\mathfrak{F}_1 \vee^{\tau}_{\omega_n} \ldots \vee^{\tau}_{\omega_n} \mathfrak{F}_{r-1}) \cup \mathfrak{F}_r) = CF_{\omega}(f),$$

where

$$f(a) = c^{\tau}_{\omega_{n-1}} \text{form} \left(\left(f_1(a) \vee^{\tau}_{\omega_{n-1}} \ldots \vee^{\tau}_{\omega_{n-1}} f_{r-1}(a) \right) \bigcup f_r(a) \right)$$

$$= f_1(a) \vee^{\tau}_{\omega_{n-1}} \ldots \vee^{\tau}_{\omega_{n-1}} f_r(a) = \left(f_1 \vee^{\tau}_{\omega_{n-1}} \ldots \vee^{\tau}_{\omega_{n-1}} f_r \right)(a)$$

for any $a \in \omega \bigcup \{\omega'\}$. Therefore, $f = f_1 \vee^{\tau}_{\omega_{n-1}} \ldots \vee^{\tau}_{\omega_{n-1}} f_r$. In view of the arbitrariness of the choice of r, the theorem is proved.

3. Main Result

To prove our main result, we need two auxiliary assertions establishing the fact that any formation with internal $c^{\tau}_{\omega_{n-1}}$-valued ω-composite satellite is τ-closed and the fact that the lattice $c^{\tau}_{\omega_n}$ is a complete sublattice of the lattice c^{ω}_{n}.

Lemma 3.1. *Let $\mathfrak{F}$ be an n-multiply ω-composite formation. If $\mathfrak{F}$ has an internal $c^{\tau}_{\omega_{n-1}}$-valued ω-composite satellite, then $\mathfrak{F}$ is a τ-closed formation.*

Proof. Let $\mathfrak{F}$ be an n-multiply ω-composite formation with internal $c^{\tau}_{\omega_{n-1}}$-valued ω-composite satellite f. We now show that $\mathfrak{F}$ is τ-closed.

Let $G \in \mathfrak{F}$ and let $H \in \tau(G)$. Since, for any $a \in \omega \bigcup \{\omega'\}$, the formation $f(a)$ is τ-closed, we have

$$H / R_{\omega}(H) = H / \left(R_{\omega}(G) \bigcap H \right) \cong H R_{\omega}(G) / R_{\omega}(G) \in \tau\left(G / R_{\omega}(G) \right).$$

Further, since $G \in \mathfrak{F}$, we get $G / R_{\omega}(G) \in f(\omega')$. Therefore, $H / R_{\omega}(H) \in f(\omega')$.

Now let $p \in \omega \bigcap \pi(\mathrm{Com}(H))$. Then $C^p(G) = G_{\mathfrak{G}_{cp}}$. By virtue of the restriction imposed on the subgroup functor τ, we conclude that $H \lhd\lhd G$. Then $H_{\mathfrak{G}_{cp}} = G_{\mathfrak{G}_{cp}} \bigcap H$. Therefore,

$$H / H_{\mathfrak{G}_{cp}} = H / \left(G_{\mathfrak{G}_{cp}} \bigcap H \right) \cong H G_{\mathfrak{G}_{cp}} / G_{\mathfrak{G}_{cp}} \in \tau\left(G / G_{\mathfrak{G}_{cp}} \right) = \tau\left(G / C^p(G) \right).$$

Since $G \in \mathfrak{F}$, we get $G / C^p(G) \in f(p)$. Hence, $H / C^p(H) = H / H_{\mathfrak{G}_{cp}} \in f(p)$. This yields $H / R_{\omega}(H) \in f(\omega')$ and, for any $p \in \omega \bigcap \pi(\mathrm{Com}(H))$, we find $H / C^p(H) \in f(p)$. Thus, $H \in \mathfrak{F}$, i.e., the formation $\mathfrak{F}$ is τ-closed.

The lemma is proved.

Lemma 3.2. *The lattice $c^{\tau}_{\omega_n}$ is a complete sublattice of the lattice c^{ω}_{n}.*

Proof. We prove the lemma by induction on n. According to Lemma 1.5, we have

$$c^{\tau}_{\omega_0}\text{form}\left(\bigcup_{i\in I}\mathfrak{M}_i\right) = \tau\,\text{form}\left(\bigcup_{i\in I}\mathfrak{M}_i\right) = \text{form}\left(\bigcup_{i\in I}\mathfrak{M}_i\right) = c^{\omega}_0\text{form}\left(\bigcup_{i\in I}\mathfrak{M}_i\right).$$

Hence, for $n = 0$, the lemma is true.

Assume that $n > 0$ and that the assertion of the lemma is true for $n-1$. Let $\{\mathfrak{M}_i \mid i \in I\}$ be an arbitrary collection of τ-closed n-multiply ω-composite formations and let m_i be the minimum $c^{\tau}_{\omega_{n-1}}$-valued ω-composite satellite of the formation $\mathfrak{M}_i$. In Theorem 2.1, the role of τ is now played by a trivial subgroup functor . Then

$$\vee^{\omega}_n(\mathfrak{M}_i \mid i \in I) = CF_{\omega}(\vee^{\omega}_{n-1}(m_i \mid i \in I)).$$

However, by the assumption, for any $p \in \omega \cap \pi\left(\text{Com}\left(\bigcup_{i\in I}\mathfrak{M}_i\right)\right)$, the formations

$$\vee^{\omega}_{n-1}(m_i \mid i \in I)(p) \quad \text{and} \quad (\vee^{\omega}_{n-1}(m_i \mid i \in I))(\omega')$$

are τ-closed. Hence, by virtue Lemma 3.1, the formation $\vee^{\omega}_n(\mathfrak{M}_i \mid i \in I)$ is τ-closed.

The lemma is thus proved.

The following theorem is the main result of the present paper:

Theorem 3.1. *The lattice of all* τ*-closed* n*-multiply* ω*-composite formations is algebraic and modular.*

Proof. First, we prove that the lattice $c^{\tau}_{\omega_n}$ is algebraic. Note that any τ-closed n-multiply ω-composite formation is a union of its one-generated τ-closed n-multiply ω-composite subformations in the lattice $c^{\tau}_{\omega_n}$. By induction on n, we show that each one-generated τ-closed n-multiply ω-composite formation $\mathfrak{F}$ is a compact element in the lattice $c^{\tau}_{\omega_n}$. Let

$$\mathfrak{F} = c^{\tau}_{\omega_n}\text{form}\, G \subseteq \mathfrak{M} = c^{\tau}_{\omega_n}\text{form}\left(\bigcup_{i\in I}\mathfrak{F}_i\right),$$

where $\mathfrak{F}_i$ is a τ-closed n-multiply ω-composite formation. Further, let $n = 0$. Then, by virtue of Lemmas 1.5 and 1.6,

$$G \in c^{\tau}_{\omega_0}\text{form}\left(\bigcup_{i\in I}\mathfrak{F}_i\right) = \text{form}\left(\bigcup_{i\in I}\mathfrak{F}_i\right) = \text{QR}_0\left(\bigcup_{i\in I}\mathfrak{F}_i\right).$$

Hence, $G \cong T/N$ for a certain group $T \in \mathrm{R}_0(\bigcup_{i\in I} \mathfrak{F}_i)$. Thus, there exist subscripts $i_1, \ldots, i_k \in I$ such that $T \in \mathrm{R}_0(\mathfrak{F}_{i_1} \cup \ldots \cup \mathfrak{F}_{i_k})$. Therefore, $G \in \mathrm{form}\,(\mathfrak{F}_{i_1} \cup \ldots \cup \mathfrak{F}_{i_k})$. Finally, by virtue of Lemma 1.5, we get

$$\mathfrak{F} \subseteq \mathrm{form}\left(\mathfrak{F}_{i_1} \cup \ldots \cup \mathfrak{F}_{i_k}\right) = \tau\,\mathrm{form}\left(\mathfrak{F}_{i_1} \cup \ldots \cup \mathfrak{F}_{i_k}\right).$$

Now let $n > 0$. Assume that the one-generated formations from $c^{\tau}_{\omega_{n-1}}$ are compact elements in the lattice $c^{\tau}_{\omega_{n-1}}$. Further, let f_i be the minimum $c^{\tau}_{\omega_{n-1}}$-valued ω-composite satellite of the formation $\mathfrak{F}_i$, let f be the minimum $c^{\tau}_{\omega_{n-1}}$-valued ω-composite satellite of the formation $\mathfrak{F}$, and let m be the minimum $c^{\tau}_{\omega_{n-1}}$-valued ω-composite satellite of the formation $\mathfrak{M}$. By virtue of Lemma 1.4,

$$f(a) = \begin{cases} c^{\tau}_{\omega_{n-1}}\,\mathrm{form}\,(G/C^{a}(G)) & \text{for } a = \pi, \\ \varnothing & \text{for } a \in \omega \setminus \pi, \\ c^{\tau}_{\omega_{n-1}}\,\mathrm{form}\,(G/R_{\omega}(G)) & \text{for } a = \{\omega'\}, \end{cases}$$

where $\pi = \omega \cap \pi(\mathrm{Com}\,(G))$.

It follows from Lemma 1.7 that $f \le m$. In view of Theorem 2.1, we have $m = \vee^{\tau}_{\omega_{n-1}} (f_i \mid i \in I)$. Thus, for any $p \in \omega \cap \pi(\mathrm{Com}\,(G))$, there exist subscripts $i_1, \ldots, i_r \in I$ such that

$$G/C^{p}(G) \in f_{i_1}(p) \vee^{\tau}_{\omega_{n-1}} \cdots \vee^{\tau}_{\omega_{n-1}} f_{i_r}(p).$$

Since $\pi(\mathrm{Com}\,(G))$ is a finite set, one can find subscripts $j_1, \ldots, j_s \in I$ such that $G \in \mathfrak{F}_{j_1} \vee^{\tau}_{\omega_n} \cdots \vee^{\tau}_{\omega_n} \mathfrak{F}_{j_s}$ and, hence, $\mathfrak{F} \subseteq \mathfrak{F}_{j_1} \vee^{\tau}_{\omega_n} \cdots \vee^{\tau}_{\omega_n} \mathfrak{F}_{j_s}$. This means that the lattice $c^{\tau}_{\omega_n}$ is algebraic and, therefore, the one-generated τ-closed *n*-multiply ω-composite formations are compact elements of this lattice.

We now prove the second assertion of the theorem. By induction on n, we show that the following identity is true for any τ-closed *n*-multiply ω-composite formations $\mathfrak{X}$, $\mathfrak{H}$, and $\mathfrak{F}$ such that $\mathfrak{X} \subseteq \mathfrak{H}$:

$$\mathfrak{H} \cap (\mathfrak{X} \vee^{\tau}_{\omega_n} \mathfrak{F}) = \mathfrak{X} \vee^{\tau}_{\omega_n} (\mathfrak{H} \cap \mathfrak{F}).$$

In view of the modularity of the lattice of all formations (see [1]), for $n = 0$, the assertion of the theorem is true for the trivial subgroup functor τ. Therefore, the lattice $c^{\omega}_0 = c_0$ is modular. According to Lemma 3.2, the lattice $c^{\tau}_{\omega_0}$ is a sublattice of c^{ω}_0. Hence, the lattice $c^{\tau}_{\omega_0}$ is modular.

Assume that $n > 0$ and that the second assertion of the theorem is true for $n - 1$. Further, let $\mathfrak{F}_i = CF_{\omega}(F_i)$, $i = 1, 2, 3$, be a τ-closed *n*-multiply ω-composite formation and let $\mathfrak{F}_2 \subseteq \mathfrak{F}_1$. It is necessary to show that

$$\mathfrak{F}_1 \cap (\mathfrak{F}_2 \vee^{\tau}_{\omega_n} \mathfrak{F}_3) = \mathfrak{F}_2 \vee^{\tau}_{\omega_n} (\mathfrak{F}_1 \cap \mathfrak{F}_3).$$

Let f_i be an ω-composite $c^{\tau}_{\omega_{n-1}}$-valued satellite of the formation $\mathfrak{F}_i$ such that $f_i(\omega') = \mathfrak{F}_i = F_i(\omega')$ and $f_i(p) = c^{\tau}_{\omega_{n-1}}\mathrm{form}(\mathfrak{F}_i(C^p))$ for all $p \in \omega$. By virtue of Lemma 1.8, we have $\mathfrak{F}_i = CF_{\omega}(f_i)$. Further, let $r_1 = f_2 \vee^{\tau}_{\omega_{n-1}} f_3$. Theorem 2.1 implies that

$$\mathfrak{F}_2 \vee^{\tau}_{\omega_n} \mathfrak{F}_3 = CF_{\omega}(f_2 \vee^{\tau}_{\omega_{n-1}} f_3) = CF_{\omega}(r_1).$$

According to Lemma 1.9, $h_1 = f_2 \cap r_1$ is an internal ω-composite $c^{\tau}_{\omega_{n-1}}$-valued satellite of the formation $\mathfrak{F}_1 \cap (\mathfrak{F}_2 \vee^{\tau}_{\omega_n} \mathfrak{F}_3)$.

It is clear that $f_2(a) \subseteq f_1(a)$ for all $a \in \omega \cup \{\omega'\}$. Therefore, by the assumption, for all $a \in \omega \cup \{\omega'\}$, we have

$$f_1(a) \cap \Big(f_2(a) \vee^{\tau}_{\omega_{n-1}} f_3(a)\Big) = f_2(a) \vee^{\tau}_{\omega_{n-1}} \big(f_1(a) \cap f_3(a)\big).$$

Hence, $h_1 = f_2 \vee^{\tau}_{\omega_{n-1}} (f_1 \cap f_3)$. However, $f_1 \cap f_3$ is the internal ω-composite $c^{\tau}_{\omega_{n-1}}$-valued satellite of the formation $\mathfrak{F}_1 \cap \mathfrak{F}_3$. Therefore, by virtue of Theorem 2.1, $\mathfrak{F}_2 \vee^{\tau}_{\omega_n} (\mathfrak{F}_1 \cap \mathfrak{F}_3) = CF_{\omega}(h_1)$. Thus, for any non-negative integers n, the lattice $c^{\tau}_{\omega_n}$ is modular.

The theorem is proved.

In the case $n = 1$, we arrive at the following corollary:

Corollary 3.1 [8]. *The lattice of all τ-closed ω-composite formations is algebraic and modular.*

If τ is a trivial subgroup functor, then, by using Corollary 1 and Remark 3 in [6], we obtain the following statement:

Corollary 3.2 [6]. *The lattice of all n-multiply $\mathfrak{L}$-composite formations is algebraic and modular.*

If $\omega = \mathbb{P}$, then the following assertion is true for the trivial subgroup functor τ:

Corollary 3.3. *The lattice of all n-multiply composite formations is algebraic and modular.*

If $n = 1$ and $\omega = \mathbb{P}$, then the following corollary is true for the trivial subgroup functor τ:

Corollary 3.4. *The lattice of all composite formations is algebraic and modular.*

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