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On the Lattices of Saturated and Solubly Saturated Formations of Finite Groups

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Abstract. It is proved that the lattice of all saturated formations of finite groups is a complete sublattice of the lattice of all solubly saturated formations of finite groups.

Keywords: Finite group; Formation of groups; Saturated formation; Solubly saturated formation; Algebraic lattice; Compact element of a lattice.

1. Introduction

Throughout this paper, all groups are finite. We write $R(G)$ to denote the largest soluble normal subgroup of the group G.

Let $\mathfrak{F}$ be a class of groups. If $1 \in \mathfrak{F}$, then we write $G^{\mathfrak{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$. The class $\mathfrak{F}$ is called a *formation* if either $\mathfrak{F} = \varnothing$ or $1 \in \mathfrak{F}$ and every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ for every group G.

The most useful for applications of the formation theory (in particular, in the theory of formal languages [7, 8, 9, 28, 14] and in the theory of lattices of group classes [10, 13, 15, 27, 33, 40]) are so-called saturated and solubly saturated formations.

Recall that the formation $\mathfrak{F}$ is said to be: *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$; *solubly saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(R(G)) \in \mathfrak{F}$.

A non-empty set Θ of formations is called a *complete lattice of formations* [33] if the intersection of every set of formations in Θ belongs to Θ and there is a formation $\mathfrak{F}$ in Θ such that $\mathfrak{M} \subseteq \mathfrak{F}$ for each other formation $\mathfrak{M}$ in Θ. In what follows, Θ denotes a complete lattice of formations.

It is clear that the sets of all formations $\mathcal{F}$, of all saturated formations $\mathcal{L}$ and of all solubly saturated formations $\mathcal{C}$ are examples of complete lattices of formations. These three lattices are algebraic and modular (see [33, 35]). Let's also note, in passing, that the modularity of these lattices has found wide applications in questions of classification of formations [27, 33, 15, 40]. Further, many other classes of algebraic and modular lattices of formations have been found (see, in particular, [1, 28, 19, 30, 21, 37, 29, 38, 39] and the recent book [40]). Nevertheless, it is necessary to note that the connections between different lattices of formations are still a little studied.

This circumstance is the main motivation for results of this paper.

Our first result is the following observation.

Theorem 1.1. *The lattice $\mathcal{L}$ is a complete sublattice of the lattice $\mathcal{C}$.*

Let's recall that the *product* $\mathfrak{M}\mathfrak{H}$ of the non-empty formations $\mathfrak{M}$ and $\mathfrak{H}$ is the class of all groups G such that $G^{\mathfrak{H}} \in \mathfrak{M}$. Such an operation on the set $\mathcal{F}$ is associative (W. Gaschütz). Moreover, the sets of all saturated formations and of all hereditary (in the sense of A.I. Mal'cev [25]) solubly saturated formations are subsemigroups of the semigroup of all formations $\mathcal{F}$. A great number of researches in the formation theory are connected with studying of factorizations of elements of these two subsemigroups (see, in particular, [31, 36, 32, 42, 2, 34, 16, 17, 18, 19, 20, 22, 11, 3, 24, 5, 6, 41] and the recent book [26]).

Every representation of the formation $\mathfrak{F}$ in the form $\mathfrak{F} = \mathfrak{F}_1 \dots \mathfrak{F}_t$, where $\mathfrak{F} \neq \mathfrak{F}_1 \dots \mathfrak{F}_{i-1}\mathfrak{F}_{i+1} \dots \mathfrak{F}_t$ for all i, is called an *irreducible factorization* of $\mathfrak{F}$.

In the book by A.N. Skiba [33] the description of all irreducible factorizations of saturated formations $\mathfrak{F}$ contained in a compact element of the lattice $\mathcal{L}$ was obtained. Further, in the work by W. Guo and K.P. Shum [20], all irreducible factorizations of a formations $\mathfrak{F}$ was described under condition that $\mathfrak{F}$ is solubly saturated and $\mathfrak{F}$ is contained in some compact element of the lattice $\mathcal{C}$. Since every saturated formation is solubly saturated, these two results are the motivation for the following question: *Suppose that a solubly saturated formation $\mathfrak{F}$ is contained in a compact element of the lattice $\mathcal{L}$. Does it true then that $\mathfrak{F}$ is contained in some compact element of the lattice $\mathcal{C}$?*

Our next result gives the positive answer to this question.

Theorem 1.2. *Every solubly saturated formation contained in a compact element* *of the lattice $\mathcal{L}$ is also contained in some compact element of the lattice $\mathcal{C}$.*

Therefore, in view of this result, the above-mentioned result of A.N. Skiba in [33] is a consequence of the main result in [20, Theorem 4.1].

All unexplained notations and terminologies are standard. The reader is refereed to [27, 10, 15, 4] if necessary.

2. Preliminaries

Recall that $\pi(G)$ denotes the set of all prime divisors of the order of a group G. For any collection of groups $\mathfrak{X}$ we denote by $\mathrm{Com}(\mathfrak{X})$ the class of all abelian groups A such that $A \cong H/K$, for some composition factor H/K of a group $G \in \mathfrak{X}$.

Recall that $C^p(G)$ is the intersection of the centralizers of all the abelian p-chief factors of G ($C^p(G) = G$ if G has no abelian p-chief factors).

The symbols $\mathfrak{G}$, $\mathfrak{G}_p$, $\mathfrak{G}_{p'}$ and $\mathfrak{S}$ denote the class of all groups, the class of all p-groups, the class of all p'-groups and the class of all soluble groups, respectively.

Let $\mathbb{P}$ be the set of all primes. Then for any formation function

$$f : \mathbb{P} \to \{\text{group formations}\}, \tag{1}$$

the symbol $LF(f)$ denotes the collection of all groups G such that either $G = 1$ or $G \neq 1$ and $G/O_{p',p}(G) \in f(p)$ for every $p \in \pi(G)$. If for a formation $\mathfrak{F}$ we have $\mathfrak{F} = LF(f)$, then f is called a *local satellite* of $\mathfrak{F}$.

In the following lemma, the symbol $\mathfrak{G}_pF(p)$ denotes the set of all groups A such that $A^{F(p)}$ is a p-group.

Lemma 2.1. [10] *For any non-empty saturated formation $\mathfrak{F}$, there is a unique formation function F such that $\mathfrak{F} = LF(F)$ and $F(p) = \mathfrak{G}_pF(p) \subseteq \mathfrak{F}$ for all primes p.*

The formation function F in Lemma 2.1 is called the *canonical local satellite* of $\mathfrak{F}$.

For any function f of the form

$$f : \mathbb{P} \cup \{0\} \to \{\text{group formations}\} \tag{2}$$

we put, following [35], $CF(f) = (G$ is a group $\mid G/R(G) \in f(0)$ and $G/C^p(G) \in f(p)$ for all $p \in \pi(\mathrm{Com}(G)))$. If for a formation $\mathfrak{F}$ we have $\mathfrak{F} = CF(f)$, then f is called a *composition satellite* of $\mathfrak{F}$.

In the papers [28, 35], the following useful facts are proved.

Lemma 2.2.

(a) *For any function f of the form* (1), *the class $LF(f)$ is a saturated formation.*

(b) *For any function f of the form* (2), *the class $CF(f)$ is a solubly saturated formation.*

(c) *For any non-empty solubly saturated formation $\mathfrak{F}$, there is a unique function F of the form* (2) *such that $\mathfrak{F} = CF(F)$, $F(p) = \mathfrak{G}_p F(p) \subseteq \mathfrak{F}$ for all primes p, and $F(0) = \mathfrak{F}$.*

If $\mathfrak{F} = LF(f)$ and $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$, then f is called an *inner local satellite* of $\mathfrak{F}$.

The function F in Lemma 2.2 is called the *canonical composition* satellite of $\mathfrak{F}$. If $\mathfrak{F} = CF(f)$ and $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$, then f is called an *inner composition satellite* of $\mathfrak{F}$.

A formation function f of the form (1) or (2) is called Θ-*valued* if all its values belong to the lattice Θ. We denote by Θ^l the set of all formations having a local Θ-valued satellite (see [27]); analogously we denote by Θ^c the set of all formations having a composition Θ-valued satellite.

The symbol $\Theta\mathrm{form}(\mathfrak{X})$ denotes the intersection of all formations in Θ containing the collection $\mathfrak{X}$ of groups. In the case, when $\Theta = \mathcal{F}$ is the lattice of all formations, we write $\mathrm{form}(\mathfrak{X})$ instead of $\Theta\mathrm{form}(\mathfrak{X})$.

For any collection $\{\mathfrak{F}_i \mid i \in I\}$ of formations in Θ we put

$$\vee_\Theta(\mathfrak{F}_i \mid i \in I) = \Theta \text{ form } \big(\bigcup_{i \in I} \mathfrak{F}_i\big).$$

In the case, when $\Theta = \mathcal{F}$, we write $\vee(\mathfrak{F}_i \mid i \in I)$ instead of $\vee_\Theta(\mathfrak{F}_i \mid i \in I)$.

The complete lattice of formations Θ^l is called *inductive* [33], if for any collection $\{\mathfrak{F}_i \mid i \in I\}$ of formations $\mathfrak{F}_i$ in Θ^l and for any collection $\{f_i \mid i \in I\}$, where f_i is an inner local satellite of $\mathfrak{F}_i$, we have $\vee_{\Theta^l}(\mathfrak{F}_i \mid i \in I) = LF\big(\vee_\Theta(f_i \mid i \in I)\big)$, where $\vee_\Theta(f_i \mid i \in I)$ is a local satellite of the formation $\vee_{\Theta^l}(\mathfrak{F}_i \mid i \in I)$ such that $f(p) = \vee_\Theta(f_i(p) \mid i \in I)$ for all $p \in \mathbb{P}$.

Lemma 2.3. [33] *The lattice $\mathcal{L}$ is inductive.*

The complete lattice of formations Θ^c is called *inductive* [33], if for any collection $\{\mathfrak{F}_i \mid i \in I\}$ of formations $\mathfrak{F}_i$ in Θ^c and for any collection $\{f_i \mid i \in I\}$, where f_i is an inner composition satellite of $\mathfrak{F}_i$, we have $\vee_{\Theta^c}(\mathfrak{F}_i \mid i \in I) = CF\big(\vee_\Theta(f_i \mid i \in I)\big)$, where $\vee_\Theta(f_i \mid i \in I)$ is a composition satellite of the formation $\vee_{\Theta^l}(\mathfrak{F}_i \mid i \in I)$ such that $f(a) = \vee_\Theta(f_i(a) \mid i \in I)$ for all $a \in \mathbb{P} \cup \{0\}$.

Lemma 2.4. [37] *The lattice $\mathcal{C}$ is inductive.*

A group class closed under taking homomorphic images is called a *semiformation* [27].

Lemma 2.5. [39] *Let $\mathfrak{M}$ be a semiformation and $A \in \mathrm{form}\,\mathfrak{M}$.*

(a) *If $O_p(A) = 1$, then $A \in \mathrm{form}(\mathfrak{M}_1)$, where $\mathfrak{M}_1 = (G/O_p(G) \mid G \in \mathfrak{M})$.*
(b) *If $R(A) = 1$, then $A \in \mathrm{form}(\mathfrak{M}_2)$, where $\mathfrak{M}_2 = (G/R(G) \mid G \in \mathfrak{M})$.*

Lemma 2.6. [35] *Let $\mathfrak{X}$ be a non-empty collection of groups and $\mathfrak{F} = \mathcal{C}\mathrm{form}(\mathfrak{X})$. Let $\pi = \pi(\mathrm{Com}(\mathfrak{X}))$. Then $\mathfrak{F} = CF(f)$, where:*

(a) *$f(p) = \mathrm{form}(G/C^p(G) \mid G \in \mathfrak{X})$ for all $p \in \pi$.*
(b) *$f(p) = \varnothing$ for all $p \in \mathbb{P} \setminus \pi$.*
(c) *$f(0) = \mathrm{form}(G/R(G) \mid G \in \mathfrak{X})$.*
(d) *$\pi = \pi(\mathrm{Com}(\mathfrak{F}))$.*

The satellite f in Lemma 2.6 is called the *minimal* composition satellite of $\mathfrak{F}$ [27].

Lemma 2.7. [28] *Let $\mathfrak{X}$ be a non-empty collection of groups and $\mathfrak{F} = \mathcal{L}\mathrm{form}(\mathfrak{X})$. Let $\pi = \pi(\mathfrak{X})$. Then $\mathfrak{F} = CF(f)$, where:*

(a) *$f(p) = \mathrm{form}(G/O_{p',p}(G) \mid G \in \mathfrak{X})$ for all $p \in \pi$*
(b) *$f(p) = \varnothing$ for all $p \in \mathbb{P} \setminus \pi$.*
(c) *$\pi = \pi(\mathfrak{F})$.*

The satellite f in Lemma 2.7 is called the *minimal* local satellite of $\mathfrak{F}$ [27].

Lemma 2.8. *Let Z_p be a group of prime order p, and G be a group with $O_p(G) = 1$. Suppose that $T = Z_p \wr G$ is the regular wreath product, where K is the base group of T. Then $K = C^p(T) = O_p(T)$.*

Proof. Let $1 = K_0 \leqslant K_1 \leqslant \ldots \leqslant K_t = K$ be a chief series of T below K. Let $C_i = C_T(K_i/K_{i-1})$ and $D = C_1 \cap \ldots \cap C_t$. Clearly, $K \leqslant D$. Consequently, $D = D \cap KG = K(D \cap G)$. Suppose $K \neq D$. Then $D \cap G$ is a non-identity group. But $D \cap G$ is a stable group of automorphisms of K. By [12, Chapter V, Corollary 3.3], $D \cap G$ is a normal p-subgroup of G, a contradiction. Thus $D = K = C^p(T) = O_p(T)$. ∎

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\{\mathfrak{F}_i \mid i \in I\}$ be a collection of saturated formations and let F_i be the canonical local satellite of $\mathfrak{F}_i$. Let $\mathfrak{F} = \vee_{\mathcal{L}}(\mathfrak{F}_i \mid i \in I)$ and $\mathfrak{H} = \vee_{\mathcal{C}}(\mathfrak{F}_i \mid i \in I)$. It is clear that $\bigcap_{i \in I} \mathfrak{F}_i$ is a saturated formation and this formation is the greatest lower bound for $\{\mathfrak{F}_i \mid i \in I\}$ in $\mathcal{L}$. On the other hand, clearly, $\mathfrak{F}$ is the least upper bound for $\{\mathfrak{F}_i \mid i \in I\}$ in $\mathcal{L}$ and $\mathfrak{H}$ is the least upper bound for $\{\mathfrak{F}_i \mid i \in I\}$ in $\mathcal{C}$. Therefore, in fact, we need only prove that $\mathfrak{F} = \mathfrak{H}$. The inclusion $\mathfrak{H} \subseteq \mathfrak{F}$ is evident. Hence, we need only show that $\mathfrak{F} \subseteq \mathfrak{H}$.

Let $\mathfrak{H}_i = CF(H_i)$, where H_i is a composition satellite such that

$$H_i(a) = \begin{cases} \mathfrak{F}_i & \text{if } a = 0, \\ F_i(a) & \text{if } a = p \in \mathbb{P}. \end{cases}$$

First we show that $\mathfrak{F}_i = \mathfrak{H}_i$ for all i.

Suppose $\mathfrak{H}_i \nsubseteq \mathfrak{F}_i$. Let G be a group of minimal order in $\mathfrak{H}_i \setminus \mathfrak{F}_i$. Then G is a monolithic group and $R = G^{\mathfrak{F}_i}$ is the monolith of G. If R is non-abelian, then $R(G) = 1$. Therefore, $G = G/1 = G/R(G) \in H_i(0) = \mathfrak{F}_i$. This contradicts the choice of G. Hence, R is an abelian p-group, where $p \in \pi(R)$. Since $\mathfrak{F}_i$ is saturated, it follows that $R \nleq \Phi(G)$. Therefore, by [10, Chapter A, Theorem 15.2], $R = C_G(R) = O_p(G)$. Hence, $R = C^p(G) = O_{p',p}(G)$. Consequently, $G/O_{p',p}(G) = G/C^p(G) \in H_i(p) = F_i(p)$. Hence, $G \in \mathfrak{F}_i$, a contradiction. Therefore, $\mathfrak{H}_i \subseteq \mathfrak{F}_i$.

Now we show that $\mathfrak{F}_i \subseteq \mathfrak{H}_i$. Assume this is false and let G be a group of minimal order in $\mathfrak{F}_i \setminus \mathfrak{H}_i$ with $R = G^{\mathfrak{H}_i}$. Let $p \in \pi(R)$. If R is non-abelian, then $O_{p',p}(G) = 1$. Hence, $G \cong G/1 = G/O_{p',p}(G) \in F_i(p) = H_i(p) \subseteq \mathfrak{H}_i$, a contradiction. Consequently, R is an abelian p-group. Let $T = R \rtimes (G/C_G(R))$. Since $G \in \mathfrak{F}_i$, using [10, Chapter IV, Proposition 1.5], we have $T \in \mathfrak{F}_i$. If $|T| < |G|$, then $T \in \mathfrak{H}_i$, by the choice of G. It follows that $G/C_G(R) \cong T/R = T/C_G(R) = T/C_T(R) = T/C^p(T) \in H_i(p)$. Hence, $G \in \mathfrak{H}_i$, a contradiction. Therefore, $|T| = |G|$, so $R = C_G(R) = O_p(G) = C^p(G) = O_{p',p}(G)$. Therefore, $G/C^p(G) = G/O_{p',p}(G) \in F_i(p) = H_i(p)$. Hence, $G \in \mathfrak{H}_i$, a contradiction. Consequently, $\mathfrak{F}_i \subseteq \mathfrak{H}_i$. Thus, $\mathfrak{F}_i = \mathfrak{H}_i$ for all $i \in I$.

Since by Lemma 2.3 the lattice $\mathcal{L}$ is inductive, we have $\mathfrak{F} = \vee_{\mathcal{L}}(\mathfrak{F}_i \mid i \in I) = LF\big(\vee(F_i \mid i \in I)\big)$. Since by Lemma 2.4 the lattice $\mathcal{C}$ is inductive, we have $\mathfrak{H} = \vee_{\mathcal{C}}(\mathfrak{F}_i \mid i \in I) = CF\big(\vee(H_i \mid i \in I)\big)$.

Now assume that $\mathfrak{F} \nsubseteq \mathfrak{H}$. Let G be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{H}$ with $R = G^{\mathfrak{H}}$. Let $p \in \pi(R)$.

If R is non-abelian, then $O_{p',p}(G) = 1$. Hence, since the canonical local satellite F_i is inner,

$$\begin{aligned} G \cong G/1 =& G/O_{p',p}(G) \in \big(\vee(F_i \mid i \in I)\big)(p) \\ =& \vee\,(F_i(p) \mid i \in I) \subseteq \vee(\mathfrak{F}_i \mid i \in I) \subseteq \vee_{\mathcal{C}}(\mathfrak{F}_i \mid i \in I) \\ =& \mathfrak{H}. \end{aligned}$$

This contradicts the choice of G. Hence, R is an abelian p-group. Let $T = R \rtimes (G/C_G(R))$. Since $G \in \mathfrak{F}$, using [10, Chapter IV, Proposition 1.5], we have $T \in \mathfrak{F}$. If $|T| < |G|$, then $T \in \mathfrak{H}$, by the choice of G. Consequently,

$$G/C_G(R) \cong T/R = T/C_G(R) = T/C_T(R) = T/C^p(T) \in (\vee\big(H_i \mid i \in I\big))(p).$$

Hence, $G \in \mathfrak{H}$, a contradiction. Thus, $|T| = |G|$, so $R = C_G(R) = O_p(G) = C^p(G) = O_{p',p}(G)$. Therefore, since $\mathfrak{F}_i = \mathfrak{H}_i$ for all $i \in I$,

$$\begin{aligned} G/C^p(G) = G/O_{p',p}(G) &\in (\vee\big(F_i \mid i \in I\big))(p) = \vee(F_i(p) \mid i \in I) \\ &= \vee(H_i(p) \mid i \in I) = (\vee\big(H_i \mid i \in I\big))(p). \end{aligned}$$

Hence, $G \in \mathfrak{H}$. Consequently, $\mathfrak{F} \subseteq \mathfrak{H}$. Thus, $\mathfrak{F} = \mathfrak{H}$, and the theorem is proved.■

4. Proof of Theorem 1.2

The formation $\mathfrak{F}$ is called a *one-generated* Θ*-formation* if $\mathfrak{F}$ is the intersection of all formations in Θ such which contain a fixed group G.

Lemma 4.1.

(a) *Every compact elements in* Θ *is a one-generated* Θ*-formation.*
(b) *Every one-generated* $\mathcal{C}$*-formation is a compact elements in* $\mathcal{C}$*.*

Proof. (a) It is clear that for any formation $\mathfrak{R} \in \Theta$, there is a set $\{G_i \mid \in I\}$ of groups $G_i \in \mathfrak{R}$ such that $\mathfrak{R} = \vee_\Theta(\Theta\text{form}(G_i) \mid i \in I)$. Therefore, if $\mathfrak{R}$ is a compact element in Θ, then there exist $i_1, \dots, i_t \in I$ such that

$$\mathfrak{R} \subseteq \Theta\text{form}(G_{i_1}) \vee_\Theta \dots \vee_\Theta \Theta\text{form}(G_{i_t}) = \Theta\text{form}(G_{i_1} \times \dots \times G_{i_t}) \subseteq \mathfrak{R}.$$

Hence $\mathfrak{R} = \Theta\text{form}(G_{i_1} \times \dots \times G_{i_t})$ is a one-generated Θ-formation, as desired.

(b) This assertion is proved in [35]. ■

Proof of Theorem 1.2. Let $\mathfrak{F}$ be a solubly saturated formation contained in the compact element $\mathfrak{H}$ of the lattice of $\mathcal{L}$. Then, by Lemma 4.1, there is a group G such that $\mathfrak{H} = \mathcal{L}\text{form}(G)$. Let $\pi = \pi(G) = \{p_1, \dots, p_t\}$ and $\mathfrak{K} = \mathcal{C}\text{form}(G^*)$, where

$$G^* = G \times (Z_{p_1} \wr (G/O_{p_1}(G))) \times \dots \times (Z_{p_t} \wr (G/O_{p_t}(G))).$$

In view of Lemma 4.1, in order to prove the result, it is enough to show that $\mathfrak{F} \subseteq \mathfrak{K}$.

Let f and k be the minimal composition satellites of $\mathfrak{F}$ and $\mathfrak{K}$ respectively, and let h be the minimal local satellite of $\mathfrak{H}$.

To prove the inclusion $\mathfrak{F} \subseteq \mathfrak{K}$ it is enough to show $f \leqslant k$, i.e., $f(0) \subseteq k(0)$ and $f(p) \subseteq k(p)$ for all $p \in \mathbb{P}$.

First we shall prove that $f(0) \subseteq k(0)$. By Lemma 2.6, $f(0) = \text{form}(A \mid A \in \mathfrak{F}$ and $R(A) = 1)$ and $k(0) = \text{form}(G^*/R(G^*))$.

Therefore, in view of Lemma 2.5, in order to prove the inclusion $f(0) \subseteq k(0)$, it is enough to show that for any group $A \in \mathfrak{F}$ with $R(A) = 1$ we have $A \in \text{form}G^*$.

Let $\text{Soc}(A) = N_1 \times \dots \times N_k$, where N_i is a minimal normal subgroup of A ($i = 1, \dots, k$). Since $R(A) = 1$, N_i is non-abelian for all $i = 1, \dots, t$. If $t = 1$ and p is a prime dividing $|N_1|$, then $O_{p',p}(A) = 1$ and so we have, at once, by Lemma 2.7, $A \cong A/O_{p',p}(A) \in h(p) = \text{form}(G/O_{p',p}(G)) \subseteq \text{form}G \subseteq \text{form}G^*$.

Now assume that $t > 1$. Let M_i be the largest normal subgroup of A containing $N_1 \times \dots \times N_{i-1} \times N_{i+1} \times \dots \times N_k$, but not containing N_i. Then N_iM_i/M_i is a unique minimal normal subgroup of G/M_i, N_iM_i/M_i is G-isomorphic to N_i, and $A/M_i \in \mathfrak{F}$ since $A/M_i \in \mathfrak{F}$. Hence, $C_A(N_iM_i/M_i) = M_i$, and so for any prime p dividing $|N_iM_i/M_i|$ we have

$$A/M_i \cong (A/M_i)/O_{p',p}(A/M_i) \in h(p) = \text{form}(G/O_{p',p}(G)) \subseteq \text{form}G \subseteq \text{form}G^*.$$

Therefore, $A \cong A/1 = A/M_1 \cap \ldots \cap M_k \in \text{form}G^*$. It follows that $A \in k(0)$. Thus, $f(0) \subseteq k(0)$.

Now we prove that $f(p) \subseteq k(p)$ for all $p \in \mathbb{P}$. If $f(p) = \varnothing$, then the inclusion is obvious. Let $f(p) \neq \varnothing$. But in this case we have $p \in \pi$. Indeed, from $f(p) \neq \varnothing$ we have $Z_{p_1} \in \mathfrak{F} \subseteq \mathfrak{H} = \mathcal{L}\text{form}G$. Hence $p \in \pi$ by Lemma 2.7. Hence, $p = p_i$ for some $i \in \{1, \ldots, t\}$.

By Lemma 2.6, $f(p) = \text{form}(A/C^p(A) \mid A \in \mathfrak{F})$. Therefore, in order to prove the inclusion $f(p) \subseteq k(p)$, it is enough to show that for any group $A \in \mathfrak{F}$ we have $\overline{A} = A/C^p(A) \in k(p)$.

First note that $\overline{A} \in \text{form}G$. Indeed, since $O_{p',p}(A) \leqslant C^p(A)$, $\overline{A} = A/C^p(A)$ is a homomorphic image of $A/O_{p',p}(A)$. On the other hand, since $A \in \mathfrak{F} \subseteq \mathfrak{H}$, $A/O_{p',p}(A) \in h(p) = \text{form}(G/O_{p',p}(G))$. Hence, $\overline{A} \in h(p) = \text{form}(G/O_{p',p}(G)) \subseteq \text{form}G$.

Since $T = Z_p \wr (G/O_p(G)) = K \rtimes (G/O_p(G)) \in \mathfrak{K}$, where K is the base group of the regular wreath product T, we have $G/O_p(G) \cong T/K = T/C^p(T) \in k(p)$ x by Lemma 2.8. Note also that in view of [10, Chapter A, Lemma 13.6], $O_p(\overline{A}) = 1$. Therefore from $\overline{A} \in \text{form}G$ we get $\overline{A} \in \text{form}(G/O_p(G))$ by Lemma 2.5. Hence, $\overline{A} \in k(p)$. Consequently, $f(p) \subseteq k(p)$.

Thus, $f(a) \subseteq k(a)$ for all $a \in \mathbb{P} \cup \{0\}$. Hence, $\mathfrak{F} \subseteq \mathfrak{K}$. This proves the theorem. ■

5. Some Open Questions

Every formation is 0-multiply saturated, by definition. For $n > 0$, a formation $\mathfrak{F}$ is called *n-multiply saturated* if $\mathfrak{F} = LF(f)$ and all non-empty values of f are $(n-1)$-multiply saturated formations [27]. If a formation $\mathfrak{F}$ is n-multiply saturated for all natural n, then $\mathfrak{F}$ is called *totally* saturated. n-Multiply solubly saturated formations and totally solubly saturated formations are defined analogously [35].

Now, we mention the following open questions in the theory of lattices of group classes.

Question 5.1. Is any complete lattice of formations algebraic?

Question 5.2. Let Θ be a complete lattice of formations. Does true then that every one-generated Θ-formation is a compact element in Θ?

Question 5.3. Does it true that the lattice $\mathcal{L}_n$ of all n-multiply saturated formations is a complete sublattice of the lattice $\mathcal{C}_n$ of all n-multiply solubly saturated formations?

Question 5.4. Does it true that the lattice $\mathcal{L}_\infty$ of all totally saturated formations is a complete sublattice of the lattice $\mathcal{C}_\infty$ of all totally solubly saturated formations?

Question 5.5. Suppose that an n-multiply solubly saturated formation $\mathfrak{F}$ is contained in a compact element of the lattice $\mathcal{L}_n$. Does it true then that $\mathfrak{F}$ is contained in some compact element of the lattice $\mathcal{C}_n$?

Question 5.6. Suppose that a totally solubly saturated formation $\mathfrak{F}$ is contained in a compact element of the lattice $\mathcal{L}_\infty$. Does it true then that $\mathfrak{F}$ is contained in some compact element of the lattice $\mathcal{C}_\infty$?

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