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# ON LAWS OF LATTICES OF PARTIALLY SATURATED FORMATIONS

Leonid A. Shemetkov*

*Department of Mathematics, Francisk Skorina Gomel State University*
*Gomel 246019, Belarus*
*shemetkov@gsu.by*

Alexander N. Skiba†

*Department of Mathematics, Francisk Skorina Gomel State University,*
*Gomel 246019, Belarus*
*alexander.skiba49@gmail.com*

Nikolay N. Vorob'ev

*Department of Mathematics, Masherov Vitebsk State University,*
*Vitebsk 210038, Belarus*
*vornic2001@yahoo.com*

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It is proved that every law of the lattice of all $\tau$-closed formations of finite groups is fulfilled in the lattice of all $\tau$-closed $n$-multiply $\omega$-saturated formations of finite groups, for every subgroup functor $\tau$ and every natural number $n$.

*Keywords*: Finite group; subgroup functor; formation of groups; $n$-multiply $\omega$-saturated formation; law of lattice; modular lattice.

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## 1. Introduction

All groups considered are finite.

In the book [4] and in the recent books [9], [10] it was demonstrated that constructions and results of lattice theory are very useful tools to study groups and

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group classes. In particular, it was proved that the lattice of all saturated formations is modular [4]. Further this result was developed in different ways. In the book [1] modularity of the lattice of all $\tau$-closed $n$-multiply saturated formations was established, for every subgroup functor $\tau$; in [5] it was shown by A. Ballester-Bolinches and L.A. Shemetkov that the lattice of all $p$-saturated formations is modular; A. N. Skiba and L. A. Shemetkov proved [2], [6] modularity of the lattice of all $n$-multiply $\omega$-saturated formations and the lattice of all $n$-multiply $\mathfrak{L}$-composition formations, respectively; I.P. Shabalina proved [7] modularity of the lattice of all $\tau$-closed $n$-multiply $\omega$-saturated formations.

Since the lattice of all formations is modular [3], all the above-mentioned results are special cases of our first theorem.

**Theorem 1.** *Let $n > 0$. Then every law of the lattice of all $\tau$-closed formations is fulfilled in the lattice of all $\tau$-closed $n$-multiply $\omega$-saturated formations.*

The second theorem give a further information about the lattice of all $\tau$-closed $n$-multiply $\omega$-saturated formations.

**Theorem 2.** *Let $n > 0$. If $\omega$ is an infinite set, then the law system of the lattice of all $\tau$-closed formations coincides with the law system of the lattice of all $\tau$-closed $n$-multiply $\omega$-saturated formations.*

All unexplained notations and terminologies are standard. The reader is referred to [8], [9] and [10] if necessary.



## 2. Proof of Theorem 1

Recall that a group class closed under taking homomorphic images and finite subdirect products is called a formation.

In each group $G$ we select a system of subgroups $\tau(G)$. It is said that $\tau$ is a subgroup functor if the following conditions hold:

1) $G \in \tau(G)$ for every group $G$;

2) for every epimorphism $\varphi : A \to B$ and all groups $H \in \tau(A)$ and $T \in \tau(B)$ we have $H^{\varphi} \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$.

A formation $\mathfrak{F}$ is called $\tau$-closed if $\tau(G) \subseteq \mathfrak{F}$ for every group $G$ of $\mathfrak{F}$ (see [1]).

Let $\omega$ be a nonempty set of primes, $\omega' = \mathbb{P} \setminus \omega$. $\pi(G)$ denotes the set of all prime divisors of the order of a group $G$. Recall that a group $G$ is called an $\omega d$-group if $\omega \cap \pi(G) \neq \varnothing$. The symbols $\mathfrak{G}$, $\mathfrak{N}_p$ and $\mathfrak{G}_{p'}$ denote, respectively, the class of all groups, the class of all $p$-groups and the class of all $p'$-groups; $\mathfrak{G}_{\omega d}$ denotes the class of all groups in which every composition factor is an $\omega d$-group. For every group class $(1) \subseteq \mathfrak{F}$, by $G_{\mathfrak{F}}$ we denote the product of all normal $\mathfrak{F}$-subgroups of group $G$. In particular, we write

$$G_{\omega d} = G_{\mathfrak{G}_{\omega d}},\ F_p(G) = G_{\mathfrak{G}_{p'}\mathfrak{N}_p}.$$

Functions of the form

$$f : \omega \bigcup \{\omega'\} \to \{\text{formations of groups}\}$$

are called $\omega$-local satellites (see [2]). For every $\omega$-local satellite $f$, we define the class

$$LF_\omega(f) = (G \mid G/G_{\omega d} \in f(\omega') \text{ and } G/F_p(G) \in f(p) \text{ for all } p \in \omega \cap \pi(G)).$$

If $\mathfrak{F}$ is a formation such that $\mathfrak{F} = LF_\omega(f)$ for an $\omega$-local satellite $f$, then the formation $\mathfrak{F}$ is said to be $\omega$-saturated, and $f$ is said to be an $\omega$-local satellite of $\mathfrak{F}$.

Every formation is 0-multiply $\omega$-saturated, by definition. For $n > 0$, a formation is called $n$-multiply $\omega$-saturated if $\mathfrak{F} = LF_\omega(f)$ and all nonempty values of $f$ are $(n-1)$-multiply $\omega$-saturated formations (see [2]). If a formation $\mathfrak{F}$ is $n$-multiply $\omega$-saturated for all natural $n$, then $\mathfrak{F}$ is called totally $\omega$-saturated.

By $l^\tau_{\omega_n}$ we denote the set of all $\tau$-closed $n$-multiply $\omega$-saturated formations. With respect to inclusion, an arbitrary nonempty subset $\{\mathcal{H}_i \mid i \in \Lambda\}$ of $l^\tau_{\omega_n}$ has a greatest lower bound, namely $\cap_{i\in\Lambda}\mathcal{H}_i$; besides, $\{\mathcal{H}_i \mid i \in \Lambda\}$ has a least upper bound, the intersections of all elements in $l^\tau_{\omega_n}$ containing $\cup_{i\in\Lambda}\mathcal{H}_i$. Thus, $l^\tau_{\omega_n}$ is a complete lattice. In particular, $l^\tau_{\omega_0}$ is the lattice of all $\tau$-closed formations.

A group class closed under taking homomorphic images is called a semiformation [4]. The symbol $l^\tau_{\omega_n}$form $\mathfrak{X}$ denotes the intersection of all $\tau$-closed $n$-multiply $\omega$-saturated formations containing a collection $\mathfrak{X}$ of groups.

By [2], Lemma 5, if $\mathfrak{F} = l^\tau_{\omega_n}$form $\mathfrak{X}$, then $\mathfrak{F} = LF_\omega(f)$ where

$$f(a) = \begin{cases} l^\tau_{\omega_{n-1}}\text{form } (G/F_p(G) \mid G \in \mathfrak{X}), & \text{if } a = p \in \omega \bigcap \pi(\mathfrak{X}), \\ \varnothing, & \text{if } a = p \in \omega \setminus \pi(\mathfrak{X}), \\ l^\tau_{\omega_{n-1}}\text{form } (G/G_{\omega d} \mid G \in \mathfrak{X}), & \text{if } a = \omega'. \end{cases}$$

The satellite $f$ is called the minimal $l^\tau_{\omega_{n-1}}$-valued $\omega$-local satellite of $\mathfrak{F}$ (see [2]).

First we prove the following lemmas.

**Lemma 1.** *Let $A$ be a monolithic group, $R$ a non-abelian socle of $A$, $\mathfrak{M}$ a semiformation and $A \in l^\tau_{\omega_n}$form $\mathfrak{M}$. Then $A \in \mathfrak{M}$.*

*Proof.* We proceed by induction on $n$. Let $n = 0$. Then

$$A \in l^\tau_{\omega_0}\text{form } \mathfrak{M} = \text{form } \mathfrak{M}.$$

Let $A \notin \mathfrak{M}$. Then, by [1], Corollary 1.2.26, there exists a group $H$ in form $\mathfrak{M}$ and normal subgroups $N, M, N_1, \dots, N_t, M_1, \dots, M_t$ $(t \geq 2)$ of $H$ such that the following statements hold:

1) $A \simeq H/N$ and $M/N = \mathrm{Soc}(H/N)$;

2) $H/N_i$ is a monolithic $\mathfrak{M}$-group and $M_i/N_i$ is the socle of $H/N_i$ which is $H$-isomorphic to $M/N$.

Clearly $C_H(M/N) = N$. Hence $N_i \subseteq N$. Therefore $A \simeq H/N \in \mathfrak{M}$, a contradiction. This completes the proof of the lemma for $n = 0$.

Let $n > 0$, and let the lemma holds for $n - 1$. Suppose $f$ is the minimal $l^\tau_{\omega_{n-1}}$-valued $\omega$-local satellite of $\mathfrak{F} = l^\tau_{\omega_n}$form $\mathfrak{M}$. If $\omega \cap \pi(R) = \varnothing$, then $A_{\omega d} = 1$, and so, by [2], Lemma 5, we have

$$A \simeq A/A_{\omega d} \in f(\omega') \subseteq l^\tau_{\omega_{n-1}}\text{form } \mathfrak{M}.$$

Consequently, $A \in \mathfrak{M}$.

Let $\omega \cap \pi(R) \neq \varnothing$ and $p \in \omega \bigcap \pi(R)$. Then $F_p(A) = 1$, and by [2], Lemma 5, we have

$$A \simeq A/F_p(A) \in f(p) \subseteq l^{\tau}_{\omega_{n-1}}\text{form } \mathfrak{M}.$$

Hence $A \in \mathfrak{M}$, and the lemma is proved.

**Lemma 2** [1], Lemma 4.1.3., *Let* $N_1 \times \ldots \times N_t = \mathrm{Soc}(G)$ *where* $t > 1$, *and* $G$ *a group with* $O_p(G) = 1$. *Let* $M_i$ *be the largest normal subgroup in* $G$ *containing* $N_1 \times \ldots \times N_{i-1} \times N_{i+1} \times \ldots \times N_t$ *but not containing* $N_i$ $(i = 1, \ldots, t)$. *Then*

1) *for every* $i \in \{1, \ldots, t\}$, $O_p(G/M_i) = 1$, $G/M_i$ *is monolithic and its socle* $N_iM_i/M_i$ *is* $G$*-isomorphic to* $N_i$ ;

2) $M_1 \bigcap \ldots \bigcap M_t = 1$.

**Lemma 3.** *Let* $\mathfrak{M}$ *be a semiformation and* $A \in l^{\tau}_{\omega_n}\text{form } \mathfrak{M}$. *Then the following statements hold:*

1) *if* $O_p(A) = 1$ *and* $p \in \omega$, *then* $A \in l^{\tau}_{\omega_n}\text{form } \mathfrak{M}_1$ *where* $\mathfrak{M}_1 = \{G/O_p(G) \mid G \in \mathfrak{M}\}$;

2) *if* $A_{\omega d} = 1$, *then* $A \in l^{\tau}_{\omega_n}\text{form } \mathfrak{M}_2$ *where* $\mathfrak{M}_2 = \{G/G_{\omega d} \mid G \in \mathfrak{M}\}$.

*Proof.* If $A \in \mathfrak{M}$, the result is clear. Hence we may suppose that $A \notin \mathfrak{M}$. Suppose that $A$ is a monolithic group and $R$ is the socle of $A$. Let $n = 0$. Then $A \in l^{\tau}_{\omega_0}\text{form } \mathfrak{M} = \text{form } \mathfrak{M}$. Hence, by [1], Corollary 1.2.26, there exists a group $H$ in form $\mathfrak{M}$, normal subgroups $N, M, N_1, \ldots, N_t, M_1, \ldots, M_t (t \geq 2)$ in $H$ such that the following statements hold: 1) $H/N \simeq A$, $M/N = \mathrm{Soc}(H/N)$; 2) $N_1 \bigcap \ldots \bigcap N_t = 1$; 3) $H/N_i$ is a monolithic $\mathfrak{M}$-group and $M_i/N_i$ is the socle of $H/N_i$ which is $H$-isomorphic to $M/N$. Since $O_p(A) = 1$, we have

$$A \in \mathrm{QR}_0\{H/N_1, \ldots, H/N_t\} \subseteq \text{form } \mathfrak{M}_1.$$

Let $n > 0$. Suppose that $O_p(A) = 1$. If $R$ is nonabelian, then Lemma 1 implies $A \in \mathfrak{M}$. This contradicts the choice of $A$. Hence $R$ is a $q$-group where $q \in \omega \setminus \{p\}$. Consequently, $F_q(A) = O_q(A)$. Since for every group $G$ we have

$$G/G_{\omega d} \simeq (G/O_p(G))/(G_{\omega d}/O_p(G))$$

$$= (G/O_p(G))/(G/O_p(G))_{\omega d},$$

by [2], Lemma 5, it follows that $f(\omega') = h(\omega')$ where $f$ and $h$ are minimal $l^{\tau}_{\omega_{n-1}}$-valued $\omega$-local satellites of $\mathfrak{F} = l^{\tau}_{\omega_n}\text{form } \mathfrak{M}$ and $\mathfrak{H} = l^{\tau}_{\omega_n}\text{form } \mathfrak{M}_1$ respectively. If $q \notin \omega$, then $A_{\omega d} = 1$ and so

$$A \simeq A/A_{\omega d} \in f(\omega') = h(\omega') \subseteq \mathfrak{H}.$$

Let $q \in \omega$. Since for every group $G$ we have

$$G/F_q(G) \simeq (G/O_p(G))/(F_q(G)/O_p(G))$$

$$= (G/O_p(G))/F_q(G/O_p(G)),$$

by [2], Lemma 5, it follows that $f(q) = h(q)$. Hence $A/O_q(A) \in \mathfrak{H}$ and

$$A/F_r(A) \simeq (A/O_q(A))/(F_r(A)/O_q(A))$$

$$= (A/O_q(A))/F_r(A/O_q(A)) \in h(r),$$

for all $r \in \omega \bigcap \pi(A)$. We deduce, that $A \in \mathfrak{H}$. Analogously $A \in l^{\tau}_{\omega_n}$form $\mathfrak{M}_2$ where $\mathfrak{M}_2 = \{G/G_{\omega d} \mid G \in \mathfrak{M}\}$.

Now suppose that $\mathrm{Soc}(A) = N_1 \times \ldots \times N_t$ where $t > 1$. Let $M_i$ be the largest normal subgroup of $A$ containing $N_1 \times \ldots \times N_{i-1} \times N_{i+1} \times \ldots \times N_t$ but not containing $N_i, i = 1, \ldots, t$. Using Lemma 2, we have $A \in \mathrm{R}_0(A/M_1, \ldots, A/M_t)$ where $A/M_i$ is monolithic, $N_iM_i/M_i$ is the socle of $A/M_i$ and $O_p(A/M_i) = 1$. Clearly $A/M_i \in l^{\tau}_{\omega_n}$form $\mathfrak{M}$. As we proved above, $A/M_i \in l^{\tau}_{\omega_n}$form $\mathfrak{M}_1$. Consequently, $A \in \mathfrak{H}$, as claimed.

Let $\{\mathfrak{F}_i \mid i \in I\}$ be an arbitrary collection of $\tau$-closed $n$-multiply $\omega$-saturated formations. We denote

$$\vee^{\tau}_{\omega_n}(\mathfrak{F}_i \mid i \in I) = l^{\tau}_{\omega_n}\text{form } (\bigcup_{i \in I} \mathfrak{F}_i).$$

In particular,

$$\mathfrak{M} \vee^{\tau}_{\omega_n} \mathfrak{H} = l^{\tau}_{\omega_n}\text{form } (\mathfrak{M} \bigcup \mathfrak{H}).$$

A function $f : \omega \bigcup \{\omega'\} \to \{\text{formations of groups}\}$ is called $l^{\tau}_{\omega_n}$-valued if all its values belong to the lattice $l^{\tau}_{\omega_n}$.

Let $\{f_i \mid i \in I\}$ be a collection of $l^{\tau}_{\omega_n}$-valued functions of the form

$$f_i : \omega \bigcup \{\omega'\} \to \{\text{formations of groups}\}.$$

In this case, by $\vee^{\tau}_{\omega_n}(f_i \mid i \in I)$ we denote a function $f$ such that $f(\omega') = l^{\tau}_{\omega_n}$form $(\bigcup_{i \in I} f_i(\omega'))$. In particular,

$$(f_1 \vee^{\tau}_{\omega_n} f_2)(\omega') = l^{\tau}_{\omega_n}\text{form } (f_1(\omega') \bigcup f_2(\omega'))$$

and for $p \in \omega$ we have $f(p) = l^{\tau}_{\omega_n}$form $(\bigcup_{i \in I} f_i(p))$. In particular,

$$(f_1 \vee^{\tau}_{\omega_n} f_2)(p) = l^{\tau}_{\omega_n}\text{form } (f_1(p) \bigcup f_2(p))$$

if at least one of the formations $f_i(p) \neq \varnothing$. If $f_i(p) = \varnothing$ for all $i \in I$, then we suppose that $f(p) = \varnothing$.

**Lemma 4.** *Let $f_i$ be the minimal $l^{\tau}_{\omega_{n-1}}$-valued $\omega$-local satellite of a $\tau$-closed $n$-multiply $\omega$-saturated formation $\mathfrak{F}_i$ where $i \in I$. Then $\vee^{\tau}_{\omega_{n-1}}(f_i \mid i \in I)$ is the minimal $l^{\tau}_{\omega_{n-1}}$-valued $\omega$-local satellite of $\mathfrak{F} = \vee^{\tau}_{\omega_n}(\mathfrak{F}_i \mid i \in I)$.*

*Proof.* Observe that

$$\pi(\bigcup_{i \in I} \mathfrak{F}_i) = \bigcup_{i \in I} \pi(\mathfrak{F}_i) = \pi(\mathfrak{F}).$$

Let $f = \vee^{\tau}_{\omega_{n-1}}(f_i \mid i \in I)$, and let $h$ be the minimal $l^{\tau}_{\omega_{n-1}}$-valued $\omega$-local satellite of $\mathfrak{F}$. Let $p \in \omega \setminus \pi(\mathfrak{F})$. In this case, for every $i \in I$, we have $f_i(p) = \varnothing$. Hence $f(p) = \varnothing$. Clearly $h(p) = \varnothing$.

Let $p \in \omega \cap \pi(\mathfrak{F})$. In this case, there is $i \in I$ such that $f_i(p) \neq \varnothing$. Using [2], Lemma 5, we have

$$h(p) = l^{\tau}_{\omega_{n-1}}\text{form } (G/F_p(G) \mid G \in \bigcup_{i \in I} \mathfrak{F}_i) =$$

$$l^{\tau}_{\omega_{n-1}}\text{form } (\bigcup_{i \in I} l^{\tau}_{\omega_{n-1}}\text{form } (G/F_p(G) \mid G \in \mathfrak{F}_i)) =$$

$$l^{\tau}_{\omega_{n-1}}\text{form } (\bigcup_{i \in I} f_i(p)) = (\vee^{\tau}_{\omega_{n-1}}(f_i \mid i \in I))(p).$$

Moreover, by [2], Lemma 5, we have

$$h(\omega') = l^{\tau}_{\omega_{n-1}}\text{form } (G/G_{\omega d} \mid G \in \bigcup_{i \in I} \mathfrak{F}_i) =$$

$$l^{\tau}_{\omega_{n-1}}\text{form } (\bigcup_{i \in I} l^{\tau}_{\omega_{n-1}}\text{form } (G/G_{\omega d} \mid G \in \mathfrak{F}_i)) =$$

$$l^{\tau}_{\omega_{n-1}}\text{form } (\bigcup_{i \in I} f_i(\omega')) = (\vee^{\tau}_{\omega_{n-1}}(f_i \mid i \in I))(\omega').$$

Thus $\vee^{\tau}_{\omega_{n-1}}(f_i \mid i \in I)$ is the minimal $l^{\tau}_{\omega_{n-1}}$-valued $\omega$-local satellite of $\mathfrak{F} = \vee^{\tau}_{\omega_n}(\mathfrak{F}_i \mid i \in I)$, and the lemma is proved.

If $\mathfrak{F} = LF_\omega(f)$ and $f(a) \subseteq \mathfrak{F}$ for all $a \in \omega \bigcup \{\omega'\}$, then $f$ is called an inner satellite of $\mathfrak{F}$.

**Lemma 5.** *If* $\{\mathfrak{F}_i = LF_\omega(f_i) \mid i \in I\}$ *is a set of* $\tau$*-closed* $\omega$*-saturated formations* $\mathfrak{F}_i$ *where* $f_i$ *is an inner* $l^{\tau}_{\omega_{n-1}}$*-valued satellite of* $\mathfrak{F}_i$*, then*

$$\vee^{\tau}_{\omega_n}(\mathfrak{F}_i \mid i \in I) = LF_\omega(\vee^{\tau}_{\omega_{n-1}}(f_i \mid i \in I)).$$

*Proof.* Let $\{\mathfrak{F}_i \mid i \in I\}$ be a set of $\tau$-closed $n$-multiply $\omega$-saturated formations and $f_i$ be an inner $l^{\tau}_{\omega_{n-1}}$-valued $\omega$-local satellite of $\mathfrak{F}_i$. Let $\mathfrak{F} = \vee^{\tau}_{\omega_n}(\mathfrak{F}_i \mid i \in I)$, $\mathfrak{M} = LF_\omega(\vee^{\tau}_{\omega_{n-1}}(f_i \mid i \in I))$ and $h_i$ be the minimal $l^{\tau}_{\omega_{n-1}}$-valued $\omega$-local satellite of $\mathfrak{F}_i$. Then by Lemma 4 we have that $h = \vee^{\tau}_{\omega_{n-1}}(h_i \mid i \in I)$ is the minimal $l^{\tau}_{\omega_{n-1}}$-valued $\omega$-local satellite of $\tau$-closed formation $\mathfrak{F}$. Clearly $h \leq f = \vee^{\tau}_{\omega_{n-1}}(f_i \mid i \in I)$. Hence $\mathfrak{F} \subseteq \mathfrak{M}$. Suppose that the converse inclusion is false. Let $G$ be a group of minimal order in $\mathfrak{M} \setminus \mathfrak{F}$. Let $R$ be the socle of $G$. Then $R = G^{\mathfrak{F}}$. Let $p \in \pi(R) \bigcap \omega$.

Suppose that $R$ is nonabelian. Then $F_p(G) = 1$. Therefore

$$G \simeq G/F_p(G) \in (\vee^{\tau}_{\omega_{n-1}}(f_i \mid i \in I))(p) = l^{\tau}_{\omega_{n-1}}\text{form } (\bigcup_{i \in I} f_i(p)).$$

Hence, by Lemma 1, Lemma 5, we have

$$G \in \bigcup_{i \in I} f_i(p) \subseteq \bigcup_{i \in I} \mathfrak{F}_i \subseteq \mathfrak{F},$$

a contradiction. Consequently, $R$ is a $p$-group. Then $O_p(G) = F_p(G)$. But $G \in \mathfrak{M} = LF_\omega(\vee^\tau_{\omega_{n-1}}(f_i \mid i \in I))$. Hence $G/O_p(G) \in l^\tau_{\omega_{n-1}}\text{form } (\bigcup_{i\in I} f_i(p))$. Since $O_p(G/O_p(G)) = 1$, Lemma 3 and [2], imply

$$G/O_p(G) \in l^\tau_{\omega_{n-1}}\text{form } (A/O_p(A) \mid A \in \bigcup_{i\in I} f_i(p))$$

$$= l^\tau_{\omega_{n-1}}\text{form } (\bigcup_{i\in I} l^\tau_{\omega_{n-1}}\text{form } (A/O_p(A) \mid A \in f_i(p)))$$

$$= l^\tau_{\omega_{n-1}}\text{form } (\bigcup_{i\in I} h_i(p)) = (\vee^\tau_{\omega_{n-1}}(h_i \mid i \in I))(p) = h(p).$$

Hence, by [2], Lemma 4, we have $G \in \mathfrak{F}$, a contradiction. Consequently, $\omega \bigcap \pi(R) = \varnothing$. Therefore $G_{\omega d} = 1$. Applying Lemma 3 and [2], Lemma 5, we have

$$G \simeq G/G_{\omega d} \in l^\tau_{\omega_{n-1}}\text{form } (A/A_{\omega d} \mid A \in \bigcup_{i\in I} f_i(\omega'))$$

$$= l^\tau_{\omega_{n-1}}\text{form } (\bigcup_{i\in I} l^\tau_{\omega_{n-1}}\text{form } (A/A_{\omega d} \mid A \in f_i(\omega')))$$

$$= l^\tau_{\omega_{n-1}}\text{form } (\bigcup_{i\in I} h_i(\omega')) = (\vee^\tau_{\omega_{n-1}}(h_i \mid i \in I))(\omega') = h(\omega').$$

Consequently, $\mathfrak{F} = \mathfrak{M}$. This proves the lemma.

A subgroup functor $\tau$ is said to be *closed* [1] if $H \in \tau(G)$ always implies $\tau(H) \subseteq \tau(G)$. If $\tau$ is a subgroup functor, we denote by $\overline{\tau}$ the intersection of all closed functors $\tau_i$ such that $\tau \leq \tau_i$.

For every collection of groups $\mathfrak{X}$, by $\mathrm{S}_\tau\mathfrak{X}$ we denote the set of all groups $H$ such that $H \in \tau(G)$ for a group $G \in \mathfrak{X}$ (see [1]).

**Lemma 6** [1] . *Let $\mathfrak{X}$ be a collection of groups. Then*

$$\tau\text{form } \mathfrak{X} = \mathrm{QR}_0\mathrm{S}_{\overline{\tau}}(\mathfrak{X}).$$

The intersection of all $\tau$-closed semiformations containing $\mathfrak{X}$ is called *the $\tau$-closed semiformation generated by* $\mathfrak{X}$ [1].

**Lemma 7** [1] . *Let $\mathfrak{F}$ be a $\tau$-closed semiformation generated by $\mathfrak{X}$. Then*

$$\mathfrak{F} = \mathrm{QS}_{\overline{\tau}}\mathfrak{X}.$$

Recall that a set of formations $\theta$ is called a complete lattice of formations (see [1]) if an intersection of every set of formations in $\theta$ belongs to $\theta$ and there is a formation $\mathfrak{F}$ in $\theta$ such that $\mathfrak{M} \subseteq \mathfrak{F}$ for every formation $\mathfrak{M}$ of $\theta$. A formation in $\theta$ is called a $\theta$-formation. By $\theta$form $G$ we denote the intersection of all $\theta$-formations containing a group $G$.

If $\theta$ is a complete lattice of formations and $\mathfrak{M}, \mathfrak{H} \in \theta$, then $\mathfrak{M} \bigcap \mathfrak{H}$ is the greatest lower bound for $\{\mathfrak{M}, \mathfrak{H}\}$ in $\theta$, and $\mathfrak{M} \vee_\theta \mathfrak{H}$ is the least upper bound for $\{\mathfrak{M}, \mathfrak{H}\}$ in $\theta$.

A complete lattice of formations $\theta$ is called $\mathfrak{X}$-*separated* if for every term $\xi(x_1,\dots,x_m)$ of the signature $\{\bigcap, \vee_\theta\}$, every $\theta$-formations $\mathfrak{F}_1,\dots,\mathfrak{F}_m$ and every group $A\in\mathfrak{X}\bigcap\xi(\mathfrak{F}_1,\dots,\mathfrak{F}_m)$ there exist $\mathfrak{X}$-groups $A_1\in\mathfrak{F}_1,\dots,A_m\in\mathfrak{F}_m$ such that $A\in\xi(\theta\text{form }A_1,\dots,\theta\text{form }A_m)$.

**Lemma 8.** *The lattice* $l^\tau_{\omega_n}$ *is* $\mathfrak{G}$*-separated.*

*Proof.* Let $\xi(x_1,\dots,x_m)$ be a term of the signature $\{\bigcap,\vee^\tau_{\omega_n}\}$, $\mathfrak{F}_1,\dots,\mathfrak{F}_m$ be formations in $l^\tau_{\omega_n}$ and $A\in\xi(\mathfrak{F}_1,\dots,\mathfrak{F}_m)$. We proceed by induction on the number $r$ of occurrences of the symbols in $\{\bigcap,\vee^\tau_{\omega_n}\}$ into the term $\xi$. We show that there exist groups $A_i\in\mathfrak{F}_i\quad(i=1,\dots,m)$ such that $A\in\xi(\mathfrak{M}_1,\dots,\mathfrak{M}_m)$ where $\mathfrak{M}_i=l^\tau_{\omega_n}\text{form }A_i$. It is obvious for $r=0$. We prove the assertion for $r=1$ by induction on $n$. Let $n=0$, i.e., either $A\in\mathfrak{F}_1\bigcap\mathfrak{F}_2$ or

$$A\in\mathfrak{F}_1\vee^\tau_{\omega_0}\mathfrak{F}_2=l^\tau_{\omega_0}\text{form }(\mathfrak{F}_1\bigcup\mathfrak{F}_2)=\text{form }(\mathfrak{F}_1\bigcup\mathfrak{F}_2).$$

In the first case $A\in\text{form }A\bigcap\text{form }A$. In the second case, by Lemma 6, we have $A\simeq H/N$ where

$$H\in\mathrm{R}_0(\mathfrak{F}_1\bigcup\mathfrak{F}_2).$$

Clearly $H^{\mathfrak{F}_1}\bigcap H^{\mathfrak{F}_2}=1$. Hence

$$A\in\text{form }(H/H^{\mathfrak{F}_1},H/H^{\mathfrak{F}_2})=\text{form }(H/H^{\mathfrak{F}_1})\vee$$

$$\text{form }(H/H^{\mathfrak{F}_2})\subseteq\mathfrak{F}_1\vee^\tau_{\omega_0}\mathfrak{F}_2.$$

Let $n>0$, $\{p_1,\dots,p_t\}=\pi(A)$ and $A\in\mathfrak{F}_1\vee^\tau_{\omega_n}\mathfrak{F}_2$. Then using [2], Lemma 5, and Lemma 4 we have

$$A/F_{p_i}(A)\in f_1(p_i)\vee^\tau_{\omega_{n-1}}f_2(p_i),\quad A/A_{\omega d}\in f_1(\omega')\vee^\tau_{\omega_{n-1}}f_2(\omega'),$$

where $f_j$ is the minimal $l^\tau_{\omega_{n-1}}$-valued $\omega$-local satellite of $\mathfrak{F}_j$ where $j=1,2$. By induction there exist groups $A_{i_1}\in f_1(p_i)$, $A_{i_2}\in f_2(p_i)$, $T_1\in f_1(\omega')$, $T_2\in f_2(\omega')$ such that

$$A/F_{p_i}(A)\in(l^\tau_{\omega_{n-1}}\text{form }A_{i_1})\vee^\tau_{\omega_{n-1}}(l^\tau_{\omega_{n-1}}\text{form }A_{i_2}),$$

$$A/A_{\omega d}\in(l^\tau_{\omega_{n-1}}\text{form }T_1)\vee^\tau_{\omega_{n-1}}(l^\tau_{\omega_{n-1}}\text{form }T_2).$$

Clearly,

$$(l^\tau_{\omega_{n-1}}\text{form }A_{i_1})\vee^\tau_{\omega_{n-1}}(l^\tau_{\omega_{n-1}}\text{form }A_{i_2})=l^\tau_{\omega_{n-1}}\text{form }(A_{i_1},A_{i_2}).$$

$$(l^\tau_{\omega_{n-1}}\text{form }T_1)\vee^\tau_{\omega_{n-1}}(l^\tau_{\omega_{n-1}}\text{form }T_2)=l^\tau_{\omega_{n-1}}\text{form }(T_1,T_2).$$

Let $\mathfrak{M}_1$ be a semiformation generated by $A_{i_1}$, and $\mathfrak{M}_2$ be a semiformation generated by $A_{i_2}$. By Lemma 7 we have $\mathfrak{M}_1=(A_1,\dots,A_t)$ and $\mathfrak{M}_2=(B_1,\dots,B_r)$ where $A_1,\dots,A_t\in\mathrm{QS}_{\overline{\tau}}(A_{i_1})$ and $B_1,\dots,B_r\in\mathrm{QS}_{\overline{\tau}}(A_{i_2})$. Clearly $\mathfrak{M}_1\bigcup\mathfrak{M}_2$ is a $\tau$-closed semiformation and

$$A/F_{p_i}(A)\in l^\tau_{\omega_{n-1}}\text{form }(A_{i_1},A_{i_2})=l^\tau_{\omega_{n-1}}\text{form }(\mathfrak{M}_1\bigcup\mathfrak{M}_2).$$

Hence, by Lemma 3, we may suppose that

$$|O_{p_i}(A_k)| = 1 = |O_{p_i}(B_l)|$$

for all $k = 1, \dots, t$ and $l = 1, \dots, r$. Applying Lemma 3 and analogous argument we may suppose that $(T_i)_{\omega d} = 1$, $i = 1, 2$.

Let $D_{i_1} = A_1 \times \ldots \times A_t$ and $D_{i_2} = B_1 \times \ldots \times B_r$. Then

$$|O_{p_i}(D_{i_1})| = 1 = |O_{p_i}(D_{i_2})|.$$

Besides,

$$A/F_{p_i}(A) \in l^{\tau}_{\omega_{n-1}}\text{form } (D_{i_1}, D_{i_2}) \subseteq l^{\tau}_{\omega_{n-1}}\text{form } (A_{i_1}, A_{i_2}).$$

Let $Z_i$ be a group of order $p_i$, $B_{i_1} = Z_i \wr D_{i_1}$ and $B_{i_2} = Z_i \wr D_{i_2}$. Using [2] we have $B_{i_1} \in \mathfrak{F}_1$, $B_{i_2} \in \mathfrak{F}_2$. Hence

$$A_1 = B_{1_1} \times B_{2_1} \times \ldots \times B_{t_1} \times T_1 \in \mathfrak{F}_1, \quad A_2 = B_{1_2} \times B_{2_2} \times \ldots \times B_{t_2} \times T_2 \in \mathfrak{F}_2.$$

We show that

$$A \in \mathfrak{F} = (l^{\tau}_{\omega_n}\text{form } A_1) \vee^{\tau}_{\omega_n} (l^{\tau}_{\omega_n}\text{form } A_2).$$

It suffices to prove $A/A_{\omega d} \in f(\omega')$ and $A/F_{p_i}(A) \in f(p_i)$ where $f$ is the minimal $l^{\tau}_{\omega_{n-1}}$-valued $\omega$-local satellite of $\mathfrak{F}$. Clearly $B_{i_1} \in \mathfrak{F}$. Hence $B_{i_1}/F_{p_i}(B_{i_1}) \in f(p_i)$. Since $O_{p_i}(D_{i_1}) = 1$, we have $B_{i_1}/F_{p_i}(B_{i_1}) \simeq D_{i_1}$, i.e. $D_{i_1} \in f(p_i)$. Analogously we deduce that $D_{i_2} \in f(p_i)$. Consequently,

$$A/F_{p_i}(A) \in l^{\tau}_{\omega_{n-1}}\text{form } (D_{i_1}, D_{i_2}) \subseteq f(p_i).$$

Clearly $T_1, T_2 \in \mathfrak{F}$. Hence, by [2], Lemma 5, we have

$$T_i \simeq T_i/(T_i)_{\omega d} \in f(\omega') = l^{\tau}_{\omega_{n-1}}\text{form } (G/G_{\omega d} \mid G \in \mathfrak{F}) = f(\omega').$$

Consequently, $l^{\tau}_{\omega_{n-1}}\text{form } (T_1, T_2) \subseteq f(\omega')$. Therefore $A/A_{\omega d} \in f(\omega')$. This completes the proof of the lemma for $r = 1$.

Let a term $\xi$ have $r > 1$ occurrences of the symbols in $\{\bigcap, \vee^{\tau}_{\omega_n}\}$. We suppose proving by induction that the lemma holds for terms with less number of occurrences. Assume that $\xi$ is of the form

$$\xi_1(x_{i_1}, \dots, x_{i_a}) \triangle \xi_2(x_{j_1}, \dots, x_{j_b}),$$

where $\triangle \in \{\bigcap, \vee^{\tau}_{\omega_n}\}$ and

$$\{x_{i_1}, \dots, x_{i_a}\} \bigcup \{x_{j_1}, \dots, x_{j_b}\} = \{x_1, \dots, x_m\}.$$

By $\mathfrak{H}_1$ we denote the formation $\xi_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a})$, and by $\mathfrak{H}_2$ the formation $\xi_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b})$. Then, as it was proved above, there are groups $A_1 \in \mathfrak{H}_1$ and $A_2 \in \mathfrak{H}_2$ such that $A \in l^{\tau}_{\omega_n}\text{form } A_1 \triangle l^{\tau}_{\omega_n}\text{form } A_2$. On the other hand, since the number of operations in the term $\xi_1$ is less than $r$, it follows by induction that there exist groups $B_1 \in \mathfrak{F}_{i_1}, \dots, B_a \in \mathfrak{F}_{i_a}$ such that $A_1 \in \xi_1(l^{\tau}_{\omega_n}\text{form } B_1, \dots, l^{\tau}_{\omega_n}\text{form } B_a)$. Analogously there exist groups $C_1 \in \mathfrak{F}_{j_1}, \dots, C_b \in \mathfrak{F}_{j_b}$ such that $A_2 \in \xi_2(l^{\tau}_{\omega_n}\text{form } C_1, \dots, l^{\tau}_{\omega_n}\text{form } C_b)$.

Let $x_{i_{t+1}}, \dots, x_{i_a} \in \{x_{j_1}, \dots, x_{j_b}\}$ and let $\{x_{i_1}, \dots, x_{i_t}\} \cap \{x_{j_1}, \dots, x_{j_b}\} = \varnothing$. Assume that

$$D_{i_k} = \begin{cases} B_k, & \text{for } k < t+1, \\ B_k \times C_q, & \text{where } x_{i_k} = x_{j_q}, \\ & \text{for some } q \in \{1, \dots, b\} \text{ provided that } k \geq t+1. \end{cases}$$

Let $D_{j_k} = C_k$ if $x_{j_k} \notin \{x_{i_{t+1}}, \dots, x_{i_a}\}$. By $\mathfrak{M}_p$ we denote the formation $l^\tau_{\omega_n}$form $D_{i_p}$ where $p = 1, \dots, a$, and by $\mathfrak{X}_c$ the formation $l^\tau_{\omega_n}$form $D_{j_c}$ where $c = 1, \dots, b$. Thus

$$A_1 \in \xi_1(l^\tau_{\omega_n}\text{form } B_1, \dots, l^\tau_{\omega_n}\text{form } B_a) \subseteq$$

$$\subseteq \xi_1(l^\tau_{\omega_n}\text{form } D_{i_1}, \dots, l^\tau_{\omega_n}\text{form } D_{i_a}) = \xi_1(\mathfrak{M}_1, \dots, \mathfrak{M}_a),$$

$$A_2 \in \xi_2(l^\tau_{\omega_n}\text{form } C_1, \dots, l^\tau_{\omega_n}\text{form } C_b) \subseteq$$

$$\subseteq \xi_2(l^\tau_{\omega_n}\text{form } D_{j_1}, \dots, l^\tau_{\omega_n}\text{form } D_{j_b}) = \xi_2(\mathfrak{X}_1, \dots, \mathfrak{X}_b).$$

Consequently, there exist formations $\mathfrak{K}_1, \dots, \mathfrak{K}_m$ such that

$$A \in \xi_1(\mathfrak{K}_{i_1}, \dots, \mathfrak{K}_{i_a}) \triangle \xi_2(\mathfrak{K}_{j_1}, \dots, \mathfrak{K}_{j_b}) = \xi(\mathfrak{K}_1, \dots, \mathfrak{K}_m)$$

where $\mathfrak{K}_i = l^\tau_{\omega_n}$form $K_i$ for $K_i \in \mathfrak{F}_i$. This proves the claim.

For every term $\xi$ of the signature $\{\bigcap, \vee^\tau_{\omega_n}\}$ we denote by $\overline{\xi}$ a term of the signature $\{\bigcap, \vee^\tau_{\omega_{n-1}}\}$ obtained from $\xi$ by replacing of every symbol $\vee^\tau_{\omega_n}$ by the symbol $\vee^\tau_{\omega_{n-1}}$.

**Lemma 9.** *Let $\xi(x_1, \dots, x_m)$ be a term of the signature $\{\bigcap, \vee^\tau_{\omega_n}\}$ and $f_i$ be an inner $l^\tau_{\omega_{n-1}}$-valued $\omega$-local satellite of a formation $\mathfrak{F}_i$ where $i = 1, \dots, m$. Then*

$$\xi(\mathfrak{F}_1, \dots, \mathfrak{F}_m) = LF_\omega(\overline{\xi}(f_1, \dots, f_m)).$$

*Proof.* We proceed by induction on the number $r$ of occurrences of the symbols in $\{\bigcap, \vee^\tau_{\omega_n}\}$ into $\xi$. Let

$$\xi(x_1, \dots, x_m) = \xi_1(x_{i_1}, \dots, x_{i_a}) \triangle \xi_2(x_{j_1}, \dots, x_{j_b})$$

where $\triangle \in \{\bigcap, \vee^\tau_{\omega_n}\}$,

$$\{x_{i_1}, \dots, x_{i_a}\} \bigcup \{x_{j_1}, \dots, x_{j_b}\} = \{x_1, \dots, x_m\}.$$

Assume that the lemma holds for the terms $\xi_1$ and $\xi_2$. Then

$$\xi_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) = LF_\omega(\overline{\xi}_1(f_{i_1}, \dots, f_{i_a})),$$

$$\xi_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b}) = LF_\omega(\overline{\xi}_2(f_{j_1}, \dots, f_{j_b})).$$

It is clear that $\overline{\xi}_1(f_{i_1}, \dots, f_{i_a})$ and $\overline{\xi}_2(f_{j_1}, \dots, f_{j_b})$ are inner $l^\tau_{\omega_{n-1}}$-valued $\omega$-local satellites of the formations $\xi_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a})$ and $\xi_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b})$, respectively. Hence by induction we have

$$\begin{aligned}\xi(\mathfrak{F}_1,\ldots,\mathfrak{F}_m) &= \xi_1(\mathfrak{F}_{i_1},\ldots,\mathfrak{F}_{i_a})\triangle\,\xi_2(\mathfrak{F}_{j_1},\ldots,\mathfrak{F}_{j_b})\\ &= LF_\omega(\overline{\xi}_1(f_{i_1},\ldots,f_{i_a})\overline{\triangle}\,\overline{\xi}_2(f_{j_1},\ldots,f_{j_b})) = LF_\omega(\overline{\xi}(f_1,\ldots,f_m))\end{aligned}$$

where $\overline{\triangle} = \bigcap$ if $\triangle = \bigcap$ and $\overline{\triangle} = \vee^\tau_{\omega_{n-1}}$ if $\triangle = \vee^\tau_{\omega_n}$, as claimed.

**Lemma 10.** *Let $\theta$ be a $\mathfrak{X}$-separated complete lattice of formations and $\eta$ be a sublattice of $\theta$ such that $\eta$ contains all one-generated $\theta$-subformations of the form $\theta$* form $A$, *where $A \in \mathfrak{X}$, of every formation $\mathfrak{F} \in \eta$. Suppose that a law $\xi_1 = \xi_2$ of the signature $\{\bigcap, \vee_\theta\}$ is true for all one-generated $\theta$-formations belonging to $\eta$. Then the law $\xi_1 = \xi_2$ is true for all $\theta$-formations belonging to $\eta$.*

*Proof.* Let $x_{i_1},\ldots,x_{i_a}$ be arguments occurring in the term $\xi_1$, let $x_{j_1},\ldots,x_{j_b}$ be arguments occurring in the term $\xi_2$, and let $\mathfrak{F}_{i_1},\ldots,\mathfrak{F}_{i_a};\mathfrak{F}_{j_1},\ldots,\mathfrak{F}_{j_b} \in \eta$. We show that

$$\mathfrak{F} = \xi_1(\mathfrak{F}_{i_1},\ldots,\mathfrak{F}_{i_a}) \subseteq \xi_2(\mathfrak{F}_{j_1},\ldots,\mathfrak{F}_{j_b}) = \mathfrak{M}.$$

Without loss of generality we may suppose that

$$x_{j_1},\ldots,x_{j_t} \in \{x_{i_1},\ldots,x_{i_a}\}$$

and

$$\{x_{j_{t+1}},\ldots,x_{j_b}\}\bigcap\{x_{i_1},\ldots,x_{i_a}\} = \varnothing.$$

Let $A \in \mathfrak{F}$. Then, by assumption there exist $\mathfrak{X}$-groups $A_{i_1},\ldots,A_{i_a}$ such that $A_{i_k} \in \mathfrak{F}_{i_k}$ (where $k = 1,\ldots,a$) and

$$A \in \xi_1(\theta\text{form } A_{i_1},\ldots,\theta\text{form } A_{i_a}).$$

Let

$$\mathfrak{H}_{i_k} = \theta\text{form } A_{i_k},$$

and let

$$\mathfrak{H}_{j_k} = \begin{cases}\mathfrak{H}_{i_c}, & \text{where } x_{j_k} = x_{i_c},\\ & \text{for some } c \in \{1,\ldots,a\} \text{ for all } k \in \{1,\ldots,t\},\\ \theta\text{form } B_{j_k}, & \text{for some group } B_{j_k} \in \mathfrak{F}_{j_k} \text{ provided that } k > t.\end{cases}$$

By assumption

$$\xi_1(\mathfrak{H}_{i_1},\ldots,\mathfrak{H}_{i_a}) = \xi_2(\mathfrak{H}_{j_1},\ldots,\mathfrak{H}_{j_b}).$$

But $\xi_2(\mathfrak{H}_{j_1},\ldots,\mathfrak{H}_{j_b}) \subseteq \mathfrak{M}$. Therefore $A \in \mathfrak{M}$. Thus $\mathfrak{F} \subseteq \mathfrak{M}$. The inverse inclusion can be proved analogously. Hence $\mathfrak{F} = \mathfrak{M}$, which completes the proof of this lemma.

*Proof of Theorem 1.* Fix a law

$$\xi_1(x_{i_1},\ldots,x_{i_a}) = \xi_2(x_{j_1},\ldots,x_{j_b}) \tag{1}$$

of the signature $\{\bigcap, \vee^{\tau}_{\omega_n}\}$. Let

$$\overline{\xi}_1(x_{i_1}, \ldots, x_{i_a}) = \overline{\xi}_2(x_{j_1}, \ldots, x_{j_b}) \tag{2}$$

be the same law of the signature $\{\bigcap, \vee^{\tau}_{\omega_{n-1}}\}$.

Suppose that law (2) is true in the lattice $l^{\tau}_{\omega_{n-1}}$. Let

$$\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}; \mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}$$

be $\tau$-closed $n$-multiply $\omega$-saturated formations. We show that

$$\xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) = \xi_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}).$$

Let $f_{i_c}$ be the minimal $l^{\tau}_{\omega_{n-1}}$-valued $\omega$-local satellite of $\mathfrak{F}_{i_c}$ (where $c = 1, \ldots, a$) and $f_{j_d}$ be the minimal $l^{\tau}_{\omega_{n-1}}$-valued $\omega$-local satellite of $\mathfrak{F}_{j_d}$ (where $d = 1, \ldots, b$). Then using Lemma 9 we have

$$\xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) = LF_{\omega}(\overline{\xi}_1(f_{i_1}, \ldots, f_{i_a})),$$

$$\xi_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}) = LF_{\omega}(\overline{\xi}_2(f_{j_1}, \ldots, f_{j_b})).$$

Besides, formations

$$f_{i_1}(\omega'), \ldots, f_{i_a}(\omega'); f_{j_1}(\omega'), \ldots, f_{j_b}(\omega')$$

and formations

$$f_{i_1}(p), \ldots, f_{i_a}(p); f_{j_1}(p), \ldots, f_{j_b}(p)$$

belong to the lattice $l^{\tau}_{\omega_{n-1}}$ for every prime $p \in \omega$. Then by assumption

$$\overline{\xi}_1(f_{i_1}, \ldots, f_{i_a})(p) = \overline{\xi}_1(f_{i_1}(p), \ldots, f_{i_a}(p)) =$$

$$\overline{\xi}_2(f_{j_1}(p), \ldots, f_{j_b}(p)) = \overline{\xi}_2(f_{j_1}, \ldots, f_{j_b})(p)$$

and

$$\overline{\xi}_1(f_{i_1}, \ldots, f_{i_a})(\omega') = \overline{\xi}_1(f_{i_1}(\omega'), \ldots, f_{i_a}(\omega')) =$$

$$\overline{\xi}_2(f_{j_1}(\omega'), \ldots, f_{j_b}(\omega')) = \overline{\xi}_2(f_{j_1}, \ldots, f_{j_b})(\omega').$$

Consequently,

$$\xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) = \xi_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}).$$

Thus law (1) is true in the lattice $l^{\tau}_{\omega_n}$, and the result is proved.

Now we give some corollaries of Theorem 1.

**Corollary 1.1** (A.N. Skiba [3]). *The lattice of all saturated formations is modular.*

**Corollary 1.2** (L.A. Shemetkov and A.N. Skiba [4]). *The lattice of all $n$-multiply saturated formations is modular.*

**Corollary 1.3** (Ballester-Bolinches and L. A. Shemetkov [5]). *The lattice of all $p$-saturated formations is modular.*

**Corollary 1.4** (A. N. Skiba [1]). *The lattice of all $\tau$-closed $n$-multiply saturated formations is modular.*

**Corollary 1.5** (A. N. Skiba and L. A. Shemetkov [2]). *The lattice of all $n$-multiply $\omega$-saturated formations is modular.*

**Corollary 1.6** (I. P. Shabalina [7]). *The lattice of all $\tau$-closed $n$-multiply $\omega$-saturated formations is modular.*

## 3. Proof of Theorem 2

*Proof.* Fix a law

$$\xi_1(x_{i_1},\ldots,x_{i_a}) = \xi_2(x_{j_1},\ldots,x_{j_b}) \tag{3}$$

of the signature $\{\bigcap, \vee^{\tau}_{\omega_n}\}$. Let

$$\overline{\xi}_1(x_{i_1},\ldots,x_{i_a}) = \overline{\xi}_2(x_{j_1},\ldots,x_{j_b}) \tag{4}$$

be the same law of the signature $\{\bigcap, \vee^{\tau}_{\omega_{n-1}}\}$.

Suppose that law (3) is true in the lattice $l^{\tau}_{\omega_n}$. We show that law (4) is true in the lattice $l^{\tau}_{\omega_{n-1}}$. By Lemma 10, it suffices to prove that if

$$\mathfrak{F}_{i_1},\ldots,\mathfrak{F}_{i_a};\mathfrak{F}_{j_1},\ldots,\mathfrak{F}_{j_b}$$

are every one-generated $\tau$-closed $(n-1)$-multiply $\omega$-saturated formations, then

$$\overline{\xi}_1(\mathfrak{F}_{i_1},\ldots,\mathfrak{F}_{i_a}) = \overline{\xi}_2(\mathfrak{F}_{j_1},\ldots,\mathfrak{F}_{j_b}).$$

Let

$$\mathfrak{F}_{i_1} = l^{\tau}_{\omega_{n-1}}\text{form } A_{i_1},\ldots,\mathfrak{F}_{i_a} = l^{\tau}_{\omega_{n-1}}\text{form } A_{i_a};$$

$$\mathfrak{F}_{j_1} = l^{\tau}_{\omega_{n-1}}\text{form } A_{j_1},\ldots,\mathfrak{F}_{j_b} = l^{\tau}_{\omega_{n-1}}\text{form } A_{j_b}.$$

We choose prime $p \in \omega$ such that $p \notin \pi(A_{i_1},\ldots, A_{i_a}, A_{j_1},\ldots,A_{j_b})$. Let

$$B_{i_1} = P \wr A_{i_1},\ldots,B_{i_a} = P \wr A_{i_a};$$

$$B_{j_1} = P \wr A_{j_1},\ldots,B_{j_b} = P \wr A_{j_b},$$

where $P$ is a group of order $p$. Since formations

$$\mathfrak{M}_{i_1} = l^{\tau}_{\omega_n}\text{form } B_{i_1},\ldots,\mathfrak{M}_{i_a} = l^{\tau}_{\omega_n}\text{form } B_{i_a};$$

$$\mathfrak{M}_{j_1} = l^{\tau}_{\omega_n}\text{form } B_{j_1},\ldots,\mathfrak{M}_{j_b} = l^{\tau}_{\omega_n}\text{form } B_{j_b}$$

belong to $l^{\tau}_{\omega_n}$, we have $\mathfrak{F} = \mathfrak{M}$ where

$$\mathfrak{F} = \xi_1(\mathfrak{M}_{i_1},\ldots,\mathfrak{M}_{i_a}) \text{ and } \mathfrak{M} = \xi_2(\mathfrak{M}_{j_1},\ldots,\mathfrak{M}_{j_b}).$$

Let $f_{i_c}$ be the minimal $l^{\tau}_{\omega_{n-1}}$-valued $\omega$-local satellite of $\mathfrak{M}_{i_c}$ (where $c = 1,\ldots,a$) and $f_{j_d}$ be the minimal $l^{\tau}_{\omega_{n-1}}$-valued $\omega$-local satellite of $\mathfrak{M}_{j_d}$ (where $d = 1,\ldots,b$). By Lemma 9 we have

$$\xi_1(\mathfrak{M}_{i_1},\ldots,\mathfrak{M}_{i_a}) = LF_{\omega}(\overline{\xi}_1(f_{i_1},\ldots,f_{i_a})),$$

$$\xi_2(\mathfrak{M}_{j_1}, \ldots, \mathfrak{M}_{j_b}) = LF_\omega(\overline{\xi}_2(f_{j_1}, \ldots, f_{j_b})).$$

Let $f$ and $m$ be the minimal $l^\tau_{\omega_{n-1}}$-valued $\omega$-local satellites of $\mathfrak{F}$ and $\mathfrak{M}$, respectively. Then using [2], Lemma 5 (3) and Lemma 4, we have

$$f(p) = \overline{\xi}_1(f_{i_1}, \ldots, f_{i_a})(p) = \overline{\xi}_1(f_{i_1}(p), \ldots, f_{i_a}(p))$$

and

$$m(p) = \overline{\xi}_2(f_{j_1}, \ldots, f_{j_b})(p) = \overline{\xi}_2(f_{j_1}(p), \ldots, f_{j_b}(p)).$$

Hence

$$\overline{\xi}_1(f_{i_1}(p), \ldots, f_{i_a}(p)) = \overline{\xi}_2(f_{j_1}(p), \ldots, f_{j_b}(p)).$$

Since $O_p(A_{i_c}) = 1$, by [2], Lemma 5 (3), we have $f_{i_c}(p) = \mathfrak{F}_{i_c}$ where $c = 1, \ldots, a$. Analogously $f_{j_d}(p) = \mathfrak{F}_{j_d}$ where $d = 1, \ldots, b$.

Consequently,

$$\overline{\xi}_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) = \overline{\xi}_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}),$$

i.e., law (4) is true in the lattice $l^\tau_{\omega_{n-1}}$. Thus every law of $l^\tau_{\omega_n}$ is true in $l^\tau_{\omega_0}$. Using Theorem 1, we have the result.

If $\omega = \mathbb{P}$, we write $l^\tau_n$ instead $l^\tau_{\omega_n}$. We have the following corollaries.

**Corollary 2.1** (A.N. Skiba [1]). *Let $n$ and $m$ be nonnegative integers. Then the law systems of lattices $l^\tau_n$ and $l^\tau_m$ coincide.*

If $\tau$ is trivial ($\tau(G) = \{G\}$ for every group $G$), we have the following result.

**Corollary 2.2** (L. A. Shemetkov and A. N. Skiba [4]). *Let $n$ and $m$ be nonnegative integers. Then the law systems of lattices $l_n$ and $l_m$ coincide.*

Finally, we note that V.G. Safonov proved modularity of the lattice of all totally saturated formations [11] and modularity of the lattice of all $\tau$-closed totally saturated formations [12].



## References


1. A. N. Skiba, *Algebra of Formations,* Belaruskaya navuka, Minsk, 1997.
2. L. A. Shemetkov, A. N. Skiba, Multiply $\omega$-local formations and Fitting classes of finite groups, *Siberian Advances in Mathematics* **10** (2) (2000) 112–141.
3. A. N. Skiba, On local formations of the length 5, in: *Arithmetical and Subgroup Structure of Finite Groups* (Works of Gomel Seminar), Nauka i Tekhnika, Minsk, 1986, pp. 135–149.
4. L. A. Shemetkov, A. N. Skiba, *Formations of algebraic systems*, Nauka, Main Editorial Board for Physical and Mathematical Literature, Moscow, 1989.
5. A. Ballester-Bolinches, L. A. Shemetkov On lattices of $p$-local formations of finite groups, *Math. Nachr.* **186** (1997) 57–65.
6. A. N. Skiba, L. A. Shemetkov, Multiply $\mathfrak{L}$-composition formations of finite groups, *Ukrainian Math. Journal* **52** (6) (2000) 898–913.
7. I. P. Shabalina, On the lattice of $\tau$-closed $n$-multiply $\omega$-local formations of finite groups, *Vesti NAN Belarusi. Ser. Fiz.-Mat. navuk* (2003) No. 1, 28–30.
8. K. Doerk, T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin–New York, 1992.

9. Wenbin Guo, *The Theory of Classes of Groups*, Science Press/Kluwer Academic Publishers, Beijing–New York–Dordrecht–Boston–London, 2000.
10. A. Ballester-Bolinches and L. M. Ezquerro, *Classes of Finite groups*, Springer, Dordrecht, 2006.
11. V. G. Safonov, On modularity of the lattice of totally saturated formations of finite groups *Comm. Algebra* **35** (11) (2007) 3495–3502.
12. V. G. Safonov, On modularity of the lattice of $\tau$-closed totally saturated formations of finite groups, *Ukrainian Math. Journal* **58** (6) (2006) 852–858.