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On Lattices of Formations of Finite Groups*

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Abstract. Let ω be a set of primes with $|\omega| > 1$, and $m > n \geq 0$ be integers. It is proved that the lattice of all τ-closed m-multiply ω-saturated formations is not a sublattice of the lattice of all τ-closed n-multiply ω-saturated formations.

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1 Introduction

All groups considered are finite.

In the book [9] and in the recent books [4, 1], it was demonstrated that constructions and results of the modular lattice theory are very useful tools for studying groups and group classes. In this connection, many authors obtained a series of modular lattices of group formations. In particular, Skiba proved that the lattice of all saturated formations is modular (see [9]). Further, this result was developed in different ways. In [12], the modularity of the lattice of all τ-closed n-multiply saturated formations was established for every subgroup functor τ; Ballester-Bolinches and Shemetkov [2] proved that the lattice of all p-saturated formations is modular;

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Skiba and Shemetkov [10, 13] proved the modularity of the lattice of all n-multiply ω-saturated formations and the lattice of all n-multiply $\mathfrak{L}$-composition formations; and Shabalina [7] proved the modularity of the lattice of all τ-closed n-multiply ω-saturated formations.

Since the lattice of all τ-closed formations is modular for all subgroup functors τ, the following natural question arises: *Is the lattice of all τ-closed n-multiply saturated formations is a sublattice of the lattice of all τ-closed formations?*

In this paper, we prove the following theorem:

Theorem. *Let ω be a set of primes with $|\omega| > 1$, and $m > n \geq 0$ be integers. Then the lattice of all τ-closed m-multiply ω-saturated formations is not a sublattice of the lattice of all τ-closed n-multiply ω-saturated formations.*

Thus, the answer to the above question is "no".

All unexplained notations and terminologies are standard. The reader is refereed to [8], [3], [4] and [1] if necessary.

2 Preliminaries

Recall that a group class closed under taking homomorphic images and finite subdirect products is called a formation.

In each group G, we select a system of subgroups $\tau(G)$. We say that τ is a subgroup functor if (i) $G \in \tau(G)$ for every group G; (ii) for every epimorphism $\varphi : A \to B$ and any $H \in \tau(A)$ and $T \in \tau(B)$, we have $H^{\varphi} \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$.

A formation $\mathfrak{F}$ is called τ-closed if $\tau(G) \subseteq \mathfrak{F}$ for every group G of $\mathfrak{F}$ (see [12]).

Let ω be a non-empty set of primes, and $\omega' = \mathbb{P} \setminus \omega$. Let $\pi(G)$ denote the set of all prime divisors of the order of a group G. Recall that a group G is called an ωd-group if $\omega \cap \pi(G) \neq \varnothing$. The symbols $\mathfrak{G}$, $\mathfrak{N}_p$ and $\mathfrak{G}_{p'}$ denote, respectively, the class of all groups, the class of all p-groups and the class of all p'-groups; $\mathfrak{G}_{\omega d}$ denotes the class of all groups in which every composition factor is an ωd-group. For every group class $\mathfrak{F} \supseteq (1)$, by $G_{\mathfrak{F}}$ we denote the product of all normal $\mathfrak{F}$-subgroups of the group G. In particular, we write $G_{\omega d} = G_{\mathfrak{G}_{\omega d}}$ and $F_p(G) = G_{\mathfrak{G}_{p'}\mathfrak{N}_p}$. A function of the form

$$f : \omega \cup \{\omega'\} \to \{\text{formations of groups}\}$$

is called an ω-local satellite (see [10]). For every ω-local satellite f, we define the class

$$LF_{\omega}(f) = \{G \mid G/G_{\omega d} \in f(\omega') \text{ and } G/F_p(G) \in f(p) \text{ for all } p \in \omega \cap \pi(G)\}.$$

If $\mathfrak{F}$ is a formation such that $\mathfrak{F} = LF_{\omega}(f)$ for an ω-local satellite f, then $\mathfrak{F}$ is said to be ω-saturated and f is said to be an ω-local satellite of $\mathfrak{F}$. If $\mathfrak{F} = LF_{\omega}(f)$ and $f(a) \subseteq \mathfrak{F}$ for all $a \in \omega \cup \{\omega'\}$, then f is called an inner satellite of $\mathfrak{F}$.

Every formation is 0-multiply ω-saturated by definition. For $n > 0$, a formation $\mathfrak{F}$ is called n-multiply ω-saturated if $\mathfrak{F} = LF_{\omega}(f)$ and all non-empty values of f are $(n-1)$-multiply ω-saturated formations (see [10]). If a formation $\mathfrak{F}$ is n-multiply ω-saturated for all natural numbers n, then $\mathfrak{F}$ is called totally ω-saturated.

Recall that a set of formations Θ is called a complete lattice of formations if the intersection of every set of formations in Θ belongs to Θ and there is a formation $\mathfrak{F}$ in Θ such that $\mathfrak{M} \subseteq \mathfrak{F}$ for every other formation $\mathfrak{M}$ of Θ (see [12]).

Let Θ be a complete lattice of formations. Then we denote by Θ^ω the set of all formations having an ω-local Θ-valued satellite (see [10]).

By $l^\tau_{\omega_n}$ we denote the set of all τ-closed n-multiply ω-saturated formations. With respect to inclusion, $l^\tau_{\omega_n}$ is a complete lattice (see [11] and [14]). In this lattice, for an arbitrary non-empty subset $\Sigma = \{\mathcal{H}_i \,|\, i \in \Lambda\}$ of $l^\tau_{\omega_n}$, $\cap_{i\in\Lambda}\mathcal{H}_i$ is the greatest lower bound for Σ in $l^\tau_{\omega_n}$; and $l^\tau_{\omega_n} \mathit{form}\,(\cup_{i\in\Lambda}\mathcal{H}_i)$ is the least upper bound for Σ in $l^\tau_{\omega_n}$. The symbol $l^\tau_{\omega_n}\mathit{form}\,\mathfrak{X}$ denotes the intersection of all τ-closed n-multiply ω-saturated formations containing a collection $\mathfrak{X}$ of groups.

A group class closed under taking homomorphic images is called a semiformation (see [9]).

A function $f : \omega \cup \{\omega'\} \to \{\text{formations of groups}\}$ is called $l^\tau_{\omega_n}$-valued if all its values belong to the lattice $l^\tau_{\omega_n}$.

By [10, Lemma 5], if $\mathfrak{F} = l^\tau_{\omega_n}\mathit{form}\,\mathfrak{X}$, then $\mathfrak{F} = LF_\omega(f)$, where

$$f(a) = \begin{cases} l^\tau_{\omega_{n-1}}\mathit{form}\,(G/F_p(G) \,|\, G \in \mathfrak{X}) & \text{if } a = p \in \omega \cap \pi(\mathfrak{X}), \\ \varnothing & \text{if } a = p \in \omega \setminus \pi(\mathfrak{X}), \\ l^\tau_{\omega_{n-1}}\mathit{form}\,(G/G_{\omega d} \,|\, G \in \mathfrak{X}) & \text{if } a = \omega'. \end{cases}$$

The satellite f is called the minimal $l^\tau_{\omega_{n-1}}$-valued ω-local satellite of $\mathfrak{F}$ (see [10]).

First we prove the following lemmas.

Lemma 1. *Let $\mathfrak{M}$ be a non-empty formation and $\mathfrak{F} = \mathfrak{N}_\omega\mathfrak{M}$ be an n-multiply ω-saturated formation. Then $\mathfrak{F} = LF_\omega(m)$, where m is an ω-local satellite such that $m(a) = \mathfrak{M}$ for every $a \in \omega \cup \{\omega'\}$.*

Proof. Let $\mathfrak{K} = LF_\omega(m)$. We show that $\mathfrak{K} \subseteq \mathfrak{F}$. Suppose $\mathfrak{K} \not\subseteq \mathfrak{F}$. Let G be a group of minimal order in $\mathfrak{K}\backslash\mathfrak{F}$. Then G is a monolithic group and $R = G^{\mathfrak{F}}$ is the socle of G. If R is an ω'-group, then $G_{\omega d} = 1$. Hence, $G \simeq G/G_{\omega d} \in m(\omega') = \mathfrak{M} \subseteq \mathfrak{N}_\omega\mathfrak{M} = \mathfrak{F}$, a contradiction. Consequently, R is an ωd-group, i.e., $\omega\cap\pi(R) \neq \varnothing$. Let $p \in \omega\cap\pi(R)$. If R is non-abelian, then $F_p(G) = 1$. Therefore,

$$G \simeq G/F_p(G) \in m(p) = \mathfrak{M} \subseteq \mathfrak{N}_\omega\mathfrak{M} = \mathfrak{F}.$$

This contradicts the choice of the group G. Hence, R is an abelian p-group. By assumption and by [10, Theorem 8], $\mathfrak{F}$ is ω-saturated. Then $R \not\subseteq \Phi(G)$. So $R = C_G(R) = F_p(G) = F(G) = O_p(G)$, and consequently,

$$G/R = G/O_p(G) = G/F_p(G) \in m(p) = \mathfrak{M}.$$

Hence, $G \in \mathfrak{N}_p\mathfrak{M} \subseteq \mathfrak{N}_\omega\mathfrak{M} = \mathfrak{F}$. This contradiction proves the inclusion $\mathfrak{K} \subseteq \mathfrak{F}$.

Now we show that $\mathfrak{F} \subseteq \mathfrak{K}$. Let $\mathfrak{F} \not\subseteq \mathfrak{K}$ and let G be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{K}$. Then G is a monolithic group with the socle $R = G^{\mathfrak{K}}$. Suppose that R is an ω'-group. Then $G_{\omega d} = 1$. Consequently, $G \simeq G/G_{\omega d} \in \mathfrak{M} \subseteq \mathfrak{K}$, a contradiction. Hence, R is an ωd-group, i.e., $\omega \cap \pi(R) \neq \varnothing$. Let $p \in \omega \cap \pi(R)$. If R is non-abelian, then $F_p(G) = 1$. Consequently, $G \simeq G/F_p(G) \in \mathfrak{M} \subseteq \mathfrak{K}$, a contradiction. Hence, R is an abelian p-group. Since $\mathfrak{K} = LF_\omega(m)$ is ω-saturated, it follows that $R \nsubseteq \Phi(G)$. Therefore, $R = C_G(R) = F_p(G) = F(G) = O_p(G)$. Consequently,

$$G/R = G/O_p(G) = G/F_p(G) \in \mathfrak{M} = m(p).$$

Using [10, Lemma 4], we have $G \in \mathfrak{K}$. The final contradiction implies the inclusion $\mathfrak{F} \subseteq \mathfrak{K}$. Thus, $\mathfrak{F} = \mathfrak{K}$, and the lemma is proved. □

By [10, Theorem 1], every ω-saturated formation $\mathfrak{F}$ has an ω-local satellite f such that $f(\omega') = \mathfrak{F}$ and $f(p) = \mathfrak{N}_p\, \mathit{form}\,(G/F_p(G) \mid G \in \mathfrak{F})$ for all $p \in \omega$. This satellite is called the canonical satellite of $\mathfrak{F}$.

Lemma 2. *Let n be a natural number. Then $(l^\tau_{\omega_{n-1}})^\omega = l^\tau_{\omega_n}$.*

Proof. Let $\mathfrak{F} \in (l^\tau_{\omega_{n-1}})^\omega$. By definition, $\mathfrak{F}$ has an ω-local satellite f such that every its non-empty value belongs to $l^\tau_{\omega_{n-1}}$, i.e., $\mathfrak{F}$ is n-multiply ω-saturated. We show that $\mathfrak{F}$ is τ-closed. Let $H \in \tau(G)$, where $G \in \mathfrak{F}$. Assume $p \in \omega \cap \pi(H)$ and $F_p = F_p(G)$. Since for every $a \in \omega \cup \{\omega'\}$ the formation $f(a)$ is τ-closed, it follows that $H/H_{\omega d} = H/(G_{\omega d} \cap H) \simeq HG_{\omega d}/G_{\omega d} \in f(\omega')$ and $H/F_p(H) = H/(F_p \cap H) \simeq HF_p/F_p \in f(p)$. Thus, $H/H_{\omega d} \in f(\omega')$, and for every $p \in \omega \cap \pi(H)$ we have $H/F_p(H) \in f(p)$. Consequently, $H \in \mathfrak{F}$. Thus, $\mathfrak{F}$ is τ-closed. We deduce $\mathfrak{F} \in l^\tau_{\omega_n}$. Consequently, $(l^\tau_{\omega_{n-1}})^\omega \subseteq l^\tau_{\omega_n}$.

Now let $\mathfrak{F} \in l^\tau_{\omega_n}$ and F be the canonical ω-local satellite of $\mathfrak{F}$. Obviously, $F(\omega') = \mathfrak{F} \in l^\tau_{\omega_{n-1}}$. We show that for every $p \in \omega \cap \pi(\mathfrak{F})$ we have $F(p) \in l^\tau_{\omega_{n-1}}$. By [10, Lemma 11], we have $F(p) \in l_{\omega_{n-1}}$. First we show that $F(p)$ is τ-closed. Let $H \in \tau(G)$, where $G \in F(p)$. We proceed by induction on $|G|$ to prove $H \in F(p)$. Let R be a minimal normal subgroup of G. Obviously, $HR/R \in \tau(G/R)$. Then by induction, $H/(R \cap H) \simeq HR/R \in F(p)$. Therefore, $O_p(G) \neq 1$. Then $H \in F(p) = \mathfrak{N}_pF(p)$. Besides, if G has two distinct minimal normal subgroups R and N, then

$$H \simeq H/1 = H/(R \cap N \cap H) \in F(p).$$

Let $O_p(G) = 1$ and R be a unique minimal normal subgroup of G. Then there exists a simple faithful $\mathbb{F}_pG$-module P. Let $T = [P]G$. Since the ω-local satellite F is inner and $G \in F(p)$, using [10, Lemma 4], we have $T \in \mathfrak{F}$. If $\varphi : T \to G$ is the natural epimorphism of T onto G, then obviously, $H^{\varphi^{-1}} = PH$. Hence, $PH \in \tau(T)$. Therefore, $PH \in \mathfrak{F}$. If $F_p = F_p(PH)$, then $PH/F_p \in F(p)$. Since $C_T(P) = P$, it follows that

$$F_p = O_p(PH) = O_p(PH) \cap PH = P(O_p(PH) \cap H) = PO_p(H).$$

Hence,

$$PH/F_p = PH/PO_p(H) \simeq H/O_p(H)(P \cap H) = H/O_p(H) \in F(p).$$

Consequently, $H \in F(p)$. Therefore, $F(p)$ is τ-closed, where $p \in \omega \cap \pi(\mathfrak{F})$.

Thus, $F(a)$ is τ-closed for all $a \in \omega \cup \{\omega'\}$. Hence, $F(a) \in l^\tau_{\omega_{n-1}}$. Consequently, $\mathfrak{F} \in (l^\tau_{\omega_{n-1}})^\omega$. Therefore, $l^\tau_{\omega_n} \subseteq (l^\tau_{\omega_{n-1}})^\omega$, as desired. □

Lemma 3. [12, Corollary 1.2.26] *Let $\mathfrak{X}$ be a τ-closed semiformation and $A \in \mathfrak{F} = \tau\, form\, \mathfrak{X}$. Suppose that A is a monolithic group and $A \notin \mathfrak{X}$. Then there exists a group H in $\mathfrak{F}$ and normal subgroups N, M, $N_1, \dots, N_t$, $M_1, \dots, M_t$ $(t \geq 2)$ of H such that the following statements hold:*

(i) $H/N \simeq A$, $M/N = \mathrm{Soc}(H/N)$;

(ii) $N_1 \cap \cdots \cap N_t = 1$;

(iii) *H/N_i is a monolithic $\mathfrak{X}$-group and M_i/N_i is the socle of H/N_i which is H-isomorphic to M/N;*

(iv) $M_1 \cap \cdots \cap M_t \subseteq M$.

Lemma 4. [12, Corollary 1.2.24] *Let $\{\mathfrak{M}_i \,|\, i \in I\}$ be a collection of τ-closed formations. Then $\tau\, form\, (\cup_{i\in I}\mathfrak{M}_i) = form\, (\cup_{i\in I}\mathfrak{M}_i)$.*

Let $\{\mathfrak{F}_i \,|\, i \in I\}$ be an arbitrary collection of τ-closed n-multiply ω-saturated formations. We denote $\vee^{\tau}_{\omega_n}\{\mathfrak{F}_i \,|\, i \in I\} = l^{\tau}_{\omega_n} form\, (\cup_{i\in I}\mathfrak{F}_i)$. In particular,

$$\mathfrak{M} \vee^{\tau}_{\omega_n} \mathfrak{H} = l^{\tau}_{\omega_n} form\, (\mathfrak{M} \cup \mathfrak{H}).$$

Let $\{f_i \,|\, i \in I\}$ be a collection of $l^{\tau}_{\omega_n}$-valued ω-local satellites. In this case, by $\vee^{\tau}_{\omega_n}(f_i \,|\, i \in I)$ we denote a function f such that $f(\omega') = l^{\tau}_{\omega_n} form\, (\cup_{i\in I} f_i(\omega'))$. In particular,

$$(f_1 \vee^{\tau}_{\omega_n} f_2)(\omega') = l^{\tau}_{\omega_n} form\, (f_1(\omega') \cup f_2(\omega')),$$

and for $p \in \omega$ we have $f(p) = l^{\tau}_{\omega_n} form\, (\cup_{i\in I} f_i(p))$. In particular,

$$(f_1 \vee^{\tau}_{\omega_n} f_2)(p) = l^{\tau}_{\omega_n} form\, (f_1(p) \cup f_2(p))$$

if at least one of formations $f_i(p) \neq \varnothing$. If $f_i(p) = \varnothing$ for all $i \in I$, then we suppose $f(p) = \varnothing$.

Lemma 5. *Let $\mathfrak{M}$ and $\mathfrak{H}$ be τ-closed n-multiply ω-saturated formations. Then*

$$(\mathfrak{N}_\omega \mathfrak{M}) \vee^{\tau}_{\omega_{n+1}} (\mathfrak{N}_\omega \mathfrak{H}) = \mathfrak{N}_\omega (\mathfrak{M} \vee^{\tau}_{\omega_n} \mathfrak{H}).$$

Proof. Let $\mathfrak{M}_1 = \mathfrak{N}_\omega \mathfrak{M}$, $\mathfrak{H}_1 = \mathfrak{N}_\omega \mathfrak{H}$ and $\mathfrak{F} = \mathfrak{N}_\omega(\mathfrak{M} \vee^{\tau}_{\omega_n} \mathfrak{H})$. By [10, Theorem 7], these formations have inner $l^{\tau}_{\omega_n}$-valued ω-local satellites m, h and f such that for every $a \in \omega \cup \{\omega'\}$ we have $m(a) = \mathfrak{M}$, $h(a) = \mathfrak{H}$ and $f(a) = \mathfrak{M} \vee^{\tau}_{\omega_n} \mathfrak{H}$.

On the other hand, by [14, Lemma 4],

$$(\mathfrak{N}_\omega \mathfrak{M}) \vee^{\tau}_{\omega_{n+1}} (\mathfrak{N}_\omega \mathfrak{H}) = LF_\omega(m \vee^{\tau}_{\omega_n} h),$$

where for every $a \in \omega \cup \{\omega'\}$ we have

$$\begin{aligned}(m \vee^{\tau}_{\omega_n} h)(a) &= m(a) \vee^{\tau}_{\omega_n} h(a) = l^{\tau}_{\omega_n} form\, (m(a) \cup h(a)) \\ &= l^{\tau}_{\omega_n} form\, (\mathfrak{M} \cup \mathfrak{H}) = \mathfrak{M} \vee^{\tau}_{\omega_n} \mathfrak{H} = f(a).\end{aligned}$$

Hence, $m \vee^{\tau}_{\omega_n} h = f$. Therefore,

$$(\mathfrak{N}_\omega \mathfrak{M}) \vee^{\tau}_{\omega_{n+1}} (\mathfrak{N}_\omega \mathfrak{H}) = LF_\omega(m \vee^{\tau}_{\omega_n} h) = LF_\omega(f) = \mathfrak{N}_\omega(\mathfrak{M} \vee^{\tau}_{\omega_n} \mathfrak{H}).$$

This proves the lemma. □

Lemma 6. *Let $|\omega| > 1$ and $\mathfrak{H}$ be a non-empty formation. Suppose that the product $\mathfrak{N}_\omega \mathfrak{H}$ is an n-multiply ω-saturated formation ($n \geq 1$). Then $\mathfrak{H}$ is $(n-1)$-multiply ω-saturated.*

Proof. We proceed by induction on n. Since every formation is 0-multiply ω-saturated by definition, it follows that for $n = 1$ the lemma is true.

We assume that $n > 1$ and the desired assertion holds for $n-1$. Lemma 1 implies that $\mathfrak{F} = \mathfrak{N}_\omega \mathfrak{H}$ has an inner ω-local satellite f such that $f(a) = \mathfrak{H}$ for every $a \in \omega \cup \{\omega'\}$. Hence, by [10, Remark 1], the canonical ω-local satellite F of $\mathfrak{F}$ has the form $F(\omega') = \mathfrak{F}$ and $F(p) = \mathfrak{N}_p \mathfrak{H}$ for all $p \in \omega$. By assumption and Lemma 2, $\mathfrak{F}$ has an $l_{\omega_{n-1}}$-valued ω-local satellite h. Then by [10, Remark 2], $F(\omega') = \mathfrak{F}$ and $F(p) = \mathfrak{N}_p(h(p) \cap \mathfrak{F}) \in l_{\omega_{n-1}} = \mathfrak{N}_p \mathfrak{H}$ for all $p \in \omega$. Therefore, the formation $\mathfrak{N}_p \mathfrak{H}$ is $(n-1)$-multiply ω-saturated for all $p \in \omega$ by [10, Corollary 9].

We show that

$$\mathfrak{N}_p l^\tau_{\omega_{n-2}} \mathit{form}\,(G/F_p(G) \,|\, G \in \mathfrak{H}) \subseteq \mathfrak{H}$$

for all $p \in \omega$. Since $\mathfrak{N}_p \mathfrak{H}$ is $(n-1)$-multiply ω-saturated, by [10, Theorem 2], we have $\mathfrak{N}_p l_{\omega_{n-2}} \mathit{form}\,(G/F_p(G) \,|\, G \in \mathfrak{N}_p \mathfrak{H}) \subseteq \mathfrak{N}_p \mathfrak{H}$ for every $p \in \omega$. Let $p \in \omega$. Suppose $q \in \omega \setminus \{p\}$. For every group $G \in \mathfrak{N}_p \mathfrak{H}$, we have $G^{\mathfrak{H}} \subseteq O_{q'}(G)$. Hence, $F_q(G/G^{\mathfrak{H}}) = F_q(G)/G^{\mathfrak{H}}$ and therefore

$$\begin{aligned} &G/F_q(G) \simeq (G/G^{\mathfrak{H}})/(F_q(G)/G^{\mathfrak{H}}) \\ =\ &(G/G^{\mathfrak{H}})/F_q(G/G^{\mathfrak{H}}) \in l^\tau_{\omega_{n-2}} \mathit{form}\,(G/F_q(G) \,|\, G \in \mathfrak{H}). \end{aligned}$$

Consequently,

$$l_{\omega_{n-2}} \mathit{form}\,(G/F_q(G) \,|\, G \in \mathfrak{N}_p \mathfrak{H}) = l_{\omega_{n-2}} \mathit{form}\,(G/F_q(G) \,|\, G \in \mathfrak{H}).$$

It follows that

$$\mathfrak{N}_q l_{\omega_{n-2}} \mathit{form}\,(G/F_q(G) \,|\, G \in \mathfrak{N}_p \mathfrak{H}) = \mathfrak{N}_q l_{\omega_{n-2}} \mathit{form}\,(G/F_q(G) \,|\, G \in \mathfrak{H}) \subseteq \mathfrak{N}_p \mathfrak{H}. \quad (1)$$

Now we show that

$$\mathfrak{N}_q l_{\omega_{n-2}} \mathit{form}\,(G/F_q(G) \,|\, G \in \mathfrak{H}) \subseteq \mathfrak{H}. \quad (2)$$

Assume false. Let A be a group of minimal order in

$$\mathfrak{N}_q l_{\omega_{n-2}} \mathit{form}\,(G/F_q(G) \,|\, G \in \mathfrak{H}) \setminus \mathfrak{H}.$$

Let $R = A^{\mathfrak{H}}$ be the socle of A. By (1), we have $A \in \mathfrak{N}_p \mathfrak{H}$. Hence, $R \subseteq O_p(A)$. Moreover, $A^{l_{\omega_{n-2}} \mathit{form}\,(G/F_q(G) \,|\, G \in \mathfrak{H})} \subseteq O_q(A)$. Therefore, $A^{l_{\omega_{n-2}} \mathit{form}\,(G/F_q(G) \,|\, G \in \mathfrak{H})} = 1$. Consequently, $A \in l_{\omega_{n-2}} \mathit{form}\,(G/F_q(G) \,|\, G \in \mathfrak{H})$. Note that

$$l_{\omega_{n-2}} \mathit{form}\,(G/F_q(G) \,|\, G \in \mathfrak{H}) \subseteq \mathfrak{H}.$$

In fact, as it was proved above, $\mathfrak{N}_p \mathfrak{H}$ is $(n-1)$-multiply ω-saturated and hence is $(n-2)$-multiply ω-saturated. By induction, $\mathfrak{H}$ is $(n-2)$-multiply ω-saturated.

Consequently, $l_{\omega_{n-2}} form\,(G/F_q(G)\,|\,G \in \mathfrak{H}) \subseteq \mathfrak{H}$. Hence, $A \in \mathfrak{H}$. This contradiction proves the inclusion (2).

Since $\mathfrak{N}_q\mathfrak{H}$ is $(n-1)$-multiply ω-saturated, it analogously follows that

$$\mathfrak{N}_p l_{\omega_{n-2}} form\,(G/F_p(G)\,|\,G \in \mathfrak{H}) \subseteq \mathfrak{H}.$$

Thus, $\mathfrak{N}_p l_{\omega_{n-2}} form\,(G/F_p(G)\,|\,G \in \mathfrak{H}) \subseteq \mathfrak{H}$ for all $p \in \omega$. Hence, by [10, Theorem 2], $\mathfrak{H}$ is $(n-1)$-multiply ω-saturated as desired. □

3 Proof of the Theorem

Since for any formations $\mathfrak{M}, \mathfrak{H} \in l^{\tau}_{\omega_n}$, we have

$$\mathfrak{M} \vee^{\tau}_{\omega_n} \mathfrak{H} \subseteq \mathfrak{M} \vee^{\tau}_{\omega_{n+1}} \mathfrak{H} \subseteq \cdots \subseteq \mathfrak{M} \vee^{\tau}_{\omega_m} \mathfrak{H},$$

it suffices to prove the theorem for $m = n+1$.

We proceed by induction on n. Let $n = 0$. We consider the formation $\mathfrak{F} = \mathfrak{N}_p\mathfrak{N}_r \vee \mathfrak{N}_p\mathfrak{N}_q$, where $p \in \omega$ but p, r and q are pairwise distinct primes. We show that $\mathfrak{F}$ is not saturated. Assume false and let f be the minimal ω-local satellite of $\mathfrak{F}$. It is easy to see that $f(p) = \mathfrak{N}_{\{r,q\}}$. Let Z_r and Z_q be groups of orders r and q, respectively. By [3, B, Corollary 10.7], the group $B = Z_r \times Z_q$ has a simple faithful module P over the field $\mathbb{F}_p$. Let $G = [P]B$. Using [10, Lemma 4], we have $G \in \mathfrak{F}$. Clearly, $G \notin \mathfrak{N}_p\mathfrak{N}_r \cup \mathfrak{N}_p\mathfrak{N}_q$. Hence, by Lemma 3, there exists a group H in $\mathfrak{F}$ and normal subgroups $N, M, N_1, \ldots, N_t, M_1, \ldots, M_t$ $(t \geq 2)$ of H such that the following statements hold: (i) $H/N \simeq G$, $M/N = \mathrm{Soc}(H/N)$; (ii) $N_1 \cap \cdots \cap N_t = 1$; (iii) H/N_i is a monolithic $(\mathfrak{N}_p\mathfrak{N}_r \cup \mathfrak{N}_p\mathfrak{N}_q)$-group and M_i/N_i is the socle of H/N_i which is H-isomorphic to M/N. Let $H/N_1 \in \mathfrak{N}_p\mathfrak{N}_r$. Since $C_G(P) = P$, it follows that $M = C_H(M/N)$. Hence, $M_1 \subseteq M$. Consequently, $B = Z_r \times Z_q \in \mathfrak{N}_p\mathfrak{N}_r$. This contradiction proves that $\mathfrak{F}$ is not ω-saturated. Since the formations $\mathfrak{N}_p$, $\mathfrak{N}_r$ and $\mathfrak{N}_q$ are s-closed and ω-saturated, it follows that both formations $\mathfrak{N}_p\mathfrak{N}_r$ and $\mathfrak{N}_p\mathfrak{N}_q$ are τ-closed and ω-saturated. By Lemma 4,

$$\begin{aligned}\mathfrak{N}_p\mathfrak{N}_r \vee^{\tau}_{\omega_1} \mathfrak{N}_p\mathfrak{N}_q &= l^{\tau}_{\omega_1} form\,(\mathfrak{N}_p\mathfrak{N}_r \cup \mathfrak{N}_p\mathfrak{N}_q)\\ &= form\,(\mathfrak{N}_p\mathfrak{N}_r \cup \mathfrak{N}_p\mathfrak{N}_q) = \mathfrak{N}_p\mathfrak{N}_r \vee \mathfrak{N}_p\mathfrak{N}_q,\end{aligned}$$

hence the lattice $l^{\tau}_{\omega_1}$ is not a sublattice of $l^{\tau}_{\omega_0}$.

Now we assume that $n > 1$ and the theorem holds for $n-1$. Then there exist τ-closed n-multiply ω-saturated formations $\mathfrak{M}$ and $\mathfrak{H}$ such that $\mathfrak{M} \vee^{\tau}_{\omega_{n-1}} \mathfrak{H} \notin l^{\tau}_{\omega_n}$. Let $\mathfrak{M}_1 = \mathfrak{N}_\omega\mathfrak{M}$ and $\mathfrak{H}_1 = \mathfrak{N}_\omega\mathfrak{H}$. By [10, Theorem 7], $\mathfrak{M}_1$ and $\mathfrak{H}_1$ have inner $l^{\tau}_{\omega_n}$-valued satellites m and h such that $m(a) = \mathfrak{M}$ and $h(a) = \mathfrak{H}$ for every $a \in \omega \cup \{\omega'\}$. Hence, both $\mathfrak{M}_1$ and $\mathfrak{H}_1$ belong to $l^{\tau}_{\omega_{n+1}}$. Suppose $\mathfrak{M}_1 \vee^{\tau}_{\omega_n} \mathfrak{H}_1 \in l^{\tau}_{\omega_{n+1}}$. By Lemma 5, we have

$$\mathfrak{M}_1 \vee^{\tau}_{\omega_n} \mathfrak{H}_1 = \mathfrak{N}_\omega(\mathfrak{M} \vee^{\tau}_{\omega_{n-1}} \mathfrak{H}),$$

and

$$\mathfrak{M}_1 \vee^{\tau}_{\omega_n} \mathfrak{H}_1 = l^{\tau}_{\omega_n} form\,(\mathfrak{M}_1 \cup \mathfrak{H}_1) = l^{\tau}_{\omega_{n+1}} form\,(\mathfrak{M}_1 \cup \mathfrak{H}_1) = \mathfrak{M}_1 \vee^{\tau}_{\omega_{n+1}} \mathfrak{H}_1.$$

Then by Lemma 6, $\mathfrak{M} \vee^{\tau}_{\omega_{n-1}} \mathfrak{H}$ is n-multiply ω-saturated. Hence, $\mathfrak{M} \vee^{\tau}_{\omega_{n-1}} \mathfrak{H} \in l^{\tau}_{\omega_n}$. This contradicts the choice of $\mathfrak{M}$ and $\mathfrak{H}$. Thus, the lattice $l^{\tau}_{\omega_{n+1}}$ is not a sublattice of $l^{\tau}_{\omega_n}$ and the theorem is proved. □

As an immediate consequence, we have:

Corollary 1. [12] *Let m and n be non-negative integers with $m > n$. Then the lattice l^{τ}_m is not a sublattice of l^{τ}_n.*

By l_m we denote the lattice of all m-multiply saturated formations.

Corollary 2. *Let m and n be non-negative integers with $m > n$. Then the lattice l_m is not a sublattice of l_n.*

Finally, we note that Safonov proved the modularity of the lattice of all totally saturated formations [6] and the modularity of the lattice of all τ-closed totally saturated formations [5]. We also note that in [11] and [14], it was proved that every law of the lattice of all τ-closed formations is fulfilled in the lattice of all τ-closed n-multiply ω-saturated formations ($n > 0$).

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