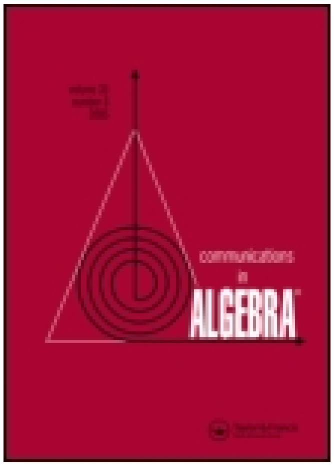

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On Factorizations of Subformations of One-Generated Saturated Finite Varieties

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ON FACTORIZATIONS OF SUBFORMATIONS OF ONE-GENERATED SATURATED FINITE VARIETIES

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The following is proved: If $\mathfrak{M}\mathfrak{H}$ ***is the noncancellable product of the formations*** $\mathfrak{M}$ ***and*** $\mathfrak{H}$ ***and*** $\mathfrak{M}\mathfrak{H} \subseteq \mathfrak{F}$ ***for some one-generated saturated finite variety*** $\mathfrak{F}$***, then*** $\mathfrak{M}$ ***is soluble.***

Key Words: Finite group; Finite variety; Formation of groups; One-generated saturated finite variety; Product of formations; Saturated formation.

AMS Mathematics Subject Classification: 20D10; 20F17.

1. INTRODUCTION

Throughout this article, all groups are finite. Let $\mathfrak{F}$ be a class of groups. If $1 \in \mathfrak{F}$ and G is a group, then we write $G^{\mathfrak{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$.

A *formation* is a class $\mathfrak{F}$ of groups with the following properties:

(i) Every homomorphic image of any group $G \in \mathfrak{F}$ belongs to $\mathfrak{F}$;
(ii) If $G/G^{\mathfrak{F}} \in \mathfrak{F}$ for all groups G.

A formation $\mathfrak{F}$ is called *soluble* if it consists of soluble groups. Nilpotent, metanilpotent and abelian formations can be defined similarly.

The formation $\mathfrak{F}$ is said to be: *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$; *one-generated saturated* formation if $\mathfrak{F}$ is the intersection of all saturated formations containing some fixed group; *hereditary* or *a finite variety* [1] if $H \in \mathfrak{F}$ whenever $H \leq G \in \mathfrak{F}$; *identity* if every group in $\mathfrak{F}$ is identity.

The *product* $\mathfrak{M}\mathfrak{H}$ *of the formations* $\mathfrak{M}$ and $\mathfrak{H}$ is the class of all groups G such that $G^{\mathfrak{H}} \in \mathfrak{M}$. This product is said to be *noncancellable* if $\mathfrak{M} \neq \mathfrak{M}\mathfrak{H} \neq \mathfrak{H}$.

In 2000, at the Gomel Algebraic seminar, A. N. Skiba posed the following question.

Question. *Let* $\mathfrak{R} = \mathfrak{M}\mathfrak{H}$ *be the product of the formations* $\mathfrak{M}$ *and* $\mathfrak{H}$ *and this factorization of* $\mathfrak{R}$ *is noncancellable. Suppose that* $\mathfrak{R}$ *is a subformation of some one-generated saturated formation* $\mathfrak{F}$*. What we can say then about* $\mathfrak{R}$*? In particular, is it true then that* $\mathfrak{M}$ *is soluble?*

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Under some additional conditions on $\mathfrak{N}$ (for example, if $\mathfrak{N}$ is saturated [2, 3]; a Baer-local formation [4–6]; $\mathfrak{X}$-saturated [7, 8], and so on) the answer to both these questions are known.

Here we prove, even under weaker conditions on $\mathfrak{N}$, the following theorem.

Theorem. *Let $\mathfrak{F}$ be a one-generated saturated finite variety; that is, the intersection of all hereditary saturated formations containing some fixed group. Suppose that $\mathfrak{M}\mathfrak{H} \subseteq \mathfrak{F}$, where $\mathfrak{M}$ and $\mathfrak{H}$ are nonidentity formations. Then the following statements hold*:

1) *Every simple group in $\mathfrak{M}$ is abelian*;
2) *If $\mathfrak{H} \neq \mathfrak{M}\mathfrak{H}$, then $\mathfrak{M}$ is soluble.*

From this theorem we get the positive answer to the second of the above two questions.

Corollary. *If $\mathfrak{M}\mathfrak{H}$ is the noncancellable product of the formations $\mathfrak{M}$ and $\mathfrak{H}$ and $\mathfrak{M}\mathfrak{H} \subseteq \mathfrak{F}$ for some one-generated saturated formation $\mathfrak{F}$, then $\mathfrak{M}$ is soluble.*

In the proof of our theorem we use some ideas in [3, 9].

All unexplained notations and terminologies are standard. The reader is refereed to [10–13] if necessary.

2. PRELIMINARIES

We use $A \wr B$ to denote the regular wreath product of the groups A and B. The symbol $\mathfrak{N}_p$ denotes the class of all p-groups. Let $\pi(G)$ denote the set of all prime divisors of the order of the group G and $\pi(\mathfrak{F})$ be the set $\cup\pi(G)$, where G runs through all groups in $\mathfrak{F}$.

For any function of the form

$$f : \mathbb{P} \to \{\text{formations of groups}\},$$

the symbol $LF(f)$ denotes the collection of all groups G such that either $G = 1$ or $G \neq 1$ and $G/C_G(H/K) \in f(p)$ for every chief factor H/K of G and every $p \in \pi(H/K)$. It is well known that

$$O_{p',p}(G) = \bigcap\{C_G(H/K) \mid H/K \text{ is a chief factor of } G \text{ and } p \in \pi(H/K)\}.$$

Therefore, $G \in \mathfrak{F} = LF(f)$ if and only if either $G = 1$ or $G \neq 1$, and $G/O_{p',p}(G) \in f(p)$ for all $p \in \pi(G)$.

We use $l^s\text{form}G$ to denote the intersection of all saturated finite varieties containing the group G. The symbol $s\text{form}G$ denotes the intersection of all hereditary formations containing the group G.

Lemma 2.1 ([10, Theorem 8.3]). *Let $\mathfrak{F} = l^s\text{form}G$ be a one-generated saturated finite variety. Then $\mathfrak{F} = LF(f)$, where*:

1) $f(p) = s\text{form}(G/O_{p',p}(G))$, *for all* $p \in \pi(G)$;
2) $f(p) = \varnothing$, *if* $p \in \mathbb{P}\backslash\pi(G)$.

Lemma 2.2 ([3, Lemma 3.1.9]). *Let $G = A \wr B = K \rtimes B$, where $K = \prod_{b\in B} A_1^b$ is the base group of the wreath product G and A_1 is the first copy of the group A in K. Then the following statements hold*:

1) *If L is a minimal normal subgroup of G, L_1 is the projection of L into A_1, and $L_1 \not\subseteq Z(A_1)$, then $L = \prod_{b\in B}(L \cap A_1)^b$*;
2) *If R is a minimal normal subgroup of A_1 and $R \not\subseteq Z(A_1)$, then $R_1 = \prod_{b\in B} R^b$ is a minimal normal subgroup of G*;
3) $\mathrm{Soc}(G) \subseteq \prod_{b\in B} M^b$, *where $M = \mathrm{Soc}(A_1)$*;
4) *If $L \trianglelefteq G$, $L \subseteq K \trianglelefteq G$, and M is the projection of L into A_1, then $(A_1/M) \wr B$ is a homomorphic image of the factor group G/L.*

Lemma 2.3 ([3, Lemma 3.1.5]). *Let $A \in s\mathrm{form}G$.*

1) $\exp(A) \leqslant \exp(G)$.
2) *The order of any chief factor of A does not exceed the maximal order of chief factors of G.*
3) *If $H \leq A$, then $c(H/H^{\mathfrak{N}}) \leqslant \max\{c(T/T^{\mathfrak{N}}) \mid T \leq G\}$.*

Lemma 2.4. *Let $\mathfrak{F} = l^s\mathrm{form}G$ be a one-generated saturated finite variety and $\mathfrak{M}\mathfrak{H} \subseteq \mathfrak{F}$, where $\mathfrak{M}$ and $\mathfrak{H}$ are nonidentity formations. If $B \in \mathfrak{H}$ and there is a prime p such that $p^{|G|} \mid \exp(B)$, then $|A| = p$ for all simple groups $A \in \mathfrak{M}$.*

Proof. By Lemma 2.1, $\mathfrak{F} = LF(f)$, where

$$f(p) = \begin{cases} s\mathrm{form}(G/O_{p',p}(G)), & \text{if } p \in \pi(G),\\ \varnothing, & \text{if } p \in \mathbb{P}\backslash\pi(G). \end{cases}$$

Now, write $B = B_1 \times \cdots \times B_{|G|}$, where $B_1 \cong \cdots \cong B_{|G|}$ are nonidentity groups in $\mathfrak{H}$.

Let $B \in \mathfrak{H}$ and $p^{|G|} \mid \exp(B)$ for some prime p. Suppose that $|A| = q \neq p$ for some simple group $A \in \mathfrak{M}$.

Let $D = A \wr B = K \rtimes B$, where K is the base group of D. By the hypotheses $p^{|G|}$ divides the exponent of B. Then B has a proper cyclic subgroup H and $|H| = p^{|G|}$. It is easy to see that $KO_q(B) = O_q(D) = O_{q',q}(D)$ and $KO_q(B) \cap H = 1$. Since $D \in \mathfrak{F}$, $KO_q(B) \cap H = 1$, and $KO_q(B) = O_q(D) = O_{q',q}(D)$, it follows that

$$\begin{aligned} H &\cong H/(KO_q(B) \cap H) \cong (HKO_q(B))/(KO_q(B))\\ &\leq D/(KO_q(B)) = D/O_q(D) = D/O_{q',q}(D) \in f(q) = s\mathrm{form}(G/O_{q',q}(G)). \end{aligned}$$

As $H \in s\mathrm{form}(G/O_{q',q}(G))$, Lemma 2.3 (1) yields $\exp(H) \mid \exp(G/O_{q',q}(G))$. Hence

$$\exp(H) \leqslant \exp(G/O_{q',q}(G)) \qquad \text{and} \qquad p^{|G|} = \exp(H) = |H| \leqslant |G|.$$

Clearly, $|H| = p^{|G|} > |G|$, a contradiction. Consequently, $q = p$. This proves the lemma. □

Lemma 2.5 ([11, A, Lemma 18.2]). *Let $W = X \wr G$. Suppose that $Y \lhd X$. Then $W/Y^{\natural} \cong (X/Y) \wr G$.*

Lemma 2.6 ([11, IV, Proposition 1.5]). *Let H/K be a chief factor of a group G, and let $G \in \mathfrak{F}$ for some formation $\mathfrak{F}$. Then $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$.*

Lemma 2.7. *Let $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ be the product of the nonidentity formations $\mathfrak{M}$ and $\mathfrak{H}$. Assume that every simple group in $\mathfrak{M}$ is abelian. If there exists a group $A \in \mathfrak{M}$ and a natural number n such that, for every group $B \in \mathfrak{H}$ with $|B| \geqslant n$, the $\mathfrak{H}$-residual of the wreath product $T = A \wr B$ is not contained subdirectly in the base group of T, then there exists a group Z_p of prime order p and a group D having an exponent greater than p^n such that $Z_p \in \mathfrak{M} \cap \mathfrak{H}$ and $D \in \mathfrak{H}$.*

Proof. Let $D_1 \cong \cdots \cong D_n$ be nonidentity groups in $\mathfrak{H}$. Let $B_1 = D_1 \times \cdots \times D_n$ and $G_1 = A \wr B_1 = K \rtimes B_1$, where K is the base group of the wreath product G_1. Since, by hypothesis, $G_1^{\mathfrak{H}}$ is not contained subdirectly in K, by Lemma 2.2 (4), we can see that there is a normal subgroup $M(B_1)$ of A such that $A/M(B_1)$ is a simple group and $B_2 = (A/M(B_1)) \wr B_1$ is a homomorphic image of the group $G_1/G_1^{\mathfrak{H}} \in \mathfrak{H}$. Analogously, we can also see that there is a normal subgroup $M(B_2)$ of A such that $A/M(B_2)$ is a simple group and the group $B_3 = (A/M(B_2)) \wr B_2$ is a homomorphic image of the group $G_2/G_2^{\mathfrak{H}} \in \mathfrak{H}$, where $G_2 = A \wr B_2$, and so on. Since $A \in \mathfrak{M}$, all groups in the sequence $A/M(B_1), A/M(B_2), \ldots, A/M(B_n), \ldots$ belong to the formation $\mathfrak{M}$. By our hypothesis, each simple group in $\mathfrak{M}$ is abelian. Since $|A| < \infty$, there exists a prime p and an infinite sequence of indices $i_1, i_1, \ldots, i_n, \ldots$ such that for all $j = 1, 2, \ldots$ and the order of the group $A/M(B_{i_j})$ is equal to p.

Let Z_p be a group of order p, and let $T_1 = Z_p$, $T_2 = Z_p \wr T_1, \ldots, T_n = Z_p \wr T_{n-1}, \ldots$. We are going to show that for any i there exists an index j such that the group T_i is isomorphic to a subgroup of B_{i_j}. If $i = 1$, then the result is evident. If $i > 1$, then we let j be an index such that the group T_{i-1} is isomorphic to a subgroup of B_{i_j}. But by Lemma 2.5, it is known that $T_i = Z_p \wr T_{i-1}$ is isomorphic to a subgroup of $B_{i_{j+1}} = (A/M(B_{i_j})) \wr B_{i_j}$. Hence for any natural number i, there exists a natural number j such that T_i is isomorphic to a subgroup of $B_j \in \mathfrak{H}$.

Now let P be a p-group and l the length of its composition series. In this case, by Lemma 2.5 and by induction on l, we see that the group P is isomorphic to a subgroup of some group $T_i \in \mathfrak{H}$. Hence, there is a group $T \in \mathfrak{H}$ such that $\exp(T) \geqslant p^n$. Finally, because $B_2 = (A/M(B_1)) \wr B_1 \in \mathfrak{H}$ and since $Z(B_2) \neq 1$, we have $Z_p \in \mathfrak{H}$ by Lemma 2.6.

Lemma 2.8. *Let $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$, where $\mathfrak{M}$ and $\mathfrak{H}$ are formations and $\mathfrak{N}_p\mathfrak{H} = \mathfrak{H}$ for some prime p. If for every simple group $A \in \mathfrak{M}$ we have $|A| = p$, then $\mathfrak{F} = \mathfrak{H}$.*

Proof. See page 555 in [5] or page 667 in [6].

Lemma 2.9. *Let p be a prime number and $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$, where every simple group in $\mathfrak{M}$ is of order p, then $G = A^{\mathfrak{H}} \wr (A/A^{\mathfrak{H}}) \in \mathfrak{F}$, for all groups $A \in \mathfrak{F}$.*

Proof. See page 554 in [5] or page 666 in [6].

Lemma 2.10 ([3, Lemma 3.5.20]). *Let G be a group and R be a minimal normal subgroup of G. If R is an elementary abelian p-group, then $G \in s\mathrm{form}(Z_p \wr (G/R))$.*

3. THE PROOF OF THEOREM

By Lemma 2.1, $\mathfrak{F} = LF(f)$, where

$$f(p) = \begin{cases} s\mathrm{form}(G/O_{p',p}(G)), & \text{if } p \in \pi(G), \\ \varnothing, & \text{if } p \in \mathbb{P}\backslash\pi(G). \end{cases}$$

Now, write $B = B_1 \times \cdots \times B_{|G|}$, where $B_1 \cong \cdots \cong B_{|G|}$ are nonidentity groups in $\mathfrak{H}$. We proceed our proof as follows.

Let A be a simple group in $\mathfrak{M}$ and $D = A \wr B = K \rtimes B$, where K is the base group of the wreath product D. Then, it is clear that $D \in \mathfrak{M}\mathfrak{H}$. Hence $D \in \mathfrak{F}$.

Assume that A is a non-abelian group. Then by Lemma 2.2 (2), (3), the group D is monolithic, and its monolith is K. Let $q \in \pi(K)$. Then, evidently, $O_{q',q}(D) = 1$. Since $D \in \mathfrak{F}$, it follows that

$$D/O_{q',q}(D) \cong D \in f(q) = s\mathrm{form}(G/O_{q',q}(G)).$$

Lemma 2.3 (2) supplies a contradiction. Thus every simple group in $\mathfrak{M}$ must be abelian.

Assume that $\mathfrak{M}$ contains some nonsoluble groups, and let A be a nonsoluble group of minimal order in $\mathfrak{M}$. Then it is obvious that A has a unique minimal normal subgroup P. Clearly, P is non-abelian and A/P is a soluble group. By Lemma 2.4, we see that $P \neq A$.

We now prove the following claims:

(1) *For every group $B \in \mathfrak{H}$ such that $|B| > |G|$, the $\mathfrak{H}$-residual of the wreath product $T = A \wr B$ is not contained subdirectly in the base group of T.*
Indeed, if we let $T = A \wr B = K \rtimes B$, where K is the base group of the wreath product T. Then, by Lemma 2.2 (2), the group T is monolithic, and its monolith L coincides with $P^{\natural} = \prod_{b \in B} P_1^b$, where P_1 is the monolith of the first copy A_1 of the group A in K. Assume that $T \in \mathfrak{M}\mathfrak{H}$. Then $T \in \mathfrak{F}$. Then $O_{p',p}(T) = 1$, and so

$$T \cong T/O_{p',p}(T) \in f(p) = s\mathrm{form}(G/O_{p',p}(G)).$$

Lemma 2.3 (2) supplies a contradiction. Hence $T \notin \mathfrak{M}\mathfrak{H}$ and thereby the $\mathfrak{H}$-residual of the wreath product $T = A \wr B$ is not contained subdirectly in the base group of T.

(2) *There exists a group Z_p of prime order p and a group B having an exponent greater than $p^{|G|}$ such that $Z_p \in \mathfrak{M} \cap \mathfrak{H}$ and $B \in \mathfrak{H}$.*
From the above, it is known that every simple group in $\mathfrak{M}$ is abelian. Now, let B be a group in $\mathfrak{H}$ such that $|B| > |G|$. Also, let $T = A \wr B = K \rtimes B$, where K is the base group of the wreath product T. Assume that $T^{\mathfrak{H}}$ is contained subdirectly in K. Then since $A \in \mathfrak{M}$, we have $T^{\mathfrak{H}} \in \mathfrak{M}$, and so $T \in \mathfrak{M}\mathfrak{H}$ which contradicts to (1). Hence $T^{\mathfrak{H}}$ is not contained subdirectly in K. Now by Lemma 2.7 and (1) above, the claim (2) holds.

(3) *For every group $T \in \mathfrak{M}\mathfrak{H}$, we have $T^{\mathfrak{H}} \wr (T/T^{\mathfrak{H}}) \in \mathfrak{F}$.*
In fact, by (2) and by Lemma 2.4, we know that $|H| = p$, for every simple group H in $\mathfrak{M}$. Now by using Lemma 2.9, our claim (3) holds.

(4) $\mathfrak{N}_p\mathfrak{H} = \mathfrak{H}$.

Assume that $\mathfrak{N}_p\mathfrak{H} \not\subseteq \mathfrak{H}$, and let B be a group of minimal order in $\mathfrak{N}_p\mathfrak{H}\backslash\mathfrak{H}$. Let $R = B^{\mathfrak{H}}$ be the monolith of B. Then, it is clear that R is an abelian p-group. Hence by Lemma 2.10, $B \in s\text{form}(Z_p \wr (B/R))$, where Z_p is a group of order p. Therefore, $Z_p \wr (B/R) \notin \mathfrak{H}$. Let $T = A \wr (B/R) = K \rtimes (B/R)$, where K is the base group of the wreath product T. Since P is non-abelian and P is a unique minimal normal subgroup of A, by Lemma 2.2 (2) the group T is monolithic and its monolith $L = P^{\natural} = \prod_{b\in B/R} P_1^b$, where P_1 is the monolith of the first copy of A in K. By Lemma 2.10 and recall that $A \in \mathfrak{M}$, we see that $T^{\mathfrak{H}}$ is contained subdirectly in $K \in \mathfrak{M}$. This shows that $T \in \mathfrak{F}$. Let $D = T^{|G|} = T_1 \times T_2 \times \cdots \times T_{|G|}$, where $T_1 \simeq T_2 \simeq \cdots \simeq T_{|G|} \simeq T$. Then, it is clear that $D \in \mathfrak{F}$, and so by (3), we have $E = D^{\mathfrak{H}} \wr (D/D^{\mathfrak{H}}) \in \mathfrak{F}$. Clearly, $D^{\mathfrak{H}} \subseteq T_1^{\mathfrak{H}} \times T_2^{\mathfrak{H}} \times \cdots \times T_{|G|}^{\mathfrak{H}}$. Hence $|D/D^{\mathfrak{H}}| \geqslant |T/T^{\mathfrak{H}}|^{|G|}$. Since $Z_p \in \mathfrak{H}$, we have $R \neq B$. This leads to $|T/T^{\mathfrak{H}}| > 1$, and so $t = |D/D^{\mathfrak{H}}| > |G|$. Clearly, $T^{\mathfrak{H}} \neq 1$. It is also clear that $\text{Soc}(D) = L_1 \times L_2 \times \cdots \times L_{|G|}$, where L_i is the monolith of the group T_i.

Now we show that every minimal normal subgroup of $D^{\mathfrak{H}}$ is non-abelian. Indeed, let Q be a minimal normal subgroup of $D^{\mathfrak{H}}$. Assume that Q is a q-group. Then $O_q(D^{\mathfrak{H}}) \neq 1$. Since $O_q(D^{\mathfrak{H}})\text{char}D^{\mathfrak{H}}$ and $D^{\mathfrak{H}} \lhd D$, we have $O_q(D^{\mathfrak{H}}) \lhd D$. This shows that D has a minimal normal subgroup N such that $N \subseteq O_q(D^{\mathfrak{H}})$, a contradiction. Thus every minimal normal subgroup of $D^{\mathfrak{H}}$ is non-abelian. It follows, by Lemma 2.2 (3), that there exists a minimal normal subgroup of E, say N, such that N is non-abelian and $|N| > t \geqslant |G|$.

Let $p \in \pi(N)$. Then $O_{p',p}(E) \cap N = 1$, and so from the E-isomorphism $N \cong NO_{p',p}(E)/O_{p',p}(E)$ we conclude that $E/O_{p',p}(E)$ has a chief factor $NO_{p',p}(E)/O_{p',p}(E)$ with $|NO_{p',p}(E)/O_{p',p}(E)| > t$.

Since $E \in \mathfrak{F}$, we have

$$E/O_{p',p}(E) \in f(p) = s\text{form}(G/O_{p',p}(G)).$$

But since $|N| \geqslant |P|^t > |G|$, this is clearly impossible, by Lemma 2.3 (2). Hence $\mathfrak{N}_p\mathfrak{H} \subseteq \mathfrak{H}$, and consequently, $\mathfrak{N}_p\mathfrak{H} = \mathfrak{H}$. Thus, claim (4) is established.

Now, by Lemma 2.8, we have $\mathfrak{M}\mathfrak{H} = \mathfrak{H}$. This contradiction shows that $\mathfrak{M}$ must be a soluble formation. The proof of the theorem now is completed.

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