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On a Question of the Theory of Partially Composition Formations*

(Dedicated to Professor K.P. Shum on the occasion of his 70th birthday)

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Abstract. Let $n > 0$ and ω be a non-empty set of primes. It is proved that the lattice of all τ-closed n-multiply ω-composition formations is inductive. It gives a positive answer to a question of Skiba asked in 2001 at the Gomel Algebraic Seminar.

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Introduction

All groups considered are finite. Throughout this paper, we will use ω to denote a non-empty set of primes and $\omega' = \mathbb{P}\backslash\omega$. Let $p \in \mathbb{P}$, and G a group. Recall that the subgroup $C^p(G)$ is the intersection of the centralizers of all the abelian p-chief factors of G, with $C^p(G) = G$ if G has no abelian p-chief factors. For any set of groups $\mathfrak{X}$ we denote by $\mathrm{Com}(\mathfrak{X})$ the class of all simple abelian groups A such that $A \cong H/K$, where H/K is a composition factor of $G \in \mathfrak{X}$. The symbol $R_\omega(G)$ denotes the $\mathfrak{S}_\omega$-radical of G, i.e., the product of all soluble normal ω-subgroups of G.

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Recall that a class of groups closed under taking homomorphic images and finite subdirect products is called a *formation*. A formation $\mathfrak{F}$ is called *ω-saturated* if $G/L \in \mathfrak{F}$, where $L \subseteq \Phi(G) \cap O_\omega(G)$, always implies $G \in \mathfrak{F}$. Last years new natural generalizations of ω-saturated formations were found (ω-composition formations [23], $\mathfrak{X}$-local formations [1] etc).

Let f be a function of the form

$$f : \omega \cup \{\omega'\} \to \{\text{formations of groups}\}. \qquad (*)$$

According to [23] we consider the class of groups

$$CF_\omega(f) = (G \mid G/R_\omega(G) \in f(\omega') \text{ and } G/C^p(G) \in f(p) \text{ for all } p \in \omega \cap \pi(\mathrm{Com}(G))).$$

If $\mathfrak{F}$ is a formation such that $\mathfrak{F} = CF_\omega(f)$ for a function f of the form $(*)$, then $\mathfrak{F}$ is said to be *ω-composition* and f is said to be an *ω-composition satellite* of $\mathfrak{F}$ [23].

Every formation is 0-multiply ω-composition by definition. For $n > 0$, a formation $\mathfrak{F}$ is called *n-multiply ω-composition* [23] if $\mathfrak{F} = CF_\omega(f)$ and all non-empty values of f are $(n-1)$-multiply ω-composition formations. With respect to inclusion $\subseteq$ the set of all n-multiply ω-composition formations c_n^ω is a complete lattice [23].

$\mathfrak{X}$-Local formations (see [1]) and ω-saturated formations (see [8, 22]) are important examples of ω-composition (n-multiply ω-composition) formations.

We note that n-multiply ω-composition formations and n-multiply ω-saturated formations are of great interest because they have a wide spectrum of applications in the theory of formations.

In the books [16, 21] and in the recent books [2, 5], it was demonstrated that constructions and results of lattice theory are very useful tools for studying groups and formations of groups. In 1986 Skiba [20] proved that the lattice of all saturated formations is modular. Further many applications of this result for the investigation of the structure of saturated formations were found (see [16, Chapter 4], [21, Chapters 4 and 5], and [5, Chapter 4]). Therefore, this result was developed in researches of many authors. In particular, in [16], the modularity of the lattice of all n-multiply saturated formations was established. After a while Ballester-Bolinches and Shemetkov [3] proved that the lattice of all p-saturated formations is modular. In [21] it was shown that the lattice of all τ-closed n-multiply saturated formations is modular but is not distributive for every subgroup functor τ. At the same time the lattice of all soluble totally saturated formations is distributive [21]. Skiba and Shemetkov [22, 23] proved the modularity of the lattice of all n-multiply ω-saturated formations and the lattice of all n-multiply $\mathfrak{L}$-composition formations. The modularity of the lattice of all τ-closed n-multiply ω-saturated formations and the lattice of all τ-closed ω-composition formations was established by Shabalina [14] and Zadorozhnyuk [29]. Safonov [11, 12, 13] proved the modularity and then the distributivity of the lattice of all totally saturated formations; Zhiznevsky [30], and independently Tsarev and Vorob'ev [28] proved the modularity of the lattice of all τ-closed n-multiply ω-composition formations. In [21] it was shown that for any natural m and n, the law system of the lattice of all τ-closed m-multiply saturated formations coincides with the law system of the lattice of all τ-closed n-multiply saturated formations. Later, Guo and Skiba [9] proved that the law system of the lattice of all m-multiply ω-saturated formations coincides with the law system of the lattice of all n-multiply ω-saturated formations for any infinite set of primes ω and any natural m and n. In [17] Shemetkov, Skiba and Vorob'ev extended this result to the lattices of τ-closed n-multiply ω-saturated formations. Vorob'ev, Skiba and Tsarev [27] proved that the law system of the lattice of all m-multiply ω-composition formations coincides with the law system of the lattice of all n-multiply ω-composition formations for any infinite set of primes ω and any natural m and n.

At the end of this short review we note that Guo and Shum [7] described non-nilpotent totally saturated formations $\mathfrak{F}$ such that the lattice $\mathfrak{F}/_\infty(\mathfrak{F}\cap\mathfrak{N})$ of all totally saturated formations between $\mathfrak{F}$ and $\mathfrak{F}\cap\mathfrak{N}$ is Boolean. Guo [6] described τ-closed n-multiply saturated formations $\mathfrak{F}$ such that the lattice $\mathfrak{F}/^\tau_n(\mathfrak{F}\cap\mathfrak{N})$ of all τ-closed n-multiply saturated formations between $\mathfrak{F}$ and $\mathfrak{F}\cap\mathfrak{N}$ is Boolean.

The analogous questions were investigated in the theory of fiber formations proposed by Vedernikov (see [19, 25, 26]).

In [21] the concept of inductive lattice of formations was introduced. This concept plays an important role in the research of law systems of formation lattices.

Recall that a set of formations Θ is called a *complete lattice of formations* if the intersection of every set of formations in Θ belongs to Θ and there is a formation $\mathfrak{F}$ in Θ such that $\mathfrak{M}\subseteq\mathfrak{F}$ for every other formation $\mathfrak{M}$ of Θ (see [21]). A formation in Θ is called a Θ-*formation*. Let Θ be a complete lattice of formations. We denote by Θ^{ω_c} the set of all formations having an ω-composition Θ-valued satellite (see [22, 23]). In [23, p. 901], it is proved that Θ^{ω_c} is a complete lattice of formations.

A complete lattice Θ^{ω_c} is called *inductive* if for any collection $\{\mathfrak{F}_i = CF_\omega(f_i) \mid i\in I\}$, where f_i is an integrated satellite of $\mathfrak{F}_i\in\Theta^{\omega_c}$, the following equality holds:

$$\vee_{\Theta^{\omega_c}}(\mathfrak{F}_i \mid i\in I) = CF_\omega(\vee_\Theta(f_i \mid i\in I)).$$

We note, the inductance of a lattice Θ^{ω_c}, in fact, means that a research of the operation $\vee_{\Theta^{\omega_c}}$ on the set Θ^{ω_c} can be reduced to a research of the operation $\vee_\Theta$ on the set Θ. Therefore, the inductance is an important property of the lattice Θ^{ω_c}. Bearing in mind this fact Skiba asked in 2001 at the Gomel Algebraic Seminar the following question: *Is the lattice of all τ-closed n-multiply ω-composition formations inductive?*

In this paper, we prove the following theorem which gives a positive answer to this question.

Theorem. *Let $n>0$ and ω be a non-empty set of primes. Then the lattice of all τ-closed n-multiply ω-composition formations $c^\tau_{\omega_n}$ is inductive.*

All unexplained notations and terminologies are standard. The reader is refereed to [2, 4, 5, 15, 16, 21, 23] if necessary.

1 Preliminaries

In each group G, we select a system of subgroups $\tau(G)$. We say that τ is a *subgroup functor* if (1) $G\in\tau(G)$ for every group G; and (2) for every epimorphism $\varphi : A\mapsto B$, any $H\in\tau(A)$ and $T\in\tau(B)$, we have $H^\varphi\in\tau(B)$ and $T^{\varphi^{-1}}\in\tau(A)$.

If $\tau(G) = \{G\}$, then the functor τ is called *trivial*. A formation $\mathfrak{F}$ is called τ-*closed* if $\tau(G) \subseteq \mathfrak{F}$ for every group G of $\mathfrak{F}$ (see [21]). We will consider only subgroup functors τ such that for any group G all subgroups of $\tau(G)$ are subnormal in G.

Let Θ be a complete lattice of formations. If $\mathfrak{M}, \mathfrak{H} \in \Theta$, then $\mathfrak{M} \cap \mathfrak{H}$ is the greatest lower bound for $\{\mathfrak{M}, \mathfrak{H}\}$ in Θ; and $\mathfrak{M} \vee_\Theta \mathfrak{H}$ is the least upper bound for $\{\mathfrak{M}, \mathfrak{H}\}$ in Θ. A satellite f is called Θ-*valued* if all its values belong to Θ (see [23]).

Lemmas 2.1 and 3.1 of [28] imply the following result.

Lemma 1. *Let n be a natural number. Then $(c^\tau_{\omega_{n-1}})^{\omega_c} = c^\tau_{\omega_n}$.*

We cite here some known results as lemmas which will be useful later on.

Let $\{f_i \mid i \in I\}$ be a collection of ω-composition satellites. By $\bigcap_{i\in I} f_i$ we denote the ω-composition satellite f such that $f(a) = \bigcap_{i\in I} f_i(a)$ for all $a \in \omega \cup \{\omega'\}$.

Lemma 2. [23, Lemma 2] *Let $\mathfrak{F} = \bigcap_{i\in I} \mathfrak{F}_i$, where $\mathfrak{F}_i = CF_\omega(f_i)$. Then $\mathfrak{F} = CF_\omega(f)$, where $f = \bigcap_{i\in I} f_i$.*

Lemma 3. [24, Lemma 2.8] *Let Z_p be a group of prime order p, and G be a group with $O_p(G) = 1$. Suppose that $T = Z_p \wr G = [K]G$ is the regular wreath product, where K is the base group of T. Then $K = C^p(T) = O_p(T)$.*

Lemma 4. [23, Lemma 4] *Let $\mathfrak{F} = CF_\omega(f)$. If $G/O_p(G) \in f(p) \cap \mathfrak{F}$ for some $p \in \omega$, then $G \in \mathfrak{F}$.*

Recall that a group class closed under taking homomorphic images is called a *semiformation* (see [16]). Let $\mathfrak{X}$ be a collection of groups. The symbol $\tau\mathrm{form}\mathfrak{X}$ denotes the τ-closed formation generated by $\mathfrak{X}$, i.e., the intersection of all τ-closed formations containing $\mathfrak{X}$.

Lemma 5. [21, Corollary 1.2.26] *Let $\mathfrak{X}$ be a τ-closed semiformation and $A \in \mathfrak{F} = \tau\mathrm{form}\mathfrak{X}$. Suppose that A is a monolithic group and $A \notin \mathfrak{X}$. Then there exists a group H in $\mathfrak{F}$ and normal subgroups $N, N_1, \dots, N_t; M, M_1, \dots, M_t$ ($t \geq 2$) of H such that the following statements hold:*

(1) $H/N \cong A$, $M/N = \mathrm{Soc}(H/N)$.
(2) $N_1 \cap \cdots \cap N_t = 1$.
(3) *H/N_i is a monolithic $\mathfrak{X}$-group and M_i/N_i is the socle of H/N_i which is H-isomorphic to M/N.*
(4) $M_1 \cap \cdots \cap M_t \subseteq M$.

Let τ be a subgroup functor. For any collection of groups $\mathfrak{X}$ the symbol S_τ denotes the set of groups H such that $H \in \tau(G)$ for some group $G \in \mathfrak{X}$. A class of groups $\mathfrak{F}$ is called τ-*closed* if $\mathrm{S}_\tau(\mathfrak{F}) = \mathfrak{F}$. We say that τ is a *closed subgroup functor* if for any groups G and $H \in \tau(G)$ we have $\tau(H) \subseteq \tau(G)$.

According to [21] we define a partial order $\leq$ on the set of all subgroup functors as follows: $\tau_1 \leq \tau_2$ if and only if $\tau_1(G) \subseteq \tau_2(G)$ for any group $G \in \mathfrak{X}$. By $\overline{\tau}$, we denote the intersection of all closed subgroup functors τ_i such that $\tau \leq \tau_i$. The functor $\overline{\tau}$ is called the *closure* of τ (see [21]).

Lemma 6. [21, Lemma 1.2.22] *Let $\mathfrak{X}$ be a collection of groups. Then*

$$\tau\mathrm{form}\mathfrak{X} = \mathrm{QR}_0\mathrm{S}_{\overline{\tau}}(\mathfrak{X}).$$

Lemma 7. [21, Lemma 4.1.3] *Let $N_1 \times \cdots \times N_t = \mathrm{Soc}(G)$, where N_i is a minimal normal subgroup of G $(i = 1, \ldots, t)$, $t > 1$, and $O_p(G) = 1$. Let M_i be the largest normal subgroup in G containing $N_1 \times \cdots \times N_{i-1} \times N_{i+1} \times \cdots \times N_t$, but not containing N_i for $i = 1, \ldots, t$. Then:*

(1) *For every $i \in \{1, \ldots, t\}$, $O_p(G/M_i) = 1$, G/M_i is monolithic and its socle N_iM_i/M_i is G-isomorphic to N_i.*

(2) $M_1 \cap \cdots \cap M_t = 1$.

2 Inductance of the Lattice $c^{\tau}_{\omega_n}$

Let $\{f_i \mid i \in I\}$ be the collection of all ω-composition $c^{\tau}_{\omega_{n-1}}$-valued satellites of a formation $\mathfrak{F}$. Since the lattice $c^{\tau}_{\omega_n}$ is complete, using Lemma 2, we conclude that $f = \bigcap_{i\in I} f_i$ is an ω-composition $c^{\tau}_{\omega_{n-1}}$-valued satellite of $\mathfrak{F}$. The satellite f is called *minimal*. If Θ is a complete lattice of formations, then $\Theta\mathrm{form}\mathfrak{X}$ is the intersection of all Θ-formations containing a collection of groups $\mathfrak{X}$. In particular, if $\mathfrak{X} = \{G\}$, we write $\Theta\mathrm{form}G$. Thus, $c^{\tau}_{\omega_n}\mathrm{form}\mathfrak{X}$ is the intersection of all τ-closed n-multiply ω-composition formations containing a collection of groups $\mathfrak{X}$.

The following lemma gives a description of the minimal $c^{\tau}_{\omega_{n-1}}$-valued satellite of a formation $\mathfrak{F} = c^{\tau}_{\omega_n}\mathrm{form}\mathfrak{X}$.

Lemma 8. *Let $\mathfrak{X}$ be a non-empty collection of groups, $\mathfrak{F} = c^{\tau}_{\omega_n}\mathrm{form}\mathfrak{X}$, where $n \geq 1$, let $\pi = \omega \cap \pi(\mathrm{Com}(\mathfrak{X}))$, and f the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{F}$. Then:*

(1) $f(\omega') = c^{\tau}_{\omega_{n-1}}\mathrm{form}(G/R_\omega(G) \mid G \in \mathfrak{X})$.

(2) $f(p) = c^{\tau}_{\omega_{n-1}}\mathrm{form}(G/C^p(G) \mid G \in \mathfrak{X})$ *for all* $p \in \pi$.

(3) $f(p) = \varnothing$ *for all* $p \in \omega\backslash\pi$.

(4) *If $\mathfrak{F} = CF_\omega(h)$, where h is a $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellite, then*

$$f(p) = c^{\tau}_{\omega_{n-1}}\mathrm{form}(G \mid G \in h(p) \cap \mathfrak{F},\, O_p(G) = 1)$$

for all $p \in \pi$ and

$$f(\omega') = c^{\tau}_{\omega_{n-1}}\mathrm{form}(G \mid G \in h(\omega') \cap \mathfrak{F},\, R_\omega(G) = 1).$$

Proof. (1)–(3) Let m be a $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellite such that

$$m(a) = \begin{cases} c^{\tau}_{\omega_{n-1}}\mathrm{form}(G/R_\omega(G) \mid G \in \mathfrak{X}) & \text{if } a = \omega', \\ c^{\tau}_{\omega_{n-1}}\mathrm{form}(G/C^p(G) \mid G \in \mathfrak{X}) & \text{if } a = p \in \pi, \\ \varnothing & \text{if } a = p \in \omega\backslash\pi. \end{cases}$$

We show that $m = f$. Let $\mathfrak{M} = CF_\omega(m)$. First we show that $\mathfrak{F} = \mathfrak{M}$. By Lemma 1, $\mathfrak{M}$ is a τ-closed n-multiply ω-composition formation.

If $A \in \mathfrak{X}$, then

$$A/R_\omega(A) \in c^\tau_{\omega_{n-1}}\mathrm{form}(G/R_\omega(G) \mid G \in \mathfrak{X}) = m(\omega'),$$
$$A/C^p(A) \in c^\tau_{\omega_{n-1}}\mathrm{form}(G/C^p(G) \mid G \in \mathfrak{X}) = m(p)$$

for all $p \in \pi$. Hence, $A \in \mathfrak{M}$. Consequently, $\mathfrak{X} \subseteq \mathfrak{M}$. Hence, $\mathfrak{F} \subseteq \mathfrak{M}$.

We prove the converse inclusion. Let f_1 be a $c^\tau_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{F}$. First we prove $m \le f_1$. Let $A \in \mathfrak{X} \subseteq \mathfrak{F} = CF_\omega(f_1)$. Then $A/R_\omega(A) \in f_1(\omega')$ and $A/C^p(A) \in f_1(p)$ for all $p \in \pi$. Consequently,

$$m(\omega') \subseteq c^\tau_{\omega_{n-1}}\mathrm{form} f_1(\omega') = f_1(\omega'),$$
$$m(p) \subseteq c^\tau_{\omega_{n-1}}\mathrm{form} f_1(p) = f_1(p)$$

for all $p \in \pi$. Hence, $m \le f_1$. Then $\mathfrak{M} \subseteq \mathfrak{F}$. Therefore, $\mathfrak{F} = \mathfrak{M}$ and $m = f$.

Now we prove (4). Let t be an ω-composition satellite such that

$$t(\omega') = c^\tau_{\omega_{n-1}}\mathrm{form}(G \mid G \in h(\omega') \cap \mathfrak{F},\ R_\omega(G) = 1),$$
$$t(p) = c^\tau_{\omega_{n-1}}\mathrm{form}(G \mid G \in h(p) \cap \mathfrak{F},\ O_p(G) = 1)$$

for all $p \in \pi$. We show that $t = f$. Let $A \in \mathfrak{X} \subseteq \mathfrak{F} = CF_\omega(h)$. Hence, $A/R_\omega(A) \in h(\omega') \cap \mathfrak{F}$. Since $R_\omega(A/R_\omega(A)) = 1$, it follows that

$$A/R_\omega(A) \in c^\tau_{\omega_{n-1}}\mathrm{form}(G \mid G \in h(\omega') \cap \mathfrak{F},\ R_\omega(G) = 1) = t(\omega').$$

Thus, $f(\omega') \subseteq t(\omega')$.

Since $A \in \mathfrak{X}$, it follows that $A/C^p(A) \in h(p) \cap \mathfrak{F}$ for all $p \in \omega \cap \pi(\mathrm{Com}(A))$. Since $O_p(A/C^p(A)) = 1$, it follows that

$$A/C^p(A) \in c^\tau_{\omega_{n-1}}\mathrm{form}(G \mid G \in h(p) \cap \mathfrak{F},\ O_p(G) = 1) = t(p)$$

for all $p \in \pi$. Hence, $f(p) \subseteq t(p)$ for all $p \in \pi$. Thus, $f \le t$.

Now we prove $t \le f$. Let $A \in (G \mid G \in h(\omega') \cap \mathfrak{F},\ R_\omega(G) = 1)$. Then $A \in f(\omega')$. It follows that $t(\omega') \subseteq c^\tau_{\omega_{n-1}}\mathrm{form} f(\omega') = f(\omega')$.

Let $A \in (G \mid G \in h(p) \cap \mathfrak{F},\ O_p(G) = 1)$, where $p \in \pi$. Let $T = Z_p \wr A = [K]A$, where K is the base group of T. By Lemma 3, $C^p(T) = K$. Applying the properties of regular wreath products we have $T/O_p(T) = T/K = T/C^p(T) \in h(p) \cap \mathfrak{F}$. Hence, by Lemma 4, $T \in \mathfrak{F}$. Therefore, $A \cong T/O_p(T) \in f(p)$. It follows that

$$t(p) \subseteq c^\tau_{\omega_{n-1}}\mathrm{form} f(p) = f(p).$$

Consequently, $t \le f$. Thus,

$$f(\omega') = c^\tau_{\omega_{n-1}}\mathrm{form}(G \mid G \in h(\omega') \cap \mathfrak{F},\ R_\omega(G) = 1),$$
$$f(p) = c^\tau_{\omega_{n-1}}\mathrm{form}(G \mid G \in h(p) \cap \mathfrak{F},\ O_p(G) = 1)$$

for all $p \in \pi$ and the lemma is proved. □

Lemma 9. *Let A be a monolithic group, R a non-abelian socle of A, $\mathfrak{M}$ a τ-closed semiformation and $A \in c^\tau_{\omega_n}\mathrm{form}\mathfrak{M}$, where $n \ge 0$. Then $A \in \mathfrak{M}$.*

Proof. We proceed by induction on n. Let $n = 0$. Then $A \in c^{\tau}_{\omega_0}\text{form}\mathfrak{M} = \tau\text{form}\mathfrak{M}$. Let $A \notin \mathfrak{M}$. Then, by Lemma 5, there exists a group H in $\tau\text{form}\mathfrak{M}$ and normal subgroups $N, N_1, \ldots, N_t; M, M_1, \ldots, M_t$ $(t \geq 2)$ of H such that (1) $H/N \cong A$, $M/N = \text{Soc}(H/N)$; (2) H/N_i is a monolithic $\mathfrak{M}$-group, M_i/N_i is the socle of H/N_i and $M_i/N_i \overset{H}{\cong} M/N$ for $i = 1, \ldots, t$.

Since the socle $R \cong M/N$ is non-abelian, it follows that $C_H(M/N) = N$. Besides, $M_i/N_i \overset{H}{\cong} M/N$. Hence, $N_i \subseteq N$. Therefore, $A \cong H/N \in \mathfrak{M}$, a contradiction. This completes the proof of the lemma for $n = 0$.

Let $n > 0$, and let the lemma holds for $n - 1$. Suppose that f is the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{F} = c^{\tau}_{\omega_n}\text{form}\mathfrak{M}$. Since R is non-abelian, it follows that $\pi(\text{Com}(R)) = \varnothing$. Hence, $R_\omega(A) = 1$. Consequently, by Lemma 8, we have $A \cong A/1 = A/R_\omega(A) \in f(\omega') = c^{\tau}_{\omega_{n-1}}\text{form}(G/R_\omega(G) \mid G \in \mathfrak{M})$. Therefore,

$$A \in c^{\tau}_{\omega_{n-1}}\text{form}(G/R_\omega(G) \mid G \in \mathfrak{M}) \subseteq c^{\tau}_{\omega_{n-1}}\text{form}\mathfrak{M}.$$

By induction, $A \in \mathfrak{M}$, as desired. □

Lemma 10. *Let $\mathfrak{M}$ be a semiformation and $A \in c^{\tau}_{\omega_n}\text{form}\mathfrak{M}$, where $n \geq 0$. Let $\mathfrak{M}_1 = (G/O_p(G) \mid G \in \mathfrak{M})$ and $\mathfrak{M}_2 = (G/R_\omega(G) \mid G \in \mathfrak{M})$.*

(1) *If $O_p(A) = 1$ and $p \in \omega$, then $A \in c^{\tau}_{\omega_n}\text{form}\mathfrak{M}_1$.*

(2) *If $R_\omega(A) = 1$, then $A \in c^{\tau}_{\omega_n}\text{form}\mathfrak{M}_2$.*

Proof. If $A \in \mathfrak{M}$, the result is clear. Hence, we may suppose $A \notin \mathfrak{M}$.

Suppose that A is a monolithic group and R is the socle of A. We proceed by induction on n. Let $n = 0$. Since $A \notin \mathfrak{M}$ and $A \in c^{\tau}_{\omega_0}\text{form}\mathfrak{M} = \tau\text{form}\mathfrak{M}$, by Lemma 5, there exists a group H in $\tau\text{form}\mathfrak{M}$, normal subgroups $N, N_1, \ldots, N_t$; $M, M_1, \ldots, M_t$ $(t \geq 2)$ of H such that $H/N \cong A$, $M/N = \text{Soc}(H/N)$; $N_1 \cap \cdots \cap N_t = 1$; H/N_i is a monolithic $\mathfrak{M}$-group and M_i/N_i is the socle of H/N_i which is H-isomorphic to M/N.

Since $O_p(A) = 1$ and $R_\omega(A) = 1$, by Lemma 5, we have

$$H \in \mathrm{R}_0(H/N_1, \ldots, H/N_t) \subseteq \mathrm{R}_0\mathfrak{M}_j,$$

where $j = 1, 2$. Hence, by Lemma 5(1) and Lemma 6,

$$A \cong H/N \in \mathrm{QR}_0(H/N_1, \ldots, H/N_t) = \text{form}(H/N_1, \ldots, H/N_t) \subseteq \tau\text{form}\mathfrak{M}_j,$$

where $j = 1, 2$.

Let $n > 0$. Suppose $O_p(A) = 1$. If R is non-abelian, then Lemma 9 implies $A \in \mathfrak{M}$. This contradicts the choice of A. Hence, R is a q-group, where $q \in \omega \backslash \{p\}$.

Let $\mathfrak{F} = c^{\tau}_{\omega_n}\text{form}\mathfrak{M}$ and $\mathfrak{H}_j = c^{\tau}_{\omega_n}\text{form}\mathfrak{M}_j$, where $j = 1, 2$. Let f and h_j $(j = 1, 2)$ be the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellites of $\mathfrak{F}$ and $\mathfrak{H}_j$ respectively. By Lemma 8,

$$\begin{aligned} f(\omega') &= c^{\tau}_{\omega_{n-1}}\text{form}(G/R_\omega(G) \mid G \in \mathfrak{M}), \\ f(q) &= c^{\tau}_{\omega_{n-1}}\text{form}(G/C^q(G) \mid G \in \mathfrak{M}) \end{aligned}$$

for all $q \in \omega \cap \pi(\mathrm{Com}(\mathfrak{M}))$;

$$\begin{aligned} h_j(\omega') &= c^{\tau}_{\omega_{n-1}}\mathrm{form}(G/R_\omega(G) \mid G \in \mathfrak{M}_j), \\ h_j(q) &= c^{\tau}_{\omega_{n-1}}\mathrm{form}(G/C^q(G) \mid G \in \mathfrak{M}_j) \end{aligned}$$

for all $q \in \omega \cap \pi(\mathrm{Com}(\mathfrak{M}_j))$ and $j = 1, 2$.

Since for every group G we have

$$G/R_\omega(G) \cong (G/O_p(G))/(R_\omega(G)/O_p(G)) = (G/O_p(G))/R_\omega(G/O_p(G)),$$

it follows that $f(\omega') = h_j(\omega')$ for $j = 1, 2$.

If $q \notin \omega$, then $R_\omega(A) = 1$. Therefore,

$$A \cong A/1 = A/R_\omega(A) \in f(\omega') = h_1(\omega') \subseteq \mathfrak{H}_1.$$

Let $q \in \omega$. We show that $A/R \in \mathfrak{H}_1$. Since $A \in \mathfrak{F}$, it follows that $A/R_\omega(A) \in f(\omega') = h_1(\omega') \subseteq \mathfrak{H}_1$.

Let $O_p(A/R) = 1$. Since $|A/R| < |A|$, by induction, $A/R \in \mathfrak{H}_1$.

Let $O_p(A/R) \neq 1$. Let $R \subseteq \Phi(A)$ and $D/R = O_p(A/R)$. Then D is nilpotent. Hence, $D = D_p \times D_q$, where D_p is a Sylow p-subgroup of D and D_q is a Sylow q-subgroup of D. Consequently, $D_p = O_p(D) = 1$, a contradiction. Hence, $R \not\subseteq \Phi(A)$. It follows that $R = C_A(R) = C^q(A)$.

Suppose $\omega \cap \pi(\mathrm{Com}(A/R)) = \varnothing$. Then $R_\omega(A/R) = 1$. Since $f(\omega') = h_1(\omega')$, we have

$$A/R \cong (A/R)/1 = (A/R)/R_\omega(A/R) \in f(\omega') = h_1(\omega') \subseteq \mathfrak{H}_1.$$

Consequently, $\omega \cap \pi(\mathrm{Com}(A/R)) \neq \varnothing$. Let $q \in \omega \cap \pi(\mathrm{Com}(A/R))$. Since for every group G we have

$$G/C^q(G) \cong (G/O_p(G))/(C^q(G)/O_p(G)) = (G/O_p(G))/C^q(G/O_p(G)),$$

it follows that $f(q) = h_1(q)$. Since $A \in \mathfrak{F}$, it follows that

$$A/R = A/C^q(A) \in f(q) = h_1(q) \subseteq \mathfrak{H}_1.$$

Thus, $A/R \in \mathfrak{H}_1$. Hence,

$$A/C^r(A) \cong (A/R)/(C^r(A)/R) = (A/R)/C^r(A/R) \in h_1(r)$$

for all $r \in \omega \cap \pi(\mathrm{Com}(A/R)) \backslash \{q\}$. Consequently, $A/C^r(A) \in h_1(r)$ for all $r \in \omega \cap \pi(\mathrm{Com}(A))$. Besides, since $A \in \mathfrak{F}$, it follows that $A/R_\omega(A) \in f(\omega') = h_1(\omega')$. Hence, $A \in \mathfrak{H}_1$. This proves (1).

We now prove (2). Since $A \in \mathfrak{F}$ and $R_\omega(A) = 1$, it follows that

$$\begin{aligned} &A \cong A/1 = A/R_\omega(A) \in f(\omega') = h_2(\omega') \\ &= c^{\tau}_{\omega_{n-1}}\mathrm{form}(G/R_\omega(G) \mid G \in \mathfrak{M}_2) \subseteq c^{\tau}_{\omega_{n-1}}\mathrm{form}\mathfrak{M}_2 \subseteq \mathfrak{H}_2. \end{aligned}$$

Thus, $A \in \mathfrak{H}_2$.

Now suppose $\mathrm{Soc}(A) = N_1 \times \cdots \times N_t$, where N_i is a minimal normal subgroup of A and $t > 1$. Let M_i be the largest normal subgroup of A containing $N_1 \times \cdots \times N_{i-1} \times N_{i+1} \times \cdots \times N_t$, but not containing N_i for $i = 1, \ldots, t$. Using Lemma 7, we have $A \in \mathrm{R}_0(A/M_1, \ldots, A/M_t)$. Since $A \in c^\tau_{\omega_n}\mathrm{form}\mathfrak{M}$, it follows that $A/M_i \in c^\tau_{\omega_n}\mathrm{form}\mathfrak{M}$. As we proved above, $A/M_i \in c^\tau_{\omega_n}\mathrm{form}\mathfrak{M}_1$. Consequently, $A \in c^\tau_{\omega_n}\mathrm{form}\mathfrak{M}_1$.

Considering the proof of Lemma 7 and replacing the condition $O_p(A) = 1$ by the condition $R_\omega(A) = 1$, we conclude that A/M_i is monolithic, N_iM_i/M_i is the socle of A/M_i and $R_\omega(A/M_i) = 1$ for any $i \in \{1, \ldots, t\}$. As we proved above, $A/M_i \in c^\tau_{\omega_n}\mathrm{form}\mathfrak{M}_2$. Consequently,

$$A \cong A/1 = A/(M_1 \cap \cdots \cap M_t) \in c^\tau_{\omega_n}\mathrm{form}\mathfrak{M}_2,$$

as claimed. □

Let Θ be a complete lattice of formations. Let $\{\mathfrak{F}_i \mid i \in I\}$ be an arbitrary collection of Θ-formations. We denote $\vee_\Theta(\mathfrak{F}_i \mid i \in I) = \Theta\mathrm{form}\big(\bigcup_{i\in I}\mathfrak{F}_i\big)$. In particular, $\vee^\tau_{\omega_n}(\mathfrak{F}_i \mid i \in I) = c^\tau_{\omega_n}\mathrm{form}\big(\bigcup_{i\in I}\mathfrak{F}_i\big)$. Let $\{f_i \mid i \in I\}$ be a collection of Θ-valued functions of the form $f_i : \omega \cup \{\omega'\} \to \{\text{formations of groups}\}$. In this case, by $\vee_\Theta(f_i \mid i \in I)$ we denote a function f such that $f(a) = \Theta\mathrm{form}\big(\bigcup_{i\in I} f_i(a)\big)$ for all $a \in \omega \cup \{\omega'\}$.

The following lemma is proved by direct calculation.

Lemma 11. *Let $n \geq 1$, and f_i be the minimal $c^\tau_{\omega_{n-1}}$-valued ω-composition satellite of a formation $\mathfrak{F}_i$ for $i \in I$. Then $\vee^\tau_{\omega_{n-1}}(f_i \mid i \in I)$ is the minimal $c^\tau_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{F} = \vee^\tau_{\omega_n}(\mathfrak{F}_i \mid i \in I)$.*

If $\mathfrak{F} = CF_\omega(f)$ and $f(a) \subseteq \mathfrak{F}$ for all $a \in \omega \cup \{\omega'\}$, then f is called an *integrated* satellite of $\mathfrak{F}$.

Proof of Theorem. Let $\{\mathfrak{F}_i \mid i \in I\}$ be a collection of τ-closed n-multiply ω-composition formations and f_i an integrated $c^\tau_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{F}_i$. Let $\mathfrak{F} = \vee^\tau_{\omega_n}(\mathfrak{F}_i \mid i \in I)$, $\mathfrak{M} = CF_\omega(\vee^\tau_{\omega_{n-1}}(f_i \mid i \in I))$ and h_i be the minimal $c^\tau_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{F}_i$. Then by Lemma 11, $h = \vee^\tau_{\omega_{n-1}}(h_i \mid i \in I)$ is the minimal $c^\tau_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{F}$. Since $h_i \leq f_i$ for all $i \in I$, we have $h \leq f = \vee^\tau_{\omega_{n-1}}(f_i \mid i \in I)$. Hence, $\mathfrak{F} \subseteq \mathfrak{M}$.

Suppose $\mathfrak{M} \not\subseteq \mathfrak{F}$. Let G be a group of minimal order in $\mathfrak{M}\backslash\mathfrak{F}$. Then G is a monolithic group and $R = G^{\mathfrak{F}}$ is the socle of G. Let $\omega \cap \pi(\mathrm{Com}(R)) = \varnothing$. Then $R_\omega(G) = 1$. Consequently,

$$\begin{aligned} G \cong G/1 = G/R_\omega(G) &\in f(\omega') = (\vee^\tau_{\omega_{n-1}}(f_i \mid i \in I))(\omega') \\ = c^\tau_{\omega_{n-1}}\mathrm{form}\big(\textstyle\bigcup_{i\in I} f_i(\omega')\big) &= c^\tau_{\omega_{n-1}}\mathrm{form}\big(\textstyle\bigcup_{i\in I}\mathfrak{F}_i\big). \end{aligned}$$

Hence, by Lemma 9, $G \in \bigcup_{i\in I}\mathfrak{F}_i \subseteq c^\tau_{\omega_{n-1}}\mathrm{form}\big(\bigcup_{i\in I}\mathfrak{F}_i\big) = \mathfrak{F}$, a contradiction. Consequently, $\omega \cap \pi(\mathrm{Com}(R)) \neq \varnothing$.

If R is non-abelian, then $\pi(\mathrm{Com}(R)) = \varnothing$. Hence, $\omega \cap \pi(\mathrm{Com}(R)) = \varnothing$, a contradiction. Consequently, R is a p-group, where $p \in \omega \cap \pi(\mathrm{Com}(R))$. Since $G \in \mathfrak{M} = CF_\omega(f)$, it follows that $G/R \in \mathfrak{M}$. Since $|G/R| < |G|$, by induction, we have $G/R \in \mathfrak{F} = CF_\omega(h)$. Hence,

$$(G/R)/R_\omega(G/R) = (G/R)/(R_\omega(G)/R) \cong G/R_\omega(G) \in h(\omega'),$$
$$(G/R)/C^q(G/R) = (G/R)/(C^q(G)/R) \cong G/C^q(G) = h(q)$$

for all $q \in \omega \cap \pi(\mathrm{Com}(G/R))\backslash\{p\}$. But $G \in \mathfrak{M} = CF_\omega(f)$. Hence, $G/C^p(G) \in f(p) = c^\tau_{\omega_{n-1}}\mathrm{form}\big(\bigcup_{i\in I} f_i(p)\big)$. Since $O_p(G/C^p(G)) = 1$, Lemmas 8 and 10 imply

$$\begin{aligned} & G/C^p(G) \in c^\tau_{\omega_{n-1}}\mathrm{form}\big(A/O_p(A) \mid A \in \textstyle\bigcup_{i\in I} f_i(p)\big) \\ = \; & c^\tau_{\omega_{n-1}}\mathrm{form}\big(\textstyle\bigcup_{i\in I}(A/O_p(A) \mid A \in f_i(p))\big) \\ = \; & c^\tau_{\omega_{n-1}}\mathrm{form}\big(\textstyle\bigcup_{i\in I} c^\tau_{\omega_{n-1}}\mathrm{form}(A/O_p(A) \mid A \in f_i(p))\big) \\ = \; & c^\tau_{\omega_{n-1}}\mathrm{form}\big(\textstyle\bigcup_{i\in I} h_i(p)\big) = (\vee^\tau_{\omega_{n-1}}(h_i \mid i \in I))(p) = h(p). \end{aligned}$$

Thus, $G/R_\omega(G) \in h(\omega')$ and $G/C^r(G) \in h(r)$ for all $r \in \omega \cap \pi(\mathrm{Com}(G))$. Hence, $G \in \mathfrak{F}$, a contradiction. Consequently, $\mathfrak{F} = \mathfrak{M}$. This proves the theorem. □

If τ is trivial, we have the following result.

Corollary 1. *Let $n > 0$ and ω be a non-empty set of primes. Then the lattice of all n-multiply ω-composition formations is inductive.*

If τ is trivial and $\omega = \mathbb{P}$, we have the following corollary.

Corollary 2. *Let $n > 0$. Then the lattice of all n-multiply composition formations is inductive.*

Finally, we note that the collection of all formations of lattice ordered groups is a complete Brouwerian lattice (see [10]). We also note that in [18], it was proved that the lattice of all τ-closed m-multiply ω-saturated formations is not a sublattice of the lattice of all τ-closed n-multiply ω-saturated formations, where ω is a set of primes with $|\omega| > 1$, and $m > n \geq 0$ are integers.

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