Journal of Algebra and Its Applications
Vol. 16, No. 2 (2018) 1850084 (17 pages)

DOI: 10.1142/S0219498818500846

Lattices of composition formations of finite groups and the laws

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Received 21 June 2016
Accepted 10 April 2017
Published 19 May 2017

Communicated by E. Vdovin

We prove that every law of the lattice of all τ-closed formations of finite groups is fulfilled in the lattice of all τ-closed n-multiply ω-composition formations of finite groups for every non-empty set of primes ω and every non-negative integer n. Let $n \geq 0$ and ω be an infinite set of primes. It is proved that the lattice of all τ-closed n-multiply ω-composition formations is not distributive.

Keywords: Subgroup functor; formation of groups; τ-closed n-multiply ω-composition formation; law of a lattice; distributive lattice; $\mathfrak{X}$-separated lattice of formations.

Mathematics Subject Classification: 20D10, 20F17

1. Introduction

Recall that a *variety of groups* may be defined as a non-empty class of groups closed under taking homomorphic images and subcartesian products (see [17, Chap. 1, Sec. 5, Remark 15.53]). It is well known that the lattice of all varieties of groups is modular but is not distributive (see [17]). In the universe of all finite groups the definition of a variety leads to the concept of a formation. Therefore the theory of varieties plays a part in the study of formations of finite groups. For example A. N. Skiba [27, p. 91] and [31, p. 137] proved: *the lattice of all locally finite varieties is a sublattice of the lattice of all hereditary formations.*

All groups considered are finite. Throughout this paper, we will use ω to denote a non-empty set of primes and $\omega' = \mathbb{P} \setminus \omega$. We will consider only subgroup functors

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τ such that for any group G all subgroups of $\tau(G)$ are subnormal in G. A *formation* is a class of groups closed under taking homomorphic images and finite subdirect products. Recall that a formation $\mathfrak{F}$ is called *ω-saturated* if the condition $G/\big(\Phi(G)\cap O_\omega(G)\big) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$. Further it was found generalizations of ω-saturated formations (ω-composition formations [32] and $\mathfrak{X}$-local formations [5]) which have some interesting applications. In particular Skiba [31], Guo Wenbin [12], Ballester-Bolinches and Ezquerro [6] established that the lattices of formations are very useful to study finite groups and group classes. Recently in the papers [7–9] it was proposed a new approach of formation theory application in the theory of formal languages. In 1986, Skiba (see [31]) proved that the lattice of all (saturated) formations is modular. Later it was found many applications of this result for research of a saturated formations structure (see [31, Chaps. 4 and 5; 12, Chap. 4].

Let m and n be non-negative integers. In [31], Skiba proved that the law system of the lattice of all τ-closed m-multiply saturated formations coincides with the law system of the lattice of all τ-closed n-multiply saturated formations (for all related definitions see Sec. 2). Guo and Skiba [14] showed that for any infinite set of primes ω the law system of the lattice of all m-multiply ω-saturated formations coincides with the law system of the lattice of all n-multiply ω-saturated formations. This result was extended to the lattices of functor-closed n-multiply ω-saturated formations [28]. In [37], the analogous fact established in the class of all n-multiply ω-composition formations.

In 2000, Skiba and Shemetkov proposed the question about the laws of the lattices of multiply $\mathfrak{L}$-composition formations (see [32]). We study the following general question.

Question 1.1. Let m and n be non-negative integers. Does it true that for any sugbroup functor τ and any non-empty set of primes ω the lattices $c^\tau_{\omega_m}$ and $c^\tau_{\omega_n}$ have the same system of laws?

To obtain the solution of this problem for an infinite set of primes ω is the aim of this paper. We notice the above-mentioned results on modular lattices of composition formations are special cases of Theorem 3.1. An important step towards this aim is $\mathfrak{G}$-separability of the lattice of all τ-closed n-multiply ω-composition formations (see Theorem 3.2). By Theorems 3.2 and 3.3 we obtain some applications of the main results. In particular, it is shown that the lattice $c^\tau_{\omega_n}$ is not distributive for an infinite set of primes ω (Theorem 4.1). All unexplained notations and terminologies are standard. The reader is refereed to [6, 10, 12, 13, 25, 27, 31, 32] if necessary.

2. Preliminaries

Let $p \in \mathbb{P}$, and G be a group. Recall that the subgroup $C^p(G)$ is the intersection of the centralizers of all the abelian p-chief factors of G, with $C^p(G) = G$ if G has no abelian p-chief factors. For any set of groups $\mathfrak{X}$ we denote by $\mathrm{Com}(\mathfrak{X})$ the class of all simple abelian groups A such that $A \cong H/K$ where H/K is a composition factor of $G \in \mathfrak{X}$. The symbols $\mathfrak{N}_p$ and $\mathfrak{G}_\omega$ denote the class of all p-groups and the class of all ω-groups. For every group class $\mathfrak{F} \supseteq (1)$ by $G^{\mathfrak{F}}$ we denote the intersection of all normal subgroups N such that $G/N \in \mathfrak{F}$, and by $G_{\mathfrak{F}}$ we denote the product of all normal $\mathfrak{F}$-subgroups of the group G. In particular the symbols $O_p(G)$ and $R_\omega(G)$ denote, respectively, the $\mathfrak{N}_p$-radical of G and the $\mathfrak{G}_\omega$-radical of G. We also use $A \wr B$ to denote the regular wreath product of groups A and B.

Let f be a function of the form

$$f : \omega \cup \{\omega'\} \to \{\text{formations of groups}\}. \tag{2.1}$$

According to [32] we consider the class of groups

$$CF_\omega(f) = (G \mid G/R_\omega(G) \in f(\omega') \text{ and } G/C^p(G) \in f(p) \text{ for all } p \in \omega \cap \pi(\mathrm{Com}(G))).$$

If $\mathfrak{F}$ is a formation such that $\mathfrak{F} = CF_\omega(f)$ for a function f of the form (2.1), then $\mathfrak{F}$ is said to be *ω-composition* and f is said to be an *ω-composition satellite* of $\mathfrak{F}$ [32].

Every formation is 0-multiply ω-composition by definition. For $n > 0$, a formation $\mathfrak{F}$ is called *n-multiply ω-composition* [32] if $\mathfrak{F} = CF_\omega(f)$ and all non-empty values of f are $(n-1)$-multiply ω-composition formations. We note that n-multiply ω-composition formations are useful because they have a wide spectrum of applications in the theory of formations.

Let Θ be a set of formations. A formation in Θ is called a *Θ-formation.* If the intersection of every set of Θ-formations belongs to Θ and there is a Θ-formation $\mathfrak{F}$ such that $\mathfrak{M} \subseteq \mathfrak{F}$ for every other Θ-formation $\mathfrak{M}$, then Θ is called a *complete lattice of formations* (see [31]).

Let Θ be a complete lattice of formations. If $\mathfrak{M}, \mathfrak{H} \in \Theta$, then $\mathfrak{M} \cap \mathfrak{H}$ is the greatest lower bound for $\{\mathfrak{M}, \mathfrak{H}\}$ in Θ; and $\mathfrak{M} \vee_\Theta \mathfrak{H}$ is the least upper bound for $\{\mathfrak{M}, \mathfrak{H}\}$ in Θ. A satellite f is called *Θ-valued* if all its values belong to Θ (see [32]). If $\mathfrak{F} = CF_\omega(f)$ and $f(a) \subseteq \mathfrak{F}$ for all $a \in \omega \cup \{\omega'\}$, then f is called an *inner* satellite of $\mathfrak{F}$.

We denote by Θ^{ω_c} the set of all formations having an ω-composition Θ-valued satellite (see [32, 33]). In [32, p. 901], Skiba and Shemetkov proved that Θ^{ω_c} is a complete lattice of formations.

In each group G, we select a system of subgroups $\tau(G)$. We say that τ is a *subgroup functor* (in the sense of Skiba) if

(1) $G \in \tau(G)$ for every group G;
(2) for every epimorphism $\varphi : A \mapsto B$ and any $H \in \tau(A)$ and $T \in \tau(B)$, we have $H^\varphi \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$.

If $\tau(G) = \{G\}$, then the functor τ is called *trivial.* A formation $\mathfrak{F}$ is called *τ-closed* if $\tau(G) \subseteq \mathfrak{F}$ for every group G of $\mathfrak{F}$ (see [31]).

The collection of all τ-closed n-multiply ω-composition formations $c^\tau_{\omega_n}$ is a complete lattice by an inclusion $\subseteq$. By $c^\tau_{\omega_0}$ and c^ω_n we denote the lattice of all τ-closed formations and the lattice of all n-multiply ω-composition formations, respectively.

Applying [38, Lemmas 2.1 and 3.1], we see that $(c^{\tau}_{\omega_{n-1}})^{\omega_c} = c^{\tau}_{\omega_n}$ for any positive integer n.

We cite here some known results as lemmas which will be useful later on. Let $\{f_i \mid i \in I\}$ be a collection of ω-composition satellites. By $\bigcap_{i\in I} f_i$ we denote the ω-composition satellite f such that $f(a) = \bigcap_{i\in I} f_i(a)$ for all $a \in \omega \cup \{\omega'\}$ (see [32]).

Lemma 2.1 ([32, Lemma 2]). *Let* $\mathfrak{F} = \bigcap_{i\in I} \mathfrak{F}_i$ *where* $\mathfrak{F}_i = CF_\omega(f_i)$. *Then* $\mathfrak{F} = CF_\omega(f)$ *where* $f = \bigcap_{i\in I,} f_i$.

Let $\mathfrak{X}$ be a class of groups. The symbol τ form $\mathfrak{X}$ denotes the τ-closed formation generated by $\mathfrak{X}$, i.e. the intersection of all τ-closed formations containing $\mathfrak{X}$.

Lemma 2.2 ([31, Corollary 1.2.24]). *Let* $\{\mathfrak{M}_i \mid i \in I\}$ *be a collection of* τ*-closed formations. Then*

$$\tau \text{ form}\left(\bigcup_{i\in I} \mathfrak{M}_i\right) = \text{form}\left(\bigcup_{i\in I} \mathfrak{M}_i\right).$$

Recall that a group class closed under taking homomorphic images is called a *semiformation* (see [27]). Let $\mathfrak{X}$ be a class of groups. By [31], the intersection of all τ-closed semiformations containing $\mathfrak{X}$ is called the *τ-closed semiformation generated by* $\mathfrak{X}$.

Lemma 2.3 ([31, Lemma 1.2.21]). *Let* $\mathfrak{F}$ *be a* τ*-closed semiformation generated by* $\mathfrak{X}$. *Then* $\mathfrak{F} = \mathrm{QS}_{\overline{\tau}}\mathfrak{X}$.

Let τ be a subgroup functor. For any collection of groups $\mathfrak{X}$ the symbol S_τ denotes the set of groups H such that $H \in \tau(G)$ for some group $G \in \mathfrak{X}$. A class of groups $\mathfrak{F}$ is called *τ-closed* if $\mathrm{S}_\tau(\mathfrak{F}) = \mathfrak{F}$. We say that τ is a *closed subgroup functor* if for any groups G and $H \in \tau(G)$ we have $\tau(H) \subseteq \tau(G)$.

According to [31] we define a partial order $\leq$ on the set of all subgroup functors as follows: $\tau_1 \leq \tau_2$ if and only if $\tau_1(G) \subseteq \tau_2(G)$ for any group $G \in \mathfrak{X}$. By $\overline{\tau}$ we denote the intersection of all closed subgroup functors τ_i such that $\tau \leq \tau_i$. The functor $\overline{\tau}$ is called the *closure* of τ (see [31]).

Lemma 2.4 ([31, Lemma 1.2.22]). *Let* $\mathfrak{X}$ *be a collection of groups. Then* τ form $\mathfrak{X} = \mathrm{QR}_0\mathrm{S}_{\overline{\tau}}(\mathfrak{X})$.

Let $\{f_i \mid i \in I\}$ be the collection of all ω-composition $c^{\tau}_{\omega_{n-1}}$-valued satellites of a formation $\mathfrak{F}$. Since the lattice $c^{\tau}_{\omega_n}$ is complete, using Lemma 2.1, we conclude that $f = \bigcap_{i\in I,} f_i$ is an ω-composition $c^{\tau}_{\omega_{n-1}}$-valued satellite of $\mathfrak{F}$. The satellite f is called *minimal*.

If Θ is a complete lattice of formations, then Θ form $\mathfrak{X}$ is the intersection of all Θ-formations containing a class of groups $\mathfrak{X}$. In particular, if $\mathfrak{X} = \{G\}$, we write Θ form G. Thus $c^{\tau}_{\omega_n}$ form $\mathfrak{X}$ is the intersection of all τ-closed n-multiply ω-composition formations containing a class of groups $\mathfrak{X}$. In the paper [35, Lemma 8] it is given a description of the minimal $c^{\tau}_{\omega_{n-1}}$-valued satellite of a formation $\mathfrak{F} = c^{\tau}_{\omega_n}$ form $\mathfrak{X}$.

Lemma 2.5 ([31, Corollary 4.2.8]). *The lattice of all τ-closed n-multiply saturated formations is modular but not distributive for any non-negative integer n.*

The symbol $\mathrm{fin}(\mathfrak{M})$ denotes the class of all finite groups such that $G \in \mathfrak{M}$ where $\mathfrak{M}$ is a variety of groups.

Lemma 2.6 ([31, Lemma 3.4.3]). *For every variety of groups $\mathfrak{M}$ the map* fin *of the form $\mathfrak{M} \to \mathrm{fin}\ \mathfrak{M}$ is an embedding of the lattice and semigroup of locally finite varieties into the algebra of all formations.*

Let Θ be a complete lattice of formations. Let $\{\mathfrak{F}_i \,|\, i \in I\}$ be an arbitrary collection of Θ-formations. We denote

$$\bigvee_{\Theta}(\mathfrak{F}_i \,|\, i \in I) = \Theta \text{ form } \left(\bigcup_{i\in I} \mathfrak{F}_i\right).$$

In particular, if $\Theta = c^{\tau}_{\omega_n}$ we have $\bigvee^{\tau}_{\omega_n}(\mathfrak{F}_i \,|\, i \in I) = c^{\tau}_{\omega_n}$ form $(\bigcup_{i\in I} \mathfrak{F}_i)$. Let $\{f_i \,|\, i \in I\}$ be a collection of Θ-valued functions of the form (2.1). In this case, by $\bigvee_{\Theta}(f_i \,|\, i \in I)$ we denote a function f such that $f(a) = \Theta$ form $(\bigcup_{i\in I} f_i(a))$ for all $a \in \omega \cup \{\omega'\}$.

Let $\mathfrak{X}$ be a non-empty class of groups. A complete lattice of formations Θ is called $\mathfrak{X}$-*separated* [31] if for any term $\xi(x_1, \ldots, x_m)$ of signature $\{\cap, \vee_{\Theta}\}$, any formations $\mathfrak{F}_1, \ldots, \mathfrak{F}_m$ of Θ, and any group $A \in \mathfrak{X} \cap \xi(\mathfrak{F}_1, \ldots, \mathfrak{F}_m)$, there exist $\mathfrak{X}$-groups $A_1 \in \mathfrak{F}_1, \ldots, A_m \in \mathfrak{F}_m$ such that $A \in \xi(\Theta \text{ form } A_1, \ldots, \Theta \text{ form } A_m)$.

Lemma 2.7 ([37, Lemma 17]). *Let Θ be an $\mathfrak{X}$-separated lattice of formations and let η be a sublattice of Θ such that η contains all one-generated Θ-subformations of the form Θ form A, where $A \in \mathfrak{X}$, of every formation $\mathfrak{F} \in \eta$. Suppose that a law $\xi_1 = \xi_2$ of signature $\{\cap, \vee_{\Theta}\}$ is true for all one-generated Θ-formations belonging to η. Then the law $\xi_1 = \xi_2$ is true for all Θ-subformations belonging to η.*

3. Main Results

For every term ξ of signature $\{\cap, \vee^{\tau}_{\omega_n}\}$ we denote by $\overline{\xi}$ the term of signature $\{\cap, \vee^{\tau}_{\omega_{n-1}}\}$ obtained from ξ by replacing every symbol $\vee^{\tau}_{\omega_n}$ by the symbol $\vee^{\tau}_{\omega_{n-1}}$.

Lemma 3.1. *Let $\xi(x_{i_1}, \ldots, x_{i_m})$ be a term of signature $\{\cap, \vee^{\tau}_{\omega_n}\}$ and let f_i be an inner $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellite of a formation $\mathfrak{F}_i$ where $i = 1, \ldots, m$ and $n \geq 1$. Then $\xi(\mathfrak{F}_1, \ldots, \mathfrak{F}_m) = CF_{\omega}(\overline{\xi}(f_1, \ldots, f_m))$.*

Proof. See the proof of [37, Lemma 16]. □

Theorem 3.1. *Let $n \geq 1$. Then every law of the lattice of all τ-closed formations c^{τ}_0 is fulfilled in the lattice of all τ-closed n-multiply ω-composition formations $c^{\tau}_{\omega_n}$.*

Proof. Fix a law

$$\xi_1(x_{i_1}, \ldots, x_{i_a}) = \xi_2(x_{j_1}, \ldots, x_{j_b}) \tag{3.1}$$

of signature $\{\cap, \vee^{\tau}_{\omega_n}\}$. Let

$$\overline{\xi}_1(x_{i_1}, \ldots, x_{i_a}) = \overline{\xi}_2(x_{j_1}, \ldots, x_{j_b}) \tag{3.2}$$

be the same law of signature $\{\cap, \vee^{\tau}_{\omega_{n-1}}\}$. Suppose that law (3.2) is true in the lattice $c^{\tau}_{\omega_{n-1}}$. Let $\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}; \mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}$ be arbitrary τ-closed n-multiply ω-composition formations. We show that $\xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) = \xi_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b})$.

Let f_{i_c} be the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{F}_{i_c}$ (where $c = 1, \ldots, a$) and let f_{j_d} be the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{F}_{j_d}$ (where $d = 1, \ldots, b$). By Lemma 3.1 we have

$$\xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) = CF_{\omega}(\overline{\xi}_1(f_{i_1}, \ldots, f_{i_a}))$$

and

$$\xi_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}) = CF_{\omega}(\overline{\xi}_2(f_{j_1}, \ldots, f_{j_b})).$$

By [35, Lemma 8] for every prime $p \in \omega$, the formations $f_{i_1}(p), \ldots, f_{i_a}(p)$; $f_{j_1}(p), \ldots, f_{j_b}(p)$ and the formations $f_{i_1}(\omega'), \ldots, f_{i_a}(\omega')$; $f_{j_1}(\omega'), \ldots, f_{j_b}(\omega')$ belong to $c^{\tau}_{\omega_{n-1}}$. By induction,

$$\begin{aligned}\overline{\xi}_1(f_{i_1}, \ldots, f_{i_a})(p) &= \overline{\xi}_1(f_{i_1}(p), \ldots, f_{i_a}(p)) \\ &= \overline{\xi}_2(f_{j_1}(p), \ldots, f_{j_b}(p)) = \overline{\xi}_2(f_{j_1}, \ldots, f_{j_b})(p)\end{aligned}$$

and

$$\begin{aligned}\overline{\xi}_1(f_{i_1}, \ldots, f_{i_a})(\omega') &= \overline{\xi}_1(f_{i_1}(\omega'), \ldots, f_{i_a}(\omega')) \\ &= \overline{\xi}_2(f_{j_1}(\omega'), \ldots, f_{j_b}(\omega')) = \overline{\xi}_2(f_{j_1}, \ldots, f_{j_b})(\omega').\end{aligned}$$

Hence $\xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) = \xi_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b})$. Thus law (3.1) is true in the lattice $c^{\tau}_{\omega_n}$, and the result is proved. □

Corollary 3.1 ([38, Corollary 2.5]). *The lattice of all τ-closed n-multiply ω-composition formations $c^{\tau}_{\omega_n}$ is modular for any non-negative integer n.*

Proof. By Lemma 2.5 the lattice of all τ-closed formations c^{τ}_0 is modular. Using Theorem 3.1 we have the lattice $c^{\tau}_{\omega_n}$ is modular for any non-negative integer n. □

The following lemma is proved by direct calculation.

Lemma 3.2. *Let $n \geq 1$, f_i be the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellite of a formation $\mathfrak{F}_i$, $i \in I$. Then $\bigvee^{\tau}_{\omega_{n-1}}(f_i \,|\, i \in I)$ is the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{F} = \bigvee^{\tau}_{\omega_n}(\mathfrak{F}_i \,|\, i \in I)$.*

Lemma 2.3 implies the following result.

Lemma 3.3. *Let $\mathfrak{R}_i$ be a τ-closed semiformation generated by a group G_i, $i = 1, 2$. Then $\mathfrak{R}_1 \cup \mathfrak{R}_2$ is a τ-closed semiformation and $\mathfrak{R}_1 = (B_1, \dots, B_t)$, $\mathfrak{R}_2 = (C_1, \dots, C_s)$ for some $B_1, \dots, B_t \in \mathrm{QS}_{\overline{\tau}}(G_1)$ and $C_1, \dots, C_s \in \mathrm{QS}_{\overline{\tau}}(G_2)$.*

Lemma 3.4. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be τ-closed n-multiply ω-composition formations and*

$$A \in c^{\tau}_{\omega_n} \text{ form}(\mathfrak{F}_1 \cup \mathfrak{F}_2), \quad n \geq 0.$$

Then there exist groups $A_i \in \mathfrak{F}_i$ $(i = 1, 2)$ such that

$$A \in (c^{\tau}_{\omega_n} \text{ form } A_1) \vee^{\tau}_{\omega_n} (c^{\tau}_{\omega_n} \text{ form } A_2).$$

Proof. We proceed by induction on n. Let $n = 0$. Since the formations $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are τ-closed, then by Lemmas 2.2 and 2.4 $A \in c^{\tau}_{\omega_0}$ form $(\mathfrak{F}_1 \cup \mathfrak{F}_2) =$ form $(\mathfrak{F}_1 \cup \mathfrak{F}_2) = \mathrm{QR}_0(\mathfrak{F}_1 \cup \mathfrak{F}_2)$. Consequently $A \cong H/N$ where $H \in \mathrm{R}_0(\mathfrak{F}_1 \cup \mathfrak{F}_2)$. Using [37, Lemma 15] we have $A \in$ form $(H/H^{\mathfrak{F}_1}) \vee$ form $(H/H^{\mathfrak{F}_2}) \subseteq \mathfrak{F}_1 \vee^{\tau}_{\omega_0} \mathfrak{F}_2$.

Let $n > 0$, $\{p_1, \dots, p_t\} = \omega \cap \pi(\mathrm{Com}(A))$ and $A \in \mathfrak{F}_1 \vee^{\tau}_{\omega_n} \mathfrak{F}_2$. Then by [35, Lemma 8] and Lemma 3.2, $A/C^{p_i}(A) \in f_1(p_i) \vee^{\tau}_{\omega_{n-1}} f_2(p_i)$ and $A/R_{\omega}(A) \in f_1(\omega') \vee^{\tau}_{\omega_{n-1}} f_2(\omega')$ where f_j is the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{F}_j$, $j = 1, 2$. By induction there exist the groups $A_{i_1} \in f_1(p_i)$, $A_{i_2} \in f_2(p_i)$, $T_1 \in f_1(\omega')$, $T_2 \in f_2(\omega')$ such that

$$A/C^{p_i}(A) \in (c^{\tau}_{\omega_{n-1}} \text{ form } A_{i_1}) \vee^{\tau}_{\omega_{n-1}} (c^{\tau}_{\omega_{n-1}} \text{ form } A_{i_2}),$$

$$A/R_{\omega}(A) \in (c^{\tau}_{\omega_{n-1}} \text{ form } T_1) \vee^{\tau}_{\omega_{n-1}} (c^{\tau}_{\omega_{n-1}} \text{ form } T_2).$$

We claim that

$$c^{\tau}_{\omega_{n-1}} \text{ form } (A_{i_1}, A_{i_2}) = (c^{\tau}_{\omega_{n-1}} \text{ form } A_{i_1}) \vee^{\tau}_{\omega_{n-1}} (c^{\tau}_{\omega_{n-1}} \text{ form } A_{i_2}),$$

$$c^{\tau}_{\omega_{n-1}} \text{ form } (T_1, T_2) = (c^{\tau}_{\omega_{n-1}} \text{ form } T_1) \vee^{\tau}_{\omega_{n-1}} (c^{\tau}_{\omega_{n-1}} \text{ form } T_2).$$

Thus $A/C^{p_i}(A) \in c^{\tau}_{\omega_{n-1}}$ form (A_{i_1}, A_{i_2}) and $A/R_{\omega}(A) \in c^{\tau}_{\omega_{n-1}}$ form (T_1, T_2).

Let $\mathfrak{R}_k$ be a τ-closed semiformation generated by the group A_{i_k} and $\mathfrak{Y}_k$ be a τ-closed semiformation generated by the group T_k where $k = 1, 2$. By Lemma 3.3 the semiformations $\mathfrak{R}_1 \cup \mathfrak{R}_2$ and $\mathfrak{Y}_1 \cup \mathfrak{Y}_2$ are τ-closed, and $\mathfrak{R}_1 = (B_1, \dots, B_t)$ and $\mathfrak{R}_2 = (C_1, \dots, C_s)$ for some $B_1, \dots, B_t \in \mathrm{QS}_{\overline{\tau}}(A_{i_1})$ and $C_1, \dots, C_s \in \mathrm{QS}_{\overline{\tau}}(A_{i_2})$; $\mathfrak{Y}_1 = (U_1, \dots, U_m)$ and $\mathfrak{Y}_2 = (V_1, \dots, V_q)$ for some $U_1, \dots, U_m \in \mathrm{QS}_{\overline{\tau}}(T_1)$ and $V_1, \dots, V_q \in \mathrm{QS}_{\overline{\tau}}(T_2)$. Since $A_{i_k} \in \mathfrak{R}_k$ $(k = 1, 2)$, then $c^{\tau}_{\omega_{n-1}}$ form $(A_{i_1}, A_{i_2}) \subseteq c^{\tau}_{\omega_{n-1}}$ form $(\mathfrak{R}_1 \cup \mathfrak{R}_2)$.

We prove the inverse inclusion. Since $\mathrm{S}_{\overline{\tau}}(A_{i_k}) \subseteq \mathrm{R}_0\mathrm{S}_{\overline{\tau}}(A_{i_k})$ $(k = 1, 2)$, then by Lemma 2.4, $\mathfrak{R}_k = \mathrm{QS}_{\overline{\tau}}(A_{i_k}) \subseteq \mathrm{QR}_0\mathrm{S}_{\overline{\tau}}(A_{i_k}) = \tau$ form (A_{i_k}) where $k = 1, 2$. It follows that τ form $(\mathfrak{R}_1 \cup \mathfrak{R}_2) \subseteq \tau$ form (A_{i_1}, A_{i_2}). Hence $c^{\tau}_{\omega_{n-1}}$ form $(\mathfrak{R}_1 \cup \mathfrak{R}_2) \subseteq$

$c^{\tau}_{\omega_{n-1}}$ form (A_{i_1}, A_{i_2}). Thus $c^{\tau}_{\omega_{n-1}}$ form $(A_{i_1}, A_{i_2}) = c^{\tau}_{\omega_{n-1}}$ form $(\mathfrak{R}_1 \cup \mathfrak{R}_2)$. Analogously $c^{\tau}_{\omega_{n-1}}$ form $(T_1, T_2) = c^{\tau}_{\omega_{n-1}}$ form $(\mathfrak{Y}_1 \cup \mathfrak{Y}_2)$. Thus

$$A/C^{p_i}(A) \in c^{\tau}_{\omega_{n-1}} \text{ form } (A_{i_1}, A_{i_2}) = c^{\tau}_{\omega_{n-1}} \text{ form } (B_1, \ldots, B_t; C_1, \ldots, C_s),$$

$$A/R_\omega(A) \in c^{\tau}_{\omega_{n-1}} \text{ form } (T_1, T_2) = c^{\tau}_{\omega_{n-1}} \text{ form } (U_1, \ldots, U_m; V_1, \ldots, V_q).$$

Since $O_{p_i}(A/C^{p_i}(A)) = 1$ and $R_\omega(A/R_\omega(A)) = 1$, then by [35, Lemma 10]

$$\begin{aligned} A/C^{p_i}(A) &\in c^{\tau}_{\omega_{n-1}} \text{ form } (G/O_{p_i}(G) \mid G \in \mathfrak{R}_1 \cup \mathfrak{R}_2) \\ &= c^{\tau}_{\omega_{n-1}} \text{ form } (B_1/O_{p_i}(B_1), \ldots, B_t/O_{p_i}(B_t); C_1/O_{p_i}(C_1), \ldots, C_s/O_{p_i}(C_s)), \\ A/R_\omega(A) &\in c^{\tau}_{\omega_{n-1}} \text{ form } (G/R_\omega(G) \mid G \in \mathfrak{Y}_1 \cup \mathfrak{Y}_2) \\ &= c^{\tau}_{\omega_{n-1}} \text{ form } (U_1/R_\omega(U_1), \ldots, U_m/R_\omega(U_m); V_1/R_\omega(V_1), \ldots, V_q/R_\omega(V_q)). \end{aligned}$$

Thus we have the inclusions

$$\begin{aligned} &c^{\tau}_{\omega_{n-1}} \text{ form } (B_1, \ldots, B_t; C_1, \ldots, C_s) \\ &\quad \subseteq c^{\tau}_{\omega_{n-1}} \text{ form } (B_1/O_{p_i}(B_1), \ldots, B_t/O_{p_i}(B_t); C_1/O_{p_i}(C_1), \ldots, C_s/O_{p_i}(C_s)), \\ &c^{\tau}_{\omega_{n-1}} \text{ form } (U_1, \ldots, U_m; V_1, \ldots, V_q) \\ &\quad \subseteq c^{\tau}_{\omega_{n-1}} \text{ form } (U_1/R_\omega(U_1), \ldots, U_m/R_\omega(U_m); V_1/R_\omega(V_1), \ldots, V_q/R_\omega(V_q)). \end{aligned}$$

On the other hand, since $\mathfrak{R}_1 \cup \mathfrak{R}_2$ and $\mathfrak{Y}_1 \cup \mathfrak{Y}_2$ are semiformations, then for any group G the following conditions hold: if $G \in \mathfrak{R}_1 \cup \mathfrak{R}_2$, then $G/O_{p_i}(G) \in \mathfrak{R}_1 \cup \mathfrak{R}_2$; if $G \in \mathfrak{Y}_1 \cup \mathfrak{Y}_2$, then $G/R_\omega(G) \in \mathfrak{Y}_1 \cup \mathfrak{Y}_2$. Consequently

$$\begin{aligned} &c^{\tau}_{\omega_{n-1}} \text{ form } (B_1/O_{p_i}(B_1), \ldots, B_t/O_{p_i}(B_t); C_1/O_{p_i}(C_1), \ldots, C_s/O_{p_i}(C_s)) \\ &\quad = c^{\tau}_{\omega_{n-1}} \text{ form } (G/O_{p_i}(G) \mid G \in \mathfrak{R}_1 \cup \mathfrak{R}_2) \\ &\quad \subseteq c^{\tau}_{\omega_{n-1}} \text{ form } (\mathfrak{R}_1 \cup \mathfrak{R}_2) = c^{\tau}_{\omega_{n-1}} \text{ form } (B_1, \ldots, B_t; C_1, \ldots, C_s), \\ &c^{\tau}_{\omega_{n-1}} \text{ form } (U_1/R_\omega(U_1), \ldots, U_m/R_\omega(U_m); V_1/R_\omega(V_1), \ldots, V_q/R_\omega(V_q)) \\ &\quad = c^{\tau}_{\omega_{n-1}} \text{ form } (G/R_\omega(G) \mid G \in \mathfrak{Y}_1 \cup \mathfrak{Y}_2) \\ &\quad \subseteq c^{\tau}_{\omega_{n-1}} \text{ form } (\mathfrak{Y}_1 \cup \mathfrak{Y}_2) = c^{\tau}_{\omega_{n-1}} \text{ form } (U_1, \ldots, U_m; V_1, \ldots, V_q). \end{aligned}$$

Thus

$$\begin{aligned} A/C^{p_i}(A) &\in c^{\tau}_{\omega_{n-1}} \text{ form } (B_1, \ldots, B_t; C_1, \ldots, C_s) \\ &= c^{\tau}_{\omega_{n-1}} \text{ form } (B_1/O_{p_i}(B_1), \ldots, B_t/O_{p_i}(B_t); C_1/O_{p_i}(C_1), \ldots, C_s/O_{p_i}(C_s)), \\ A/R_\omega(A) &\in c^{\tau}_{\omega_{n-1}} \text{ form } (U_1, \ldots, U_m; V_1, \ldots, V_q) \\ &= c^{\tau}_{\omega_{n-1}} \text{ form } (U_1/R_\omega(U_1), \ldots, U_m/R_\omega(U_m); V_1/R_\omega(V_1), \ldots, V_q/R_\omega(V_q)). \end{aligned}$$

Hence we may suppose that $O_{p_i}(B_k) = 1 = O_{p_i}(C_l)$ and $R_\omega(U_x) = 1 = R_\omega(V_z)$ for all $k = 1, \ldots, t$ and $l = 1, \ldots, s$; $x = 1, \ldots, m$ and $z = 1, \ldots, w$.

Let $D_{i_1} = B_1 \times \cdots \times B_t$ and $D_{i_2} = C_1 \times \cdots \times C_s$; $U = U_1 \times \cdots \times U_m$ and $V = V_1 \times \cdots \times V_q$. Then $O_{p_i}(D_{i_1}) = 1 = O_{p_i}(D_{i_2})$ and $R_\omega(U) = 1 = R_\omega(V)$.

It is clear that $D_{i_1} \in \mathrm{D}_0\mathfrak{R}_1 \subseteq \mathrm{D}_0(c^\tau_{\omega_{n-1}}\text{ form }A_{i_1})$. Since $\mathrm{D}_0 \leq \mathrm{R}_0$ (see [10, II, c. 267]), then $D_{i_1} \in \mathrm{R}_0(c^\tau_{\omega_{n-1}}\text{ form }A_{i_1}) = c^\tau_{\omega_{n-1}}\text{ form }A_{i_1}$. Analogously $D_{i_2} \in c^\tau_{\omega_{n-1}}\text{ form }A_{i_2}$. Consequently $c^\tau_{\omega_{n-1}}\text{ form }(D_{i_1}, D_{i_2}) \subseteq c^\tau_{\omega_{n-1}}\text{ form }(A_{i_1}, A_{i_2})$. Since $B_k \leq D_{i_1}$ for all $k = 1, \ldots, t$, then $\mathfrak{R}_1 \subseteq c^\tau_{\omega_{n-1}}\text{ form }(D_{i_1}, D_{i_2})$. Analogously $\mathfrak{R}_2 \subseteq c^\tau_{\omega_{n-1}}\text{ form }(D_{i_1}, D_{i_2})$. Consequently $\mathfrak{R}_1 \cup \mathfrak{R}_2 \subseteq c^\tau_{\omega_{n-1}}\text{ form }(D_{i_1}, D_{i_2})$. Thus $A/C^{p_i}(A) \in c^\tau_{\omega_{n-1}}\text{ form }(\mathfrak{R}_1 \cup \mathfrak{R}_2) \subseteq c^\tau_{\omega_{n-1}}\text{ form }(A_{i_1}, A_{i_2})$.

Let Z_{p_i} be a group of order p_i, $W_{i_1} = Z_{p_i} \wr D_{i_1}$ and $W_{i_2} = Z_{p_i} \wr D_{i_2}$. We show that $W_{i_1} \in \mathfrak{F}_1$. Let $B = Z^\natural_{p_i}$ be the base group of the wreath product W_{i_1}. By the wreath product properties $W_{i_1}/O_{p_i}(W_{i_1}) = W_{i_1}/B = (Z_{p_i} \wr D_{i_1})/B \cong D_{i_1}$. Since $A_{i_1} \in f_1(p_i)$ and $\mathrm{QS}_{\overline{\tau}}(A_{i_1}) = (B_1, \ldots, B_t)$ where $B_1, \ldots, B_t \in \mathrm{QS}_{\overline{\tau}}(A_{i_1})$, then $\mathrm{QS}_{\overline{\tau}}(A_{i_1}) \subseteq \mathrm{QS}_{\overline{\tau}}(f_1(p_i)) = f_1(p_i)$ and $B_1, \ldots, B_t \in f_1(p_i)$. Consequently

$$D_{i_1} = B_1 \times \cdots \times B_t \in \mathrm{D}_0 f_1(p_i) = f_1(p_i).$$

Since f_1 is the minimal $c^\tau_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{F}_1$, then $D_{i_1} \in f_1(p_i) \cap \mathfrak{F}_1$. Thus $W_{i_1}/O_{p_i}(W_{i_1}) \cong D_{i_1} \in f_1(p_i) \cap \mathfrak{F}_1$ for all $p_i \in \omega \cap \pi(\mathrm{Com}(A))$. By [32, Lemma 4] $W_{i_1} \in \mathfrak{F}_1$. Analogously $W_{i_2} \in \mathfrak{F}_2$. Since $T_1 \in f_1(\omega')$ and f_1 is an inner ω-composition satellite of $\mathfrak{F}_1$, then $T_1 \in \mathfrak{F}_1$. Hence $U_x \in \mathfrak{F}_1$ for all $x = 1, \ldots, m$. Analogously $V_z \in \mathfrak{F}_2$ for all $z = 1, \ldots, q$.

Let $A_1 = W_{1_1} \times W_{2_1} \times \cdots \times W_{t_1} \times U$ and $A_2 = W_{1_2} \times W_{2_2} \times \cdots \times W_{t_2} \times V$. Then $A_1 \in \mathfrak{F}_1$ and $A_2 \in \mathfrak{F}_2$. Using [37, Lemma 15] we see that

$$A \in \mathfrak{F} = (c^\tau_{\omega_n}\text{ form }A_1) \vee^\tau_{\omega_n} (c^\tau_{\omega_n}\text{ form }A_2).$$

This proves the lemma. □

Now we prove the following result, which plays an essential role in the proof of our main results.

Theorem 3.2. *Let $n \geq 0$. Then the lattice of all τ-closed n-multiply ω-composition formations is $\mathfrak{G}$-separated.*

Proof. Let $\xi(x_1, \ldots, x_m)$ be a term of signature $\{\cap, \vee^\tau_{\omega_n}\}$, $\mathfrak{F}_1, \ldots, \mathfrak{F}_m$ be τ-closed n-multiply ω-composition formations and $A \in \xi(\mathfrak{F}_1, \ldots, \mathfrak{F}_m)$. We proceed by induction on the number r of occurrences of the symbols in $\{\cap, \vee^\tau_{\omega_n}\}$ into the term ξ. We show that there exist groups $A_i \in \mathfrak{F}_i$ $(i = 1, \ldots, m)$ such that $A \in \xi(c^\tau_{\omega_n}\text{ form }A_1, \ldots, c^\tau_{\omega_n}\text{ form }A_m)$. Let $r = 0$. It is clear that $A \in c^\tau_{\omega_n}\text{ form }A$.

We prove the assertion for $r = 1$. There exist two cases: either $A \in \mathfrak{F}_1 \cap \mathfrak{F}_2$ or

$$A \in \mathfrak{F}_1 \vee^\tau_{\omega_0} \mathfrak{F}_2 = c^\tau_{\omega_0}\text{ form }(\mathfrak{F}_1 \cup \mathfrak{F}_2) = \tau\text{ form }(\mathfrak{F}_1 \cup \mathfrak{F}_2).$$

In the first case we have $A \in \tau\text{ form }A \cap \tau\text{ form }A$. In the second case by Lemma 3.4 there exist groups $A_i \in \mathfrak{F}_i$ $(i = 1, 2)$ such that $A \in (c^\tau_{\omega_0}\text{ form }A_1) \vee^\tau_{\omega_0} (c^\tau_{\omega_0}\text{ form }A_2) = (\tau\text{ form }A_1) \vee^\tau_{\omega_0} (\tau\text{ form }A_2)$. The assertion of the theorem for $r = 1$ is true.

Let a term ξ have $r > 1$ occurrences of the symbols in $\{\cap, \vee^{\tau}_{\omega_n}\}$. We suppose proving by induction that the theorem holds for term with less number of occurrences. Assume that ξ is of the form $\xi_1(x_{i_1}, \ldots, x_{i_a}) \triangle \xi_2(x_{j_1}, \ldots, x_{j_b})$ where $\triangle \in \{\cap, \vee^{\tau}_{\omega_n}\}$ and $\{x_{i_1}, \ldots, x_{i_a}\} \cup \{x_{j_1}, \ldots, x_{j_b}\} = \{x_1, \ldots, x_m\}$. By $\mathfrak{H}_1$ we denote the formation $\xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a})$, and by $\mathfrak{H}_2$ the formation $\xi_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b})$. There exist groups $A_1 \in \mathfrak{H}_1$, $A_2 \in \mathfrak{H}_2$ such that $A \in c^{\tau}_{\omega_n}$ form $A_1 \triangle c^{\tau}_{\omega_n}$ form A_2. On the other hand, by induction, there exist groups $B_1, \ldots, B_a$; $C_1, \ldots, C_b$ such that $B_k \in \mathfrak{F}_{i_k}$, $C_k \in \mathfrak{F}_{j_k}$, $A_1 \in \xi_1(c^{\tau}_{\omega_n}$ form $B_1, \ldots, c^{\tau}_{\omega_n}$ form $B_a)$, $A_2 \in \xi_2(c^{\tau}_{\omega_n}$ form $C_1, \ldots, c^{\tau}_{\omega_n}$ form $C_b)$.

Suppose that $x_{i_1}, \ldots, x_{i_t}$ are not contained in ξ_2, but $x_{i_{t+1}}, \ldots, x_{i_a}$ are contained in ξ_2. Let $D_{i_k} = B_k$ if $k < t+1$, $D_{i_k} = B_k \times C_q$ where q satisfies $x_{i_k} = x_{j_q}$ for all $k \geq t+1$. Let $D_{j_k} = C_k$ if $x_{j_k} \notin \{x_{i_{t+1}}, \ldots, x_{i_a}\}$. We denote by $\mathfrak{R}_p$ the formation $c^{\tau}_{\omega_n}$ form D_{i_p} and by $\mathfrak{X}_c$ we denote the formation $c^{\tau}_{\omega_n}$ form D_{j_c}, $p = 1, \ldots, a$; $c = 1, \ldots, b$. It follows that $A_1 \in \xi_1(\mathfrak{R}_1, \ldots, \mathfrak{R}_a)$, $A_2 \in \xi_2(\mathfrak{X}_1, \ldots, \mathfrak{X}_b)$. There exist the formations $\mathfrak{H}_1, \ldots, \mathfrak{H}_m$ such that $A \in \xi_1(\mathfrak{H}_{i_1}, \ldots, \mathfrak{H}_{i_a}) \triangle \xi_2(\mathfrak{H}_{j_1}, \ldots, \mathfrak{H}_{j_b}) = \xi(\mathfrak{H}_1, \ldots, \mathfrak{H}_m)$ where $\mathfrak{H}_i = c^{\tau}_{\omega_n}$ form K_i, $K_i \in \mathfrak{F}_i$. Thus the lattice $c^{\tau}_{\omega_n}$ is $\mathfrak{G}$-separated. This proves the theorem. □

If τ is trivial subgroup functor by [32, Corollary 1, Remark 3] we have the following.

Corollary 3.2. *Let $n \geq 0$. Then the lattice of all n-multiply $\mathfrak{L}$-composition formations is $\mathfrak{G}$-separated.*

For trivial subgroup functor τ we have the following.

Corollary 3.3 ([37, Proposition]). *Let $n \geq 0$. Then the lattice of all n-multiply ω-composition formations is $\mathfrak{G}$-separated.*

In 1985, Förster introduced the concept of $\mathfrak{X}$-local formation [11] where $\mathfrak{X}$ is a class of simple groups with a completeness property. Ballester-Bolinches, Esteban-Romero and Calvo studied products of such formations [1–5]. Let $\mathfrak{X}$ be a class of abelian simple groups with $\mathrm{Char}(\mathfrak{X}) = \pi(\mathfrak{X})$. Then the $\mathfrak{X}$-local formation $\mathfrak{F}$ is an ω-composition formation where $\omega = \pi(\mathfrak{X})$, and by [26, Theorem 5.1] we have the following.

Corollary 3.4. *Let $\mathfrak{X}$ be a class of abelian simple groups such that* $\mathrm{Char}(\mathfrak{X}) = \pi(\mathfrak{X})$. *Then the lattice of all $\mathfrak{X}$-local formations is $\mathfrak{G}$-separated.*

Theorem 3.3. *Let $n \geq 1$. If ω is an infinite set, then the law system of the lattice of all τ-closed formations c^{τ}_0 coincides with the law system of the lattice of all τ-closed n-multiply ω-composition formations $c^{\tau}_{\omega_n}$.*

Proof. Fix a law

$$\xi_1(x_{i_1}, \ldots, x_{i_a}) = \xi_2(x_{j_1}, \ldots, x_{j_b}) \tag{3.3}$$

of signature $\{\cap, \vee^{\tau}_{\omega_n}\}$. Let

$$\overline{\xi}_1(x_{i_1}, \ldots, x_{i_a}) = \overline{\xi}_2(x_{j_1}, \ldots, x_{j_b}) \tag{3.4}$$

be the same law of signature $\{\cap, \vee^{\tau}_{\omega_{n-1}}\}$. Suppose that law (3.3) is true in the lattice $c^{\tau}_{\omega_n}$. We show that law (3.4) is true in the lattice $c^{\tau}_{\omega_{n-1}}$. By Lemma 2.7 and Theorem 3.2, it suffices to prove that if $\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}$; $\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}$ are arbitrary one-generated τ-closed $(n-1)$-multiply ω-composition formations, then $\overline{\xi}_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) = \overline{\xi}_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b})$. Let $\mathfrak{F}_{i_1} = c^{\tau}_{\omega_{n-1}}$ form $A_{i_1}, \ldots, \mathfrak{F}_{i_a} = c^{\tau}_{\omega_{n-1}}$ form A_{i_a}, $\mathfrak{F}_{j_1} = c^{\tau}_{\omega_{n-1}}$ form $A_{j_1}, \ldots, \mathfrak{F}_{j_b} = c^{\tau}_{\omega_{n-1}}$ form A_{j_b}. We choose prime $p \in \omega$ such that $p \notin \pi(A_{i_1}, \ldots, A_{i_a}; A_{j_1}, \ldots, A_{j_b})$.

Let $B_{i_1} = Z_p \wr A_{i_1}, \ldots, B_{i_a} = Z_p \wr A_{i_a}$; $B_{j_1} = Z_p \wr A_{j_1}, \ldots, B_{j_b} = Z_p \wr A_{j_b}$ where Z_p is a group of order p. Since formations $\mathfrak{M}_{i_1} = c^{\tau}_{\omega_n}$ form $B_{i_1}, \ldots, \mathfrak{M}_{i_a} = c^{\tau}_{\omega_n}$ form B_{i_a}; $\mathfrak{M}_{j_1} = c^{\tau}_{\omega_n}$ form $B_{j_1}, \ldots, \mathfrak{M}_{j_b} = c^{\tau}_{\omega_n}$ form B_{j_b} belong to the lattice $c^{\tau}_{\omega_n}$, then $\mathfrak{F} = \mathfrak{H}$ where $\mathfrak{F} = \xi_1(\mathfrak{M}_{i_1}, \ldots, \mathfrak{M}_{i_a})$ and $\mathfrak{H} = \xi_2(\mathfrak{M}_{j_1}, \ldots, \mathfrak{M}_{j_b})$. Let f_{i_c} be the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{M}_{i_c}$ (where $c = 1, \ldots, a$); f_{j_d} be the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{M}_{j_d}$ (where $d = 1, \ldots, b$). By Lemma 3.1, $\xi_1(\mathfrak{M}_{i_1}, \ldots, \mathfrak{M}_{i_a}) = CF_{\omega}(\overline{\xi}_1(f_{i_1}, \ldots, f_{i_a}))$; $\xi_2(\mathfrak{M}_{j_1}, \ldots, \mathfrak{M}_{j_b}) = CF_{\omega}(\overline{\xi}_2(f_{j_1}, \ldots, f_{j_b}))$.

Let f and h be the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellites of $\mathfrak{F}$ and $\mathfrak{H}$, respectively. Then by [35, Lemma 8] and Lemma 3.2, $f(p) = \overline{\xi}_1(f_{i_1}, \ldots, f_{i_a})(p) = \overline{\xi}_1(f_{i_1}(p), \ldots, f_{i_a}(p))$ and $h(p) = \overline{\xi}_2(f_{j_1}, \ldots, f_{j_b})(p) = \overline{\xi}_2(f_{j_1}(p), \ldots, f_{j_b}(p))$.

Hence $\overline{\xi}_1(f_{i_1}(p), \ldots, f_{i_a}(p)) = \overline{\xi}_2(f_{j_1}(p), \ldots, f_{j_b}(p))$. Since $O_p(A_{i_c}) = 1$, then by [35, Lemma 8] $f_{i_c}(p) = \mathfrak{F}_{i_c}$ where $c = 1, \ldots, a$. Analogously $f_{j_d}(p) = \mathfrak{F}_{j_d}$ where $d = 1, \ldots, b$. It follows that $\overline{\xi}_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) = \overline{\xi}_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b})$, i.e. the law (3.4) is true in the lattice $c^{\tau}_{\omega_{n-1}}$. Thus every law of the lattice $c^{\tau}_{\omega_n}$ is true in the lattice of all τ-closed formations c^{τ}_0. Using Theorem 3.1, we have the result. □

Corollary 3.5. *Let ω be an infinite set. Let m and n be non-negative integers. Then the law systems of lattices $c^{\tau}_{\omega_m}$ and $c^{\tau}_{\omega_n}$ coincide.*

Proof. Let a law be true in $c^{\tau}_{\omega_n}$. By Theorem 3.3 the law is true in c^{τ}_0. Hence by Theorem 3.1 the law is true in $c^{\tau}_{\omega_m}$, as claimed. □

Let m and n be non-negative integers. In the theory of foliated formations by Vedernikov (see [30]) it is proved that the law system of the lattice of all m-multiply canonical formations coincides with the law system of the lattice of all n-multiply canonical formations.

4. Some Applications of the Results

Recall that a lattice of formations Θ is called *distributive* if for any Θ-formations $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{F}_3$ we have $\mathfrak{F}_1 \cap (\mathfrak{F}_2 \vee_{\Theta} \mathfrak{F}_3) = (\mathfrak{F}_1 \cap \mathfrak{F}_2) \vee_{\Theta} (\mathfrak{F}_1 \cap \mathfrak{F}_3)$. We show that the lattice of all τ-closed n-multiply ω-composition formations is not distributive for an infinite set of primes ω.

Lemma 4.1. *Let $n \geq 1$ and $\mathfrak{F}_i = c^{\tau}_{\omega_n}$ form B_i $(i = 1, 2)$ where $B_i = Z_p \wr A_i$, $p \in \omega$ and $p \notin \pi(A_1, A_2)$. Let f_i and f be the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellites of $\mathfrak{F}_i$ and $\mathfrak{F} = \mathfrak{F}_1 \cap \mathfrak{F}_2$, respectively. Then $f(p) = f_1(p) \cap f_2(p)$.*

Proof. Let $h = f_1 \cap f_2$. By Lemma 2.1 we have $\mathfrak{F} = CF_{\omega}(h)$. Let $p \in \omega$ and $p \notin \pi(A_1, A_2)$. We show that $f(p) = h(p)$.

By [35, Lemma 8] $f(p) = c^{\tau}_{\omega_{n-1}}$ form $(A \,|\, A \in h(p), O_p(A) = 1)$. Since $p \notin \pi(A_1, A_2)$, then $O_p(A_1) = 1 = O_p(A_2)$. By [34, Lemma 2.8] $B_i/C^p(B_i) = B_i/O_p(B_i) \cong A_i$ where $i = 1, 2$. Hence by [35, Lemma 8] $f_i(p) = c^{\tau}_{\omega_{n-1}}$ form $(B_i/C^p(B_i)) = c^{\tau}_{\omega_{n-1}}$ form A_i. Since $A_i \in \mathfrak{G}_{p'}$ we have $f_i(p) = c^{\tau}_{\omega_{n-1}}$ form $A_i \subseteq \mathfrak{G}_{p'}$.

Hence for any group $A \in f_i(p)$ we have $O_p(A) = 1$. Consequently $f(p) = c^{\tau}_{\omega_{n-1}}$ form $(A \,|\, A \in f_1(p) \cap f_2(p), O_p(A) = 1) = c^{\tau}_{\omega_{n-1}}$ form $(f_1(p) \cap f_2(p)) = f_1(p) \cap f_2(p) = h(p)$ and the lemma is proved. □

Lemma 4.2. *Let $n \geq 1$ and $\mathfrak{F}_i = c^{\tau}_{\omega_n}$ form B_i $(i = 1, \ldots, m)$ where $B_i = Z_p \wr A_i$, $p \in \omega$ and $p \notin \pi(A_1, \ldots, A_m)$. Let f_i be the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{F}_i$. Suppose that $\xi(x_1, \ldots, x_m)$ is a term of signature $\{\cap, \vee^{\tau}_{\omega_n}\}$ and f is the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellite of $\mathfrak{F} = \xi(\mathfrak{F}_1, \ldots, \mathfrak{F}_m)$. Then $f(p) = \overline{\xi}(f_1, \ldots, f_m)(p)$.*

Proof. Let $h = \overline{\xi}(f_1, \ldots, f_m)$. By Lemma 3.1 $\xi(\mathfrak{F}_1, \ldots, \mathfrak{F}_m) = CF_{\omega}(h)$. Let $p \in \omega$ and $p \notin \pi(A_1, \ldots, A_m)$. We show that $h(p) = f(p)$. We proceed by induction on the number r of occurrences of the symbols in $\{\cap, \vee^{\tau}_{\omega_n}\}$ into ξ. For the base of induction (the case $r = 1$) we have using Lemmas 3.2 and 4.1.

Let the term ξ have $r > 1$ occurrences of the symbols in $\{\cap, \vee^{\tau}_{\omega_n}\}$. Let ξ have the form

$$\xi(x_1, \ldots, x_m) = \xi_1(x_{i_1}, \ldots, x_{i_a}) \triangle \xi_2(x_{j_1}, \ldots, x_{j_b}),$$

where $\triangle \in \{\cap, \vee^{\tau}_{\omega_n}\}$, $\{x_{i_1}, \ldots, x_{i_a}\} \cup \{x_{j_1}, \ldots, x_{j_b}\} = \{x_1, \ldots, x_m\}$ and the assertion is true for ξ_1 and ξ_2. By induction $h_1(p) = \overline{\xi}_1(f_{i_1}, \ldots, f_{i_a})(p)$ and $h_2(p) = \overline{\xi}_2(f_{j_1}, \ldots, f_{j_b})(p)$ where h_1 and h_2 are the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellites of $\xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a})$ and $\xi_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b})$, respectively. By induction

$$\begin{aligned} f(p) &= h_1(p) \,\overline{\triangle}\, h_2(p) = \overline{\xi}_1(f_{i_1}, \ldots, f_{i_a})(p) \,\overline{\triangle}\, \overline{\xi}_2(f_{j_1}, \ldots, f_{j_b})(p) \\ &= \overline{\xi}_1(f_{i_1}(p), \ldots, f_{i_a}(p)) \,\overline{\triangle}\, \overline{\xi}_2(f_{j_1}(p), \ldots, f_{j_b}(p)) \\ &= \overline{\xi}(f_1(p), \ldots, f_m(p)) = \overline{\xi}(f_1, \ldots, f_m)(p) = h(p), \end{aligned}$$

as claimed. □

By [35, Lemma 8] it is easily to show the following assertion.

Lemma 4.3. *Let $n \geq 1$, f_1 and f_2 be the minimal ω-composition $c^{\tau}_{\omega_{n-1}}$-valued satellites of formations $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively. Then $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ if and only if $f_1 \leq f_2$.*

Theorem 4.1. *Let ω be an infinite set. Then the lattice of all τ-closed n-multiply ω-composition formations $c^{\tau}_{\omega_n}$ is not distributive for any non-negative integer n.*

Proof. We proceed by induction on n. Let $n = 0$. Then by Lemma 2.5 the lattice of all τ-closed formations $c^{\tau}_{\omega_0}$ is not distributive.

Let $n > 0$ and let the lattice $c^{\tau}_{\omega_{n-1}}$ be not distributive. Suppose that $c^{\tau}_{\omega_n}$ is distributive. Let A_1, A_2, A_3 be groups. Let $\mathfrak{F}_i = c^{\tau}_{\omega_n}$ form B_i $(i = 1, 2, 3)$ where $B_i = Z_p \wr A_i$, $p \in \omega$ and $p \notin \pi(A_1, A_2, A_3)$. Then $\mathfrak{F}_1 \cap (\mathfrak{F}_2 \vee^{\tau}_{\omega_n} \mathfrak{F}_3) = (\mathfrak{F}_1 \cap \mathfrak{F}_2) \vee^{\tau}_{\omega_n} (\mathfrak{F}_1 \cap \mathfrak{F}_3)$. Let f, h and f_i be the minimal $c^{\tau}_{\omega_{n-1}}$-valued ω-composition satellites of $\mathfrak{F} = \mathfrak{F}_1 \cap (\mathfrak{F}_2 \vee^{\tau}_{\omega_n} \mathfrak{F}_3)$, $\mathfrak{H} = (\mathfrak{F}_1 \cap \mathfrak{F}_2) \vee^{\tau}_{\omega_n} (\mathfrak{F}_1 \cap \mathfrak{F}_3)$ and $\mathfrak{F}_i$, respectively. Since $\mathfrak{F} = \mathfrak{H}$ then using Lemma 4.3 we have $f(p) = h(p)$. Since $p \notin \pi(A_1, A_2, A_3)$, then $O_p(A_i) = 1$ for all $i = 1, 2, 3$. By [34, Lemma 2.8] we have $B_i/C^p(B_i) = B_i/O_p(B_i) \cong A_i$ for all $i = 1, 2, 3$. By [35, Lemma 8] we have $f_i(p) = c^{\tau}_{\omega_{n-1}}$ form $(B_i/C^p(B_i)) = c^{\tau}_{\omega_{n-1}}$ form A_i.

Using Lemma 4.2 we have

$$\begin{aligned} f(p) &= c^{\tau}_{\omega_{n-1}} \text{ form } A_1 \cap ((c^{\tau}_{\omega_{n-1}} \text{ form } A_2) \vee^{\tau}_{\omega_{n-1}} (c^{\tau}_{\omega_{n-1}} \text{ form } A_3)) \\ &= (c^{\tau}_{\omega_{n-1}} \text{ form } A_1 \cap c^{\tau}_{\omega_{n-1}} \text{ form } A_2) \vee^{\tau}_{\omega_{n-1}} (c^{\tau}_{\omega_{n-1}} \text{ form } A_1 \cap c^{\tau}_{\omega_{n-1}} \text{ form } A_3) = h(p). \end{aligned}$$

By Theorem 3.2 and Lemma 2.7 we conclude that the lattice $c^{\tau}_{\omega_{n-1}}$ is distributive, a contradiction. Thus for any non-negative integer n the lattice $c^{\tau}_{\omega_n}$ is not distributive. This proves the theorem. □

If τ is a trivial subgroup functor we have the following.

Corollary 4.1. *Let ω be an infinite set. Then the lattice of all n-multiply ω-composition formations is not distributive for any non-negative integer n.*

For $n = 1$ and for a trivial subgroup functor τ we have the following.

Corollary 4.2. *Let ω be an infinite set. Then the lattice of all ω-composition formations is not distributive.*

For $\omega = \mathbb{P}$ and a trivial subgroup functor τ we have the following.

Corollary 4.3. *The lattice of all n-multiply composition formations is not distributive for any non-negative integer n.*

For $n = 1$, $\omega = \mathbb{P}$ and for a trivial subgroup functor τ we have the following.

Corollary 4.4. *The lattice of all composition formations is not distributive.*

5. Final Remarks

In conclusion, let us take readers attention to some open questions related to lattices of composition formations. Note that some of them are analogues of the corresponding problems in [27–29, 31–33].

Question 5.1. Does it true that for all non-negative integers n, the lattice of all n-multiply composition formations and the lattice of all τ-closed n-multiply composition formations have the same system of laws?

Recall that a formation is called totally ω-composition if it is n-multiply ω composition for all natural n (see [32]). The symbol $c^{\tau}_{\omega_\infty}$ denotes the lattice of all τ-closed n-multiply ω-composition formations. Let ω be the set of all primes. Then we write c^{τ}_{∞} instead of $c^{\tau}_{\omega_\infty}$.

Question 5.2. Does it true that the lattices c_∞ and c^{τ}_{∞} have the same system of laws?

Theorem 3.1 gives a motivation to the following two questions.

Question 5.3. Does it true that every law of the lattice of all τ-closed formations c^{τ}_0 is fulfilled in the lattice of all τ-closed totally ω-composition formations $c^{\tau}_{\omega_\infty}$?

Question 5.4. Let m and n be non-negative integers with $m > n$. Does it true that the lattice of all τ-closed m-multiply ω-composition formations is not a sublattice of the lattice of all τ-closed n-multiply ω-composition formations?

Note that the answer to analogue of Question 5.4 is positive for the lattices of all τ-closed multiply ω-saturated formations (see [29]). However in [34] it was shown that the lattice of all saturated formations is a complete sublattice of the lattice of all composition formations. Safonov proved that the lattice of all τ-closed totally saturated formations is a complete sublattice of the lattice of totally saturated formations (see [24]).

A complete lattice Θ^{ω_c} is called *inductive* (cf. [31, Definition 4.1] and see [37]) if for any collection $\{\mathfrak{F}_i = CF_\omega(f_i) \,|\, i \in I\}$ where f_i is an integrated satellite of $\mathfrak{F}_i \in \Theta^{\omega_c}$, the following equality holds:

$$\vee_{\Theta^{\omega_c}}(\mathfrak{F}_i \,|\, i \in I) = CF_\omega(\vee_\Theta(f_i \,|\, i \in I)).$$

We note that the lattice of all τ-closed n-multiply ω-composition formations $c^{\tau}_{\omega_n}$ is inductive (see [35, Theorem 2.1; 37, Theorem]). In [36], it was established that the lattice of all soluble totally saturated formations is algebraic and distributive. Independently Reifferscheid solved the problem of distributivity of the lattice of all soluble totally saturated formations (see [18]). Safonov proved that the lattice of all τ-closed totally saturated formations is $\mathfrak{G}$-separated [23], algebraic [20] and modular [19, 21] (moreover this lattice is distributive [22]). However the following questions are still open now.

Question 5.5. Let $\mathcal{L}$ be the lattice of all τ-closed totally ω-composition formations. Does it true that:

(1) $\mathcal{L}$ is algebraic? (see [32, Problem 1])
(2) $\mathcal{L}$ is inductive?
(3) $\mathcal{L}$ is $\mathfrak{G}$-separated?
(4) $\mathcal{L}$ is distributive (or modular at least)?

Finally, we note that Jakubík proved that the collection of all formations of lattice ordered groups is a complete Brouwerian lattice (see [15]). We also note that the collection of all formations of finite monounary algebras forms a complete lattice (see [16]).

Acknowledgments

This work was partially supported by the Belarusian Republican Foundation for Fundamental Research (grant F15RM-025) and the programme of scientific research Konvergentsiya — 2020 (Belarus).

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