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ON THE NUMBER OF HOMOTOPY TYPES OF FIBRES OF A DEFINABLE MAP

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Abstract

In this paper we prove a single exponential upper bound on the number of possible homotopy types of the fibres of a Pfaffian map in terms of the format of its graph. In particular, we show that if a semi-algebraic set $S \subset R^{m+n}$, where R is a real closed field, is defined by a Boolean formula with s polynomials of degree less than d, and $\pi : R^{m+n} \to R^n$ is the projection on a subspace, then the number of different homotopy types of fibres of π does not exceed $s^{2(m+1)n}(2^m nd)^{O(nm)}$. As applications of our main results we prove single exponential bounds on the number of homotopy types of semi-algebraic sets defined by fewnomials, and by polynomials with bounded additive complexity. We also prove single exponential upper bounds on the radii of balls guaranteeing local contractibility for semi-algebraic sets defined by polynomials with integer coefficients.

1. *Introduction*

Let $S \subset \mathbb{R}^k$ be a set definable in an o-minimal structure over the reals (see [10]) and let $\pi_S : S \to \mathbb{R}^n$ be a definable map. For example, S can be a semi-algebraic or a restricted sub-Pfaffian [11] set. For the purpose of this paper, we will assume without any loss of generality that π_S is the restriction to S of the projection map $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^n$, where $k = m + n$.

The following statement is a version of Hardt's triviality theorem.

Theorem 1.1 [8, 16]. *There exists a finite partition of $\mathbb{R}^n$ into definable sets $\{T_i\}_{i \in I}$ such that S is definably trivial over each T_i.*

Theorem 1.1 implies that for each $i \in I$ and any point $\mathbf{y} \in T_i$, the pre-image $\pi_S^{-1}(T_i)$ is definably homeomorphic to $\pi_S^{-1}(\mathbf{y}) \times T_i$ by a fibre preserving homeomorphism. In particular, for each $i \in I$, all fibres $\pi_S^{-1}(\mathbf{y}), \mathbf{y} \in T_i$ are definably homeomorphic.

Hardt's theorem is a corollary of the existence of *cylindrical cell decompositions* of definable sets. Since the decompositions can be effectively computed in semi-algebraic and restricted sub-Pfaffian cases (see [2, 11]), this implies a double exponential (in mn) upper bound on the cardinality of I and hence on the number of homeomorphism types of the fibres of the map π_S. Apparently, no better bounds than the double exponential bound are known, even though it seems reasonable to conjecture a single exponential upper bound on the number of homeomorphism types of the fibres of the map π_S.

In this paper, we consider the weaker problem of bounding the number of distinct *homotopy types*, occurring amongst the set of all fibres of π_S. The main results of the paper are *single exponential* upper bounds on this number in cases when S is semi-algebraic, and when S is semi-Pfaffian. Our results on semi-algebraic sets are formulated and proved with the ground field being any (possibly non-archimedean) real closed field R. In the Pfaffian setting we assume that $R = \mathbb{R}$.

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Even though the precise nature of the bounds that we prove is different in each of the above two cases, the proofs are quite similar. We will first give a proof in the semi-algebraic case, and then provide additional details necessary for the Pfaffian case.

The rest of the paper is organized as follows. In Section 2 we state our main result in the semi-algebraic case. In Section 3 the main theorem is proved. In Section 4 we state the result in the Pfaffian case, and discuss the modifications needed in its proof. In Section 5 we use the result of Section 4 to state and prove a single exponential upper bound on the number of homotopy types of semi-algebraic sets defined by fewnomials, as well as those defined by polynomials with bounded additive complexity. Finally, in Section 6 we prove some metric upper bounds, related to homotopy types, for semi-algebraic sets defined by polynomials with integer coefficients.

2. *Semi-algebraic case*

Let R be a real closed field, $\mathcal{P} \subset R[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$, and let ϕ be a Boolean formula with atoms of the form $P = 0$, $P > 0$, or $P < 0$, where $P \in \mathcal{P}$. We call ϕ a *$\mathcal{P}$-formula*, and the semi-algebraic set $S \subset R^{m+n}$ defined by ϕ, a *$\mathcal{P}$-semi-algebraic set*. Note that for a given $\mathcal{P}$ there is a finite number of different $\mathcal{P}$-semi-algebraic sets. Also, it is clear from the definition that every semi-algebraic set is a $\mathcal{P}$-semi-algebraic set for an appropriate $\mathcal{P}$.

If the Boolean formula ϕ contains no negations, and its atoms are of the form $P = 0$, $P \geqslant 0$, or $P \leqslant 0$, with $P \in \mathcal{P}$, then we call ϕ a *$\mathcal{P}$-closed formula*, and the semi-algebraic set $S \subset R^{m+n}$ defined by ϕ, a *$\mathcal{P}$-closed semi-algebraic set*.

2.1. *Main result*

We prove the following theorem.

THEOREM 2.1. *Let $\mathcal{P} \subset R[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$, with $\deg(P) \leqslant d$ for each $P \in \mathcal{P}$ and cardinality $\#\mathcal{P} = s$. Then there exists a finite set $A \subset R^n$, with*

$$\#A \leqslant (2^m snd)^{O(nm)},$$

such that for every $\mathbf{y} \in R^n$ there exists $\mathbf{z} \in A$ such that for every $\mathcal{P}$-semi-algebraic set $S \subset R^{m+n}$, the set $\pi_S^{-1}(\mathbf{y})$ is semi-algebraically homotopy equivalent to $\pi_S^{-1}(\mathbf{z})$. In particular, for any fixed $\mathcal{P}$-semi-algebraic set S, the number of different homotopy types of fibres $\pi_S^{-1}(\mathbf{y})$ for various $\mathbf{y} \in \pi(S)$ is also bounded by

$$(2^m snd)^{O(nm)}.$$

REMARK 1. We actually prove a more precise bound,

$$\#A \leqslant s^{2(m+1)n}(2^m nd)^{O(nm)};$$

(see Section 3.5).

Notice that the bound in Theorem 2.1 is single exponential in mn. The following example shows that the single exponential dependence on m is unavoidable.

EXAMPLE 1. Let $P \in R[X_1, \ldots, X_m] \hookrightarrow R[X_1, \ldots, X_m, Y]$ be the polynomial defined by

$$P := \sum_{i=1}^{m} \prod_{j=0}^{d-1} (X_i - j)^2.$$

The algebraic set defined by $P = 0$ in R^{m+1} with coordinates $X_1, \ldots, X_m, Y$, consists of d^m lines all parallel to the Y axis. Consider now the semi-algebraic set $S \subset R^{m+1}$ defined by

$$(P = 0) \ \wedge \ (0 \leqslant Y \leqslant X_1 + dX_2 + d^2X_3 + \ldots + d^{m-1}X_m).$$

It is easy to verify that, if $\pi : R^{m+1} \to R$ is the projection map on the Y co-ordinate, then the fibres $\pi_S^{-1}(y)$ for $y \in \{0, 1, 2, \ldots, d^m - 1\} \subset R$ are 0-dimensional and of different cardinality, and hence have different homotopy types.

The above example does not exhibit exponential dependence of the number of homotopy types of fibres on n, which is equal to 1 in the example. In fact, we cannot hope to produce examples where the number of homotopy types of the fibres grows with n (with the parameters s, d, and m fixed) since this number can be bounded by a function of s, d and m independent of n, as we show next.

Suppose that ϕ is a Boolean formula with atoms $\{a_i, b_i, c_i \mid 1 \leqslant i \leqslant s\}$. For an ordered list $\mathcal{P} = (P_1, \ldots, P_s)$ of polynomials $P_i \in R[X_1, \ldots, X_m]$, we denote by $\phi_{\mathcal{P}}$ the formula obtained from ϕ by replacing for each i, $1 \leqslant i \leqslant s$, the atom a_i by $P_i = 0$, the atom b_i by $P_i > 0$ and the atom c_i by $P_i < 0$.

Definition 1. We say that two ordered lists $\mathcal{P} = (P_1, \ldots, P_s)$, $\mathcal{Q} = (Q_1, \ldots, Q_s)$ of polynomials $P_i, Q_i \in R[X_1, \ldots, X_m]$ have the same *homotopy type* if for any Boolean formula ϕ, the semi-algebraic sets defined by $\phi_{\mathcal{P}}$ and $\phi_{\mathcal{Q}}$ are homotopy equivalent.

Let $\mathcal{S}_{m,s,d}$ be the family of all ordered lists $(P_1, \ldots, P_s)$, $P_i \in R[X_1, \ldots, X_m]$, with $\deg(P_i) \leqslant d$ for $1 \leqslant i \leqslant s$.

Corollary 2.2. *The number of different homotopy types of ordered lists in $\mathcal{S}_{m,s,d}$ does not exceed*

$$\left(s\binom{m+d}{d}\right)^{O\left(s\binom{m+d}{d}m\right)} = (sd)^{O(sd^m)}. \tag{2.1}$$

In particular, the number of different homotopy types of semi-algebraic sets defined by a fixed formula $\phi_{\mathcal{P}}$, where $\mathcal{P}$ varies over $\mathcal{S}_{m,s,d}$, does not exceed (2.1).

Proof. We introduce one new variable for each of the possible $\binom{m+d}{d}$ monomials of degree at most d in m variables, so that every polynomial P_i is uniquely represented by a point in $R^{\binom{m+d}{d}}$. Thus, every ordered list in $\mathcal{S}_{m,s,d}$ is uniquely represented by a point in $R^{s\binom{m+d}{d}}$. Consider the ordered list $\mathcal{Q} = (Q_1, \ldots, Q_s)$ of polynomials

$$Q_i \in \mathbb{Z}[Y_1, \ldots, Y_{s\binom{m+d}{d}}, X_1, \ldots, X_m],$$

$1 \leqslant i \leqslant s$, of degrees at most $d + 1$, such that for each $\mathbf{y} \in R^{s\binom{m+d}{d}}$ the list

$$(Q_1(\mathbf{y}, X_1, \ldots, X_m), \ldots, Q_s(\mathbf{y}, X_1, \ldots, X_m)) \in \mathcal{S}_{m,s,d}.$$

Now apply Theorem 2.1 to $\mathcal{Q}$ and the projection map $\pi' : R^{s\binom{m+d}{d}+m} \to R^{s\binom{m+d}{d}}$. $\square$

Corollary 2.2 implies that for every $\mathcal{P}$-semi-algebraic set $S \subset R^{m+n}$, where $\mathcal{P} \in \mathcal{S}_{m+n,s,d}$, the number of different homotopy types of fibres $\pi_S^{-1}(\mathbf{y})$ does not exceed (2.1) (as well as the bound from Theorem 2.1). In particular, this number has an upper bound not depending on n.

3. *Proof of Theorem* 2.1

3.1. *Main ideas*

We first summarize the main ideas behind the proof of Theorem 2.1.

Let R be a real closed field and $\mathcal{P} = \{P_1, \dots, P_s\} \subset R[X_1, \dots, X_m, Y_1, \dots, Y_n]$, with $\deg(P_i) \leqslant d,\ 1 \leqslant i \leqslant s$. We fix a finite set of points $B \subset R^n$ such that for every $\mathbf{y} \in R^n$ there exists $\mathbf{z} \in B$ such that for every $\mathcal{P}$-semi-algebraic set S, the set $\pi_S^{-1}(\mathbf{y})$ is semi-algebraically homotopy equivalent to $\pi_S^{-1}(\mathbf{z})$. The existence of a set B with this property follows from Hardt's triviality theorem (Theorem 1.1) and the fact that the number of $\mathcal{P}$-semi-algebraic sets is finite.

Observe that it is possible to choose $\omega \in R,\ \omega > 0$, sufficiently large (depending on $\mathcal{P}$ and B) such that, for any $\mathcal{P}$-semi-algebraic set S:

(1) the intersection of S with the set defined by the conjunction of the $2(m+n)$ inequalities $-\omega < X_i < \omega,\ -\omega < Y_j < \omega,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n$, has the same homotopy type as S (denote this intersection by S_0);
(2) the set B lies in $(-\omega, \omega)^n$, and B preserves its defining property with respect to S_0; that is, for every $\mathbf{y} \in R^n$ there exists $\mathbf{z} \in B$ such that the set $\pi_{S_0}^{-1}(\mathbf{y})$ is semi-algebraically homotopy equivalent to $\pi_{S_0}^{-1}(\mathbf{z})$.

We will henceforth assume that $\mathcal{P}$ contains the $2(m+n)$ polynomials, $X_i \pm \omega,\ Y_j \pm \omega,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n$, and restrict our attention to the bounded $\mathcal{P}$-semi-algebraic sets, which are contained in the cube $(-\omega, \omega)^{m+n}$. Note that in order to bound the number of homotopy types of the fibres $\pi_S^{-1}(\mathbf{y}),\ \mathbf{y} \in \pi(S)$, where S is an arbitrary $\mathcal{P}$-semi-algebraic set, it is sufficient to bound the number of homotopy types of the fibres $\pi_{S_0}^{-1}(\mathbf{y}),\ \mathbf{y} \in \pi(S_0)$, of the projection of the bounded set S_0.

We replace $\mathcal{P}$ by another family $\mathcal{P}' \subset R[X_1, \dots, X_m, Y_1, \dots, Y_n]$ having the following properties.

(1) The cardinality $\#\mathcal{P}' = 2s^2$, and $\deg(P) \leqslant d$, for each $P \in \mathcal{P}'$.
(2) For each bounded $\mathcal{P}$-semi-algebraic set S, there exists a bounded and $\mathcal{P}'$-closed semi-algebraic set S', such that $S' \simeq S$ (where $\simeq$ stands for semi-algebraic homotopy equivalence).
(3) $\pi_{S'}^{-1}(\mathbf{y}) \simeq \pi_S^{-1}(\mathbf{y})$ for every $\mathbf{y} \in B$.
(4) The zero sets of the polynomials in $\mathcal{P}'$ are non-singular hypersurfaces, intersecting transversally, and the same is true for the restriction of $\mathcal{P}'$ on each $\pi^{-1}(\mathbf{y}),\ \mathbf{y} \in B$. It follows that the partition of R^{m+n} into $\mathcal{P}'$-sign invariant subsets is a Whitney stratification, denoted by $\mathcal{W}$, of R^{m+n}, which is compatible with every $\mathcal{P}'$-semi-algebraic set.

Let G_1 be the set of all critical values of π, restricted to various strata of the stratification $\mathcal{W}$ of dimensions not less than n, and let G_2 be the union of the projections of all strata of $\mathcal{W}$ of dimensions less than n. Let $G := G_1 \cup G_2$. We prove that $B \subset R^n \setminus G$, and hence in order to bound the number of homotopy types of fibres $\pi_S^{-1}(\mathbf{y})$ for any bounded $\mathcal{P}$-semi-algebraic set S, it is sufficient to bound the number of homotopy types of fibres, $\pi_{S'}^{-1}(\mathbf{y}),\ \mathbf{y} \in R^n \setminus G$. We then prove, using Thom's first isotopy lemma, that the homeomorphism type of the fibre $\pi_{S'}^{-1}(\mathbf{y})$ does not change as $\mathbf{y}$ varies over any fixed connected component of $R^n \setminus G$. This, along with property (3) of $\mathcal{P}'$ stated above, shows that in order to bound the number of homotopy types of the fibres $\pi_S^{-1}(\mathbf{y})$ it suffices to bound the number of connected components of $R^n \setminus G$. We obtain the set A (in Theorem 2.1) by choosing one point in each connected component of $R^n \setminus G$. In order to bound the cardinality of A, we use a recent result from [**13**] bounding the Betti numbers of projections of semi-algebraic sets in terms of those of certain fibred products.

3.2. *Notation*

Let R be a real closed field. For an element $a \in R$, introduce

$$\operatorname{sign}(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a > 0, \\ -1 & \text{if } a < 0. \end{cases}$$

If $\mathcal{P} \subset R[X_1, \ldots, X_k]$ is finite, we write the *set of zeros* of $\mathcal{P}$ in R^k as

$$\mathrm{Z}(\mathcal{P}) := \left\{ \mathbf{x} \in R^k \;\middle|\; \bigwedge_{P \in \mathcal{P}} P(\mathbf{x}) = 0 \right\}.$$

A *sign condition* σ on $\mathcal{P}$ is an element of $\{0, 1, -1\}^{\mathcal{P}}$. The *realization of the sign condition* σ is the basic semi-algebraic set

$$\mathcal{R}(\sigma) := \left\{ \mathbf{x} \in R^k \;\middle|\; \bigwedge_{P \in \mathcal{P}} \operatorname{sign}(P(\mathbf{x})) = \sigma(P) \right\}.$$

A sign condition σ is *realizable* if $\mathcal{R}(\sigma) \neq \emptyset$. We denote by $\operatorname{Sign}(\mathcal{P})$ the set of realizable sign conditions on $\mathcal{P}$. For $\sigma \in \operatorname{Sign}(\mathcal{P})$ we define the *level of* σ as the cardinality $\#\{P \in \mathcal{P} \mid \sigma(P) = 0\}$. For each level ℓ, $0 \leqslant \ell \leqslant \#\mathcal{P}$, we denote by $\operatorname{Sign}_\ell(\mathcal{P})$ the subset of $\operatorname{Sign}(\mathcal{P})$ of elements of level ℓ.

Finally, for a sign condition σ let

$$\mathcal{Z}(\sigma) := \left\{ \mathbf{x} \in R^k \;\middle|\; \bigwedge_{P \in \mathcal{P},\ \sigma(P) = 0} P(\mathbf{x}) = 0 \right\}.$$

3.3. *Replacing a bounded set by a homotopy equivalent closed bounded set*

In this section, we describe a modification of the construction from [**12**] (see also [**3**]) for replacing any given bounded semi-algebraic set by a closed bounded semi-algebraic set which has the same homotopy type as the original set.

Definition 2. Let $\mathcal{F}(x)$ be a predicate defined over R_+ and $y \in R_+$. The notation for all $(0 < x \ll y)\ \mathcal{F}(x)$ stands for the statement

$$\exists z \in (0, y)\ \forall x \in R_+\ (\text{if } x < z, \text{ then } \mathcal{F}(x)),$$

and can be read 'for all positive x sufficiently smaller than y, $\mathcal{F}(x)$ is true'.

Definition 3. For points

$$\bar{\varepsilon} = (\varepsilon_{2s,1}, \ldots, \varepsilon_{2s,s}, \varepsilon_{2s-1,1}, \ldots, \varepsilon_{2s-1,s}, \ldots, \varepsilon_{1,1}, \ldots, \varepsilon_{1,s}) \in R_+^{2s^2},$$

and a predicate $\mathcal{F}(\bar{\varepsilon})$ over $R_+^{2s^2}$ we say 'for all sufficiently small $\bar{\varepsilon}$, $\mathcal{F}(\bar{\varepsilon})$ is true' if

$$\forall (0 < \varepsilon_{2s,1} \ll 1) \forall (0 < \varepsilon_{2s,2} \ll \varepsilon_{2s,1}) \ldots \forall (0 < \varepsilon_{1,s} \ll \varepsilon_{1,s-1}) \mathcal{F}(\bar{\varepsilon}).$$

Consider a finite set of polynomials

$$\mathcal{P} = \{P_1, \ldots, P_s\} \subset R[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$$

and a sequence of elements in R

$$1 > \varepsilon_{2s,1} > \ldots > \varepsilon_{2s,s} > \varepsilon_{2s-1,1} > \ldots > \varepsilon_{2s-1,s} > \ldots > \varepsilon_{1,s} > 0.$$

Given $\sigma \in \mathrm{Sign}_\ell(\mathcal{P})$ we denote by $\mathcal{R}(\sigma_+)$ the closed semi-algebraic set defined by the conjunction of the inequalities:

$$\begin{aligned} -\varepsilon_{2\ell,i} \leqslant P_i \leqslant \varepsilon_{2\ell,i}, &\quad \text{for each } P_i \in \mathcal{P} \text{ such that } \sigma(P_i) = 0,\\ P \geqslant 0, &\quad \text{for each } P \in \mathcal{P} \text{ such that } \sigma(P) = 1,\\ P \leqslant 0, &\quad \text{for each } P \in \mathcal{P} \text{ such that } \sigma(P) = -1. \end{aligned}$$

We denote by $\mathcal{R}(\sigma_-)$ the open semi-algebraic set defined by the conjunction of the inequalities:

$$\begin{aligned} -\varepsilon_{2\ell-1,i} < P_i < \varepsilon_{2\ell-1,i}, &\quad \text{for each } P_i \in \mathcal{P} \text{ such that } \sigma(P_i) = 0,\\ P > 0, &\quad \text{for each } P \in \mathcal{P} \text{ such that } \sigma(P) = 1,\\ P < 0, &\quad \text{for each } P \in \mathcal{P} \text{ such that } \sigma(P) = -1. \end{aligned}$$

Notice that, $\mathcal{R}(\sigma) \subset \mathcal{R}(\sigma_+)$ and $\mathcal{R}(\sigma) \subset \mathcal{R}(\sigma_-)$.

Now suppose that the bounded $\mathcal{P}$-semi-algebraic set S is defined by

$$S = \bigcup_{\sigma \in \Sigma_S} \mathcal{R}(\sigma),$$

for some $\Sigma_S \subset \mathrm{Sign}(\mathcal{P})$.

Definition 4. Let

$$\Sigma_{S,\ell} = \Sigma_S \cap \mathrm{Sign}_\ell(\mathcal{P})$$

and define a sequence of sets, $S_\ell(\bar{\varepsilon}) \subset R^{m+n}$, $0 \leqslant \ell \leqslant s$, inductively as follows.

- $S_0(\bar{\varepsilon}) := S$.
- For $0 \leqslant \ell \leqslant s$, let

$$S_{\ell+1}(\bar{\varepsilon}) = \left(S_\ell(\bar{\varepsilon}) \setminus \bigcup_{\sigma \in \mathrm{Sign}_\ell(\mathcal{P}) \setminus \Sigma_{S,\ell}} \mathcal{R}(\sigma_-) \right) \cup \bigcup_{\sigma \in \Sigma_{S,\ell}} \mathcal{R}(\sigma_+).$$

We denote $S'(\bar{\varepsilon}) := S_{s+1}(\bar{\varepsilon})$ and $\mathcal{P}' := \{P_j \pm \varepsilon_{i,j} \mid 1 \leqslant i \leqslant 2s, 1 \leqslant j \leqslant s\}$.

The following statement is proved in [3] and is a strengthening of a result in [12] (where it is shown that for all sufficiently small $\bar{\varepsilon}$ the sums of the Betti numbers of S and $S'(\bar{\varepsilon})$ are equal).

Proposition 3.1 [3]. *For all sufficiently small $\bar{\varepsilon}$ (see Definition 3), $S'(\bar{\varepsilon}) \simeq S$.*

We now show that it is possible to rewrite the Boolean formula for $S'(\bar{\varepsilon})$, which originally (in its negation-free form) contains inequalities of the kind $P \geqslant 0$ or $P \leqslant 0$ for $P \in \mathcal{P}$, so that it still defines $S'(\bar{\varepsilon})$ but involves only inequalities of the kind $Q \geqslant 0$ or $Q \leqslant 0$, for $Q \in \mathcal{P}'$.

Lemma 3.2. *For all sufficiently small $\bar{\varepsilon}$, $S'(\bar{\varepsilon})$ is a $\mathcal{P}'$-closed semi-algebraic set.*

Proof. In the negation-free Boolean formula for $S'(\bar{\varepsilon})$, replace every inequality of the kind $P_j \geqslant 0$, where $P_j \in \mathcal{P}$, by $P_j \geqslant \varepsilon_{2,j}$, and every such inequality of the kind $P_j \leqslant 0$ by $P_j \leqslant -\varepsilon_{2,j}$. Denote by $S''(\bar{\varepsilon})$ the set defined by the new formula. Obviously $S''(\bar{\varepsilon})$ is a $\mathcal{P}'$-closed semi-algebraic set and moreover it is clear from the definition that $S''(\bar{\varepsilon}) \subset S'(\bar{\varepsilon})$.

We now prove that $S'(\bar{\varepsilon}) \subset S''(\bar{\varepsilon})$. Let $\mathbf{z} \in S'(\bar{\varepsilon})$. Let $\mathcal{R}(\sigma_+)$, where $\sigma \in \Sigma_\ell$, be the maximal (with respect to ℓ) non-empty set in the definition of $S'(\bar{\varepsilon})$ containing $\mathbf{z}$. Assume that

$$\mathcal{R}(\sigma_+) = \{(\mathbf{x}, \mathbf{y}) \mid -\varepsilon_{2\ell,i_1} \leqslant P_{i_1} \leqslant \varepsilon_{2\ell,i_1}, \ldots, -\varepsilon_{2\ell,i_\ell} \leqslant P_{i_\ell} \leqslant \varepsilon_{2\ell,i_\ell},\ P_{i_{\ell+1}} \geqslant 0, \ldots, P_{i_k} \geqslant 0\}.$$

We show that

$$\mathbf{z} \in \{(\mathbf{x}, \mathbf{y}) \mid -\varepsilon_{2\ell,i_1} \leqslant P_{i_1} \leqslant \varepsilon_{2\ell,i_1}, \ldots, -\varepsilon_{2\ell,i_\ell} \leqslant P_{i_\ell} \leqslant \varepsilon_{2\ell,i_\ell},\ P_{i_{\ell+1}} \geqslant \varepsilon_{2,i_{\ell+1}}, \ldots, P_{i_k} \geqslant \varepsilon_{2,i_k}\},$$

and hence $\mathbf{z} \in S''(\bar{\varepsilon})$. Indeed, suppose that $P_{i_r}(\mathbf{z}) < \varepsilon_{2,i_r}$ for some r, $\ell+1 \leqslant r \leqslant k$. Then $-\varepsilon_{2(\ell+1),i_r} \leqslant P_{i_r}(\mathbf{z}) \leqslant \varepsilon_{2(\ell+1),i_r}$, since $0 \leqslant P_{i_r}(\mathbf{z}) < \varepsilon_{2,i_r} \leqslant \varepsilon_{2(\ell+1),i_r}$. Therefore $\mathbf{z} \in \mathcal{R}(\sigma'_+)$, where $\sigma' \in \Sigma_{\ell+1}$, which contradicts the maximality of σ with respect to ℓ. □

In particular, we see that $S'(\bar{\varepsilon})$ is a $\mathcal{P}'$-semi-algebraic set, and we define $\Sigma'_S \subset \mathrm{Sign}(\mathcal{P}')$ by

$$S'(\bar{\varepsilon}) = \bigcup_{\sigma \in \Sigma'_S} \mathcal{R}(\sigma).$$

We now note some extra properties of the family $\mathcal{P}'$.

LEMMA 3.3. *For all sufficiently small $\bar{\varepsilon}$, the following holds. If $\sigma \in \mathrm{Sign}_\ell(\mathcal{P}')$, then $\ell \leqslant m+n$ and $\mathcal{R}(\sigma) \subset R^{m+n}$ is a non-singular $(m+n-\ell)$-dimensional manifold such that at every point $(\mathbf{x}, \mathbf{y}) \in \mathcal{R}(\sigma)$, the $(\ell \times (m+n))$-Jacobi matrix,*

$$\left(\frac{\partial P}{\partial X_i}, \frac{\partial P}{\partial Y_j}\right)_{P \in \mathcal{P}',\ \sigma(P)=0,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n}$$

has the maximal rank ℓ.

Proof. Suppose without loss of generality that

$$\{P \in \mathcal{P}' \mid \sigma(P) = 0\} = \{P_{i_1} - \varepsilon_{j_1,i_1}, \ldots, P_{i_\ell} - \varepsilon_{j_\ell,i_\ell}\}$$

(cf. the definition of $\mathcal{P}'$), where $1 > \varepsilon_{j_1,i_1} > \ldots > \varepsilon_{j_\ell,i_\ell} > 0$. Let $\ell \leqslant m+n$. Consider the semi-algebraic map $P_{i_1,\ldots,i_\ell} : R^{m+n} \to R^\ell$ defined by

$$(\mathbf{x}, \mathbf{y}) \longmapsto (P_{i_1}(\mathbf{x}, \mathbf{y}), \ldots, P_{i_\ell}(\mathbf{x}, \mathbf{y})).$$

By the semi-algebraic version of Sard's theorem (see [**5**]), the set of critical values of $P_{i_1,\ldots,i_\ell}$ is a semi-algebraic subset of R^ℓ of dimension strictly less than ℓ. It follows that *for all sufficiently small* $\bar{\varepsilon}$, the point $(\varepsilon_{j_1,i_1}, \ldots, \varepsilon_{j_\ell,i_\ell})$ is not a critical value of the map $P_{i_1,\ldots,i_\ell}$.

It follows that the algebraic set

$$\{(\mathbf{x}, \mathbf{y}) \mid P_{i_1} = \varepsilon_{j_1,i_1}, \ldots, P_{i_\ell} = \varepsilon_{j_\ell,i_\ell}\}$$

is a smooth $(m+n-\ell)$-dimensional manifold such that at every point on it the $(\ell \times (m+n))$-Jacobi matrix,

$$\left(\frac{\partial P}{\partial X_i}, \frac{\partial P}{\partial Y_j}\right)_{P \in \{P_{i_1},\ldots,P_{i_\ell}\},\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n}$$

has the maximal rank ℓ. The same is true for the basic semi-algebraic set $\mathcal{R}(\sigma)$.

We now prove that $\ell \leqslant m+n$. Suppose that $\ell > m+n$. As we have just proved, $\{P_{i_1} = \varepsilon_{j_1,i_1}, \ldots, P_{i_{m+n}} = \varepsilon_{j_{m+n},i_{m+n}}\}$ is a finite set of points. Polynomial $P_{i_\ell} - \varepsilon_{j_\ell,i_\ell}$ vanishes on each of these points. Choosing a value of $\varepsilon_{j_\ell,i_\ell}$ smaller than the value of P_{i_ℓ} at any point, we get a contradiction. □

Recall that $B \subset R^n$ is a fixed finite set of points such that for every $\mathbf{y} \in R^n$ there exists $\mathbf{z} \in B$ such that for every $\mathcal{P}$-semi-algebraic set S, we have $\pi_S^{-1}(\mathbf{y}) \simeq \pi_S^{-1}(\mathbf{z})$.

LEMMA 3.4. *For every $\mathbf{y} \in B$, bounded $\mathcal{P}$-semi-algebraic set S, for all sufficiently small $\bar{\varepsilon}$, and $\sigma \in \mathrm{Sign}_\ell(\mathcal{P}'_{\mathbf{y}})$, where $\mathcal{P}'_{\mathbf{y}} := \{P(X_1, \dots, X_m, \mathbf{y}) \mid P \in \mathcal{P}'\}$, the following statements hold.*

(i) *$0 \leqslant \ell \leqslant m$, and $\mathcal{R}(\sigma) \cap \pi^{-1}(\mathbf{y})$ is a non-singular $(m-\ell)$-dimensional manifold such that at every point $(\mathbf{x}, \mathbf{y}) \in \mathcal{R}(\sigma) \cap \pi^{-1}(\mathbf{y})$, the $(\ell \times m)$-Jacobi matrix,*

$$\left(\frac{\partial P}{\partial X_i}\right)_{P \in \mathcal{P}'_{\mathbf{y}},\ \sigma(P)=0,\ 1 \leqslant i \leqslant m}$$

has the maximal rank ℓ.

(ii) *$\pi^{-1}(\mathbf{y}) \cap S'(\bar{\varepsilon}) \simeq \pi_S^{-1}(\mathbf{y})$.*

Proof. Note that $P_{\mathbf{y}} \in R[X_1, \dots, X_m]$ for each $P \in \mathcal{P}$ and $\mathbf{y} \in R^n$. The proof of part (i) is now identical to the proof of Lemma 3.3. Part (ii) of the lemma is a consequence of Proposition 3.1. □

3.4. *Whitney stratification of R^n compatible with $\mathcal{P}'$*

LEMMA 3.5. *For any bounded $\mathcal{P}$-semi-algebraic set S and for all sufficiently small $\bar{\varepsilon}$, the partitions*

$$R^n = \bigcup_{\sigma \in \mathrm{Sign}(\mathcal{P}')} \mathcal{R}(\sigma),$$
$$S'(\bar{\varepsilon}) = \bigcup_{\sigma \in \Sigma'_S} \mathcal{R}(\sigma),$$

are compatible Whitney stratifications of R^n and $S'(\bar{\varepsilon})$, respectively.

Proof. This follows directly from the definition of Whitney stratification (see [**9**, **14**]), and Lemma 3.3. □

Fix some sign condition $\sigma \in \mathrm{Sign}(\mathcal{P}')$. Recall that $(\mathbf{x}, \mathbf{y}) \in \mathcal{R}(\sigma)$ is a *critical point* of the map $\pi_{\mathcal{R}(\sigma)}$ if the Jacobi matrix,

$$\left(\frac{\partial P}{\partial X_i}\right)_{P \in \mathcal{P}',\ \sigma(P)=0,\ 1 \leqslant i \leqslant m}$$

at $(\mathbf{x}, \mathbf{y})$ is not of the maximal possible rank. The projection $\pi(\mathbf{x}, \mathbf{y})$ of a critical point is a *critical value* of $\pi_{\mathcal{R}(\sigma)}$.

Let $C_1(\bar{\varepsilon}) \subset R^{m+n}$ be the set of critical points of $\pi_{\mathcal{R}(\sigma)}$ over all sign conditions

$$\sigma \in \bigcup_{\ell \leqslant m} \mathrm{Sign}_\ell(\mathcal{P}')$$

(that is, over all $\sigma \in \mathrm{Sign}_\ell(\mathcal{P}')$ with $\dim(\mathcal{R}(\sigma)) \geqslant n$). For a bounded $\mathcal{P}$-semi-algebraic set S, let $C_1(S, \bar{\varepsilon}) \subset S'(\bar{\varepsilon})$ be the set of critical points of $\pi_{\mathcal{R}(\sigma)}$ over all sign conditions

$$\sigma \in \bigcup_{\ell \leqslant m} \mathrm{Sign}_\ell(\mathcal{P}') \cap \Sigma'_S$$

(that is, over all $\sigma \in \Sigma'_S$ with $\dim(\mathcal{R}(\sigma)) \geqslant n$).

Let $C_2(\bar{\varepsilon}) \subset R^{m+n}$ be the union of $\mathcal{R}(\sigma)$ over all

$$\sigma \in \bigcup_{\ell > m} \mathrm{Sign}_\ell(\mathcal{P}')$$

(that is, over all $\sigma \in \mathrm{Sign}_\ell(\mathcal{P}')$ with $\dim(\mathcal{R}(\sigma)) < n$). For a bounded $\mathcal{P}$-semi-algebraic set S, let $C_2(S, \bar{\varepsilon}) \subset S'(\bar{\varepsilon})$ be the union of $\mathcal{R}(\sigma)$ over all

$$\sigma \in \bigcup_{\ell > m} \mathrm{Sign}_\ell(\mathcal{P}') \cap \Sigma'_S$$

(that is, over all $\sigma \in \Sigma'_S$ with $\dim(\mathcal{R}(\sigma)) < n$).

Denote $C(\bar{\varepsilon}) := C_1(\bar{\varepsilon}) \cup C_2(\bar{\varepsilon})$, and $C(S, \bar{\varepsilon}) := C_1(S, \bar{\varepsilon}) \cup C_2(S, \bar{\varepsilon})$. Notice that $C(S, \bar{\varepsilon}) = C(\bar{\varepsilon}) \cap S'(\bar{\varepsilon})$.

Lemma 3.6. *For each bounded $\mathcal{P}$-semi-algebraic S and for all sufficiently small $\bar{\varepsilon}$, the set $C(S, \bar{\varepsilon})$ is closed and bounded in R^{m+n}.*

Proof. The set $C(S, \bar{\varepsilon})$ is bounded since $S'(\bar{\varepsilon})$ is bounded. The union $C_2(S, \bar{\varepsilon})$ of strata of dimensions less than n is closed since $S'(\bar{\varepsilon})$ is closed.

Let $\sigma_1 \in \mathrm{Sign}_{\ell_1}(\mathcal{P}') \cap \Sigma'_S$, $\sigma_2 \in \mathrm{Sign}_{\ell_2}(\mathcal{P}') \cap \Sigma'_S$, where $\ell_1 \leqslant m$, $\ell_1 < \ell_2$, and if $\sigma_1(P) = 0$, then $\sigma_2(P) = 0$ for any $P \in \mathcal{P}'$. It follows that stratum $\mathcal{R}(\sigma_2)$ lies in the closure of the stratum $\mathcal{R}(\sigma_1)$. Let $\mathcal{J}$ be the finite family of $(\ell_1 \times \ell_1)$-minors such that $Z(\mathcal{J}) \cap \mathcal{R}(\sigma_1)$ is the set of all critical points of $\pi_{\mathcal{R}(\sigma_1)}$. Then $Z(\mathcal{J}) \cap \mathcal{R}(\sigma_2)$ is either contained in $C_2(S, \bar{\varepsilon})$ (when $\dim(\mathcal{R}(\sigma_2)) < n$), or contained in the set of all critical points of $\pi_{\mathcal{R}(\sigma_2)}$ (when $\dim(\mathcal{R}(\sigma_2)) \geqslant n$). It follows that the closure of $Z(\mathcal{J}) \cap \mathcal{R}(\sigma_1)$ lies in the union of the following sets:

(i) $Z(\mathcal{J}) \cap \mathcal{R}(\sigma_1)$;
(ii) sets of critical points of some strata of dimensions less than $m + n - \ell_1$;
(iii) some strata of dimension less than n.

Using induction on descending dimensions in case (ii), we conclude that the closure of $Z(\mathcal{J}) \cap \mathcal{R}(\sigma_1)$ is contained in $C(S, \bar{\varepsilon})$. Hence, $C(S, \bar{\varepsilon})$ is closed. □

Definition 5. We denote $G_i(\bar{\varepsilon}) := \pi(C_i(\bar{\varepsilon}))$, $i = 1, 2$, and $G(\bar{\varepsilon}) := G_1(\bar{\varepsilon}) \cup G_2(\bar{\varepsilon})$. Similarly, for each bounded $\mathcal{P}$-semi-algebraic set S, we denote $G_i(S, \bar{\varepsilon}) := \pi(C_i(S, \bar{\varepsilon}))$, $i = 1, 2$, and $G(S, \bar{\varepsilon}) := G_1(S, \bar{\varepsilon}) \cup G_2(S, \bar{\varepsilon})$.

Lemma 3.7. *For all sufficiently small $\bar{\varepsilon}$, $B \cap G(\bar{\varepsilon}) = \emptyset$. In particular, $B \cap G(S, \bar{\varepsilon}) = \emptyset$ for every bounded $\mathcal{P}$-semi-algebraic set S.*

Proof. By Lemma 3.4, for all $\mathbf{y} \in B$, for all sufficiently small $\bar{\varepsilon}$, and $\sigma \in \mathrm{Sign}_\ell(\mathcal{P}'_{\mathbf{y}})$:

(i) $0 \leqslant \ell \leqslant m$, and
(ii) $\mathcal{R}(\sigma) \cap \pi^{-1}(\mathbf{y})$ is a non-singular $(m - \ell)$-dimensional manifold such that at every point $(\mathbf{x}, \mathbf{y}) \in \mathcal{R}(\sigma) \cap \pi^{-1}(\mathbf{y})$, the $(\ell \times m)$-Jacobi matrix,

$$\left(\frac{\partial P}{\partial X_i} \right)_{P \in \mathcal{P}'_{\mathbf{y}},\ \sigma(P)=0,\ 1 \leqslant i \leqslant m}$$

has the maximal rank ℓ.

If a point $\mathbf{y}$ is contained in $B \cap G_1(\bar{\varepsilon}) = B \cap \pi(C_1(\bar{\varepsilon}))$, then there exists $\mathbf{x} \in R^m$ such that $(\mathbf{x}, \mathbf{y})$ is a critical point of $\pi_{\mathcal{R}(\sigma)}$ for some $\sigma \in \bigcup_{\ell \leqslant m} \mathrm{Sign}_\ell(\mathcal{P}')$, and this is impossible by (ii).

Similarly, $\mathbf{y} \in B \cap G_2(\bar{\varepsilon}) = B \cap \pi(\mathrm{C}_2(\bar{\varepsilon}))$ implies that there exists $\mathbf{x} \in R^m$ such that $(\mathbf{x}, \mathbf{y}) \in \mathcal{R}(\sigma)$ for some $\sigma \in \bigcup_{\ell > m} \mathrm{Sign}_\ell(\mathcal{P}')$, and this is impossible by (i). □

For all sufficiently small $\bar{\varepsilon}$, let $D(\bar{\varepsilon})$ be a connected component of $R^n \setminus G(\bar{\varepsilon})$, and for a bounded $\mathcal{P}$-semi-algebraic set S, let $D(S, \bar{\varepsilon})$ be a connected component of $\pi(S'(\bar{\varepsilon})) \setminus G(S, \bar{\varepsilon})$.

Lemma 3.8. *For every bounded $\mathcal{P}$-semi-algebraic set S and for all sufficiently small $\bar{\varepsilon}$, all fibres $\pi^{-1}(\mathbf{y}) \cap S'(\bar{\varepsilon})$, $\mathbf{y} \in D(\bar{\varepsilon})$, are homeomorphic.*

Proof. Lemma 3.5 implies that

$$\widehat{S} := \pi^{-1}_{S'(\bar{\varepsilon})}(\pi(S'(\bar{\varepsilon})) \setminus G(S, \bar{\varepsilon}))$$

is a Whitney stratified set having strata of dimension at least n. Moreover, $\pi_{\widehat{S}}$ is a proper stratified submersion. By Thom's first isotopy lemma (in the semi-algebraic version, over real closed fields [9]) the map $\pi_{\widehat{S}}$ is a locally trivial fibration. In particular, all fibres $\pi^{-1}_{S'(\bar{\varepsilon})}(\mathbf{y})$, $\mathbf{y} \in D(S, \bar{\varepsilon})$, are homeomorphic for every connected component $D(S, \bar{\varepsilon})$. The lemma follows, since the inclusion $G(S, \bar{\varepsilon}) \subset G(\bar{\varepsilon})$ implies that either $D(\bar{\varepsilon}) \subset D(S, \bar{\varepsilon})$ for some connected component $D(S, \bar{\varepsilon})$, or $D(\bar{\varepsilon}) \cap \pi(S'(\bar{\varepsilon})) = \emptyset$. □

LEMMA 3.9. *For each $\mathbf{y} \in B$ and for all sufficiently small $\bar{\varepsilon}$, there exists a connected component $D(\bar{\varepsilon})$ of $R^n \setminus G(\bar{\varepsilon})$, such that $\pi^{-1}_S(\mathbf{y}) \simeq \pi^{-1}_{S'(\bar{\varepsilon})}(\mathbf{y}_1)$ for every bounded $\mathcal{P}$-semi-algebraic set S and for every $\mathbf{y}_1 \in D(\bar{\varepsilon})$.*

Proof. By Lemma 3.7, $\mathbf{y}$ belongs to some connected component $D(\bar{\varepsilon})$ for all sufficiently small $\bar{\varepsilon}$. Lemma 3.4(ii) implies that $\pi^{-1}(\mathbf{y}) \cap S \simeq \pi^{-1}(\mathbf{y}) \cap S'(\bar{\varepsilon})$. Finally, by Lemma 3.8, $\pi^{-1}_{S'(\bar{\varepsilon})}(\mathbf{y}) \simeq \pi^{-1}_{S'(\bar{\varepsilon})}(\mathbf{y}_1)$ for every $\mathbf{y}_1 \in D(\bar{\varepsilon})$. □

3.5. *Proof of Theorem* 2.1

The following proposition gives a bound on the Betti numbers of the projection $\pi(V)$ of a closed and bounded semi-algebraic set V in terms of the number and degrees of polynomials defining V.

PROPOSITION 3.10 [13]. *Let $V \subset R^{m+n}$ be a closed and bounded semi-algebraic set defined by a Boolean formula with s distinct polynomials of degree not exceeding d. Then the kth Betti number of the projection $b_k(\pi(V))$ is at most $(ksd)^{O(n+km)}$.*

Proof. See [13]. □

For the proof of Theorem 2.1 we need the following inequalities, which can be derived from the Mayer–Vietoris exact sequence (see, for instance, [2, Proposition 7.33]).

PROPOSITION 3.11 (Mayer–Vietoris inequalities). *Let the subsets $W_1, \dots, W_t \subset R^n$ be all open or all closed. Then*

$$b_k\left(\bigcup_{1\leqslant j\leqslant t} W_j\right) \leqslant \sum_{J\subset\{1,\dots,t\}} b_{k-(\#J)+1}\left(\bigcap_{j\in J} W_j\right) \tag{3.1}$$

and

$$b_k\left(\bigcap_{1\leqslant j\leqslant t} W_j\right) \leqslant \sum_{J\subset\{1,\dots,t\}} b_{k+(\#J)-1}\left(\bigcup_{j\in J} W_j\right), \tag{3.2}$$

where b_k is the kth Betti number.

Proof of Theorem 2.1. To make clearer the idea of the proof, we first show a weaker bound, $\#A \leqslant (s^m nd)^{O(nm)}$, which requires fewer technical details.

Let

$$\mathcal{P}' = \{Q_1, \dots, Q_{2s^2}\}$$

(see Definition 4 for the definition of the set $\mathcal{P}'$). Recall that for all sufficiently small $\bar{\varepsilon}$, $G(\bar{\varepsilon}) = G_1(\bar{\varepsilon}) \cup G_2(\bar{\varepsilon})$, where $G_1(\bar{\varepsilon})$ is the union of sets of critical values of $\pi_{\mathcal{R}(\sigma)}$ over all strata $\mathcal{R}(\sigma)$ of dimension at least n, and $G_2(\bar{\varepsilon})$ is the union of projections of all strata of dimension less than n.

Observe that the set of critical points $C_1(\bar{\varepsilon})$ is a $\widehat{\mathcal{P}}$-semi-algebraic set, where

$$\widehat{\mathcal{P}} = \mathcal{P}' \cup \mathcal{Q},$$

$$\mathcal{Q} = \left\{ \det\left(M^{i_1,\ldots,i_\ell}_{j_1,\ldots,j_\ell}\right) \ \mid\ 1 \leqslant \ell \leqslant m, 1 \leqslant i_1 < \ldots < i_\ell \leqslant 2s^2, 1 \leqslant j_1 < \ldots < j_\ell \leqslant m \right\},$$

and

$$M^{i_1,\ldots,i_\ell}_{j_1,\ldots,j_\ell} = \begin{pmatrix} \dfrac{\partial Q_{i_1}}{\partial X_{j_1}} & \cdots & \dfrac{\partial Q_{i_1}}{\partial X_{j_\ell}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial Q_{i_\ell}}{\partial X_{j_1}} & \cdots & \dfrac{\partial Q_{i_\ell}}{\partial X_{j_\ell}} \end{pmatrix}.$$

Since $\#\mathcal{P}' \leqslant 2s^2$, we conclude that $\#\widehat{\mathcal{P}} \leqslant s^{O(m)}$. The degrees of polynomials from $\widehat{\mathcal{P}}$ do not exceed $O(md)$. The union $C_2(\bar{\varepsilon})$ of all strata of dimension less than n is a $\mathcal{P}'$-semi-algebraic set. It follows that $C(\bar{\varepsilon})$ is a $\widehat{\mathcal{P}}$-semi-algebraic set which is closed and bounded in R^{m+n} due to Lemma 3.6.

Then Proposition 3.10 implies that $b_{n-1}(G(\bar{\varepsilon})) \leqslant (s^m nd)^{O(nm)}$. By Alexander's duality, the homology groups $\widetilde{H}_0(R^n \setminus G(\bar{\varepsilon}))$ and $H_{n-1}(G(\bar{\varepsilon}))$ are isomorphic, and hence the number of connected components is as follows:

$$b_0(R^n \setminus G(\bar{\varepsilon})) \leqslant (s^m nd)^{O(nm)}.$$

We choose for the set A, one point in each connected component of $R^n \setminus G(\bar{\varepsilon})$. Now the bound $\#A \leqslant (s^m nd)^{O(nm)}$ follows from Lemma 3.9.

Now we proceed to the proof of the bound

$$b_0(R^n \setminus G(\bar{\varepsilon})) \leqslant s^{2(m+1)n}(2^m nd)^{O(nm)},$$

which implies the same bound for $\#A$.

Recall (Section 3.1) that the set S is contained in $(-\omega, \omega)^{n+m}$ for a sufficiently large $\omega > 0$, and therefore $S'(\bar{\varepsilon})$ lies in the closed box $[-\omega - 1, \omega + 1]^{n+m}$. Denote by $\mathcal{E}_1(\bar{\varepsilon})$ the family of closed sets each of which is the intersection of $[-\omega - 1, \omega + 1]^{n+m}$ with the set of critical points of $\pi_{\mathcal{Z}(\sigma)}$, over all sign conditions σ such that strata $\mathcal{R}(\sigma)$ have dimension at least n (the notation $\mathcal{Z}(\sigma)$ was introduced in Section 3.2). Let $\mathcal{E}_2(\bar{\varepsilon})$ be the family of closed sets each of which is the intersection of $[-\omega - 1, \omega + 1]^{n+m}$ with $\mathcal{Z}(\sigma)$, over all sign conditions σ such that strata $\mathcal{R}(\sigma)$ have dimension equal to $n - 1$. Let $\mathcal{E}(\bar{\varepsilon}) := \mathcal{E}_1(\bar{\varepsilon}) \cup \mathcal{E}_2(\bar{\varepsilon})$. Denote by $E(\bar{\varepsilon})$ the image under the projection π of the union of all sets in the family $\mathcal{E}(\bar{\varepsilon})$.

Because of the transversality condition, every stratum of the stratification of $S'(\bar{\varepsilon})$, having dimension less than $n + m$, lies in the closure of a stratum having the next higher dimension. In particular, this is true for strata of dimension less than $n - 1$. It follows that $G(\bar{\varepsilon}) \subset E(\bar{\varepsilon})$, and thus every connected component of the complement $R^n \setminus E(\bar{\varepsilon})$, is contained in a connected component of $R^n \setminus G(\bar{\varepsilon})$. Since $\dim(E(\bar{\varepsilon})) < n$, every connected component of $R^n \setminus G(\bar{\varepsilon})$ contains a connected component of $R^n \setminus E(\bar{\varepsilon})$. Therefore, it is sufficient to estimate from above the Betti number $b_0(R^n \setminus E(\bar{\varepsilon}))$ which is equal to $b_{n-1}(E(\bar{\varepsilon}))$ by the Alexander's duality.

The total number of sets $\mathcal{Z}(\sigma)$, such that $\sigma \in \mathrm{Sign}(\mathcal{P}')$ and $\dim(\mathcal{Z}(\sigma)) \geqslant n - 1$, is $O(s^{2(m+1)})$ because each $\mathcal{Z}(\sigma)$ is defined by a conjunction of at most $m + 1$ of the possible $2s^2$ polynomial equations. Thus, the cardinality $\#\mathcal{E}(\bar{\varepsilon})$, as well as the number of images under the projection π of sets in $\mathcal{E}(\bar{\varepsilon})$, is $O(s^{2(m+1)})$. According to (3.1) in Proposition 3.11, $b_{n-1}(E(\bar{\varepsilon}))$ does not exceed the sum of certain Betti numbers of sets of the type

$$\Phi := \bigcap_{1 \leqslant i \leqslant p} \pi(U_i),$$

where every $U_i \in \mathcal{E}(\bar{\varepsilon})$ and $1 \leqslant p \leqslant n$. More precisely, we have

$$b_{n-1}(E(\bar{\varepsilon})) \leqslant \sum_{1 \leqslant p \leqslant n} \sum_{\{U_1, \ldots, U_p\} \subset \mathcal{E}(\bar{\varepsilon})} b_{n-p} \left(\bigcap_{1 \leqslant i \leqslant p} \pi(U_i) \right).$$

Obviously, there are $O(s^{2(m+1)n})$ sets of the kind Φ.

Using inequality (3.2) in Proposition 3.11, we see, for each Φ as above, that the Betti number $b_{n-p}(\Phi)$ does not exceed the sum of certain Betti numbers of unions of the kind

$$\Psi := \bigcup_{1 \leqslant j \leqslant q} \pi(U_{i_j}) = \pi \left(\bigcup_{1 \leqslant j \leqslant q} U_{i_j} \right),$$

with $1 \leqslant q \leqslant p$. More precisely,

$$b_{n-p}(\Phi) \leqslant \sum_{1 \leqslant q \leqslant p} \sum_{1 \leqslant i_1 < \ldots < i_q \leqslant p} b_{n-p+q-1} \left(\pi \left(\bigcup_{1 \leqslant j \leqslant q} U_{i_j} \right) \right).$$

It is clear that there are at most $2^p \leqslant 2^n$ sets of the kind Ψ.

If a set $U \in \mathcal{E}_1(\bar{\varepsilon})$, then it is defined by $O(n+m)$ polynomials of degree at most d (including the linear polynomials defining $[-\omega - 1, \omega + 1]^{n+m}$). If a set $U \in \mathcal{E}_2(\bar{\varepsilon})$, then it is defined by $O(n + 2^m)$ polynomials of degree $O(md)$, since the critical points on strata of dimension at least n are defined by $O(2^m)$ determinantal equations, the corresponding matrices have orders $O(m)$, and the entries of these matrices are polynomials of degree at most d.

It follows that the closed and bounded set

$$\bigcup_{1 \leqslant j \leqslant q} U_{i_j}$$

is defined by $O(n(n + 2^m))$ polynomials of degree $O(md)$. By Proposition 3.10, $b_{n-p+q-1}(\Psi) \leqslant (2^m nd)^{O(nm)}$ for all $1 \leqslant p \leqslant n$, $1 \leqslant q \leqslant p$. Then $b_{n-p}(\Phi) \leqslant (2^m nd)^{O(nm)}$ for every $1 \leqslant p \leqslant n$. Since there are $O(s^{2(m+1)n})$ sets of the kind Φ, we get the claimed bound

$$b_{n-1}(E(\bar{\varepsilon})) \leqslant s^{2(m+1)n} (2^m nd)^{O(nm)}. \qquad \square$$

4. *Semi-Pfaffian case*

Pfaffian functions, introduced in [**18**], are real analytic functions satisfying triangular systems of Pfaffian (first-order partial differential) equations with polynomial coefficients.

DEFINITION 6 [**11**, **18**]. A *Pfaffian chain* of the order $r \geqslant 0$ and degree $\alpha \geqslant 1$ in an open domain $U \subset \mathbb{R}^n$ is a sequence of real analytic functions $f_1, \ldots, f_r$ in U satisfying differential equations

$$df_j(\mathbf{x}) = \sum_{1 \leqslant i \leqslant n} g_{ij}(\mathbf{x}, f_1(\mathbf{x}), \ldots, f_j(\mathbf{x})) dx_i \tag{4.1}$$

for $1 \leqslant j \leqslant r$. Here $g_{ij}(\mathbf{x}, y_1, \ldots, y_j)$ are polynomials in $\mathbf{x} = (x_1, \ldots, x_n)$, $y_1, \ldots, y_j$ of degree not exceeding α. A function $f(\mathbf{x}) = P(\mathbf{x}, f_1(\mathbf{x}), \ldots, f_r(\mathbf{x}))$, where the polynomial $P(\mathbf{x}, y_1, \ldots, y_r)$ has degree not exceeding $\beta \geqslant 1$, is called a *Pfaffian function* of order r and degree (α, β).

The class of Pfaffian functions includes polynomials, real algebraic functions and elementary transcendental functions defined in appropriate domains (see [11]). One of the important subclasses is formed by *fewnomials* [18].

Let $\mathcal{P}$ be a finite set of Pfaffian functions in the open cube $U := (-1,1)^{m+n} \subset \mathbb{R}^{m+n}$.

Definition 7. A set $S \subset U$ is called $\mathcal{P}$-semi-Pfaffian in U (or just *semi-Pfaffian* when some $\mathcal{P}$ is fixed) if it is defined by a Boolean formula with atoms of the form $P > 0$, $P < 0$, $P = 0$ for $P \in \mathcal{P}$. A $\mathcal{P}$-semi-Pfaffian set S is *restricted* if its closure in U is compact.

Semi-Pfaffian sets share many of the finiteness properties of semi-algebraic sets [11]. The proofs of Proposition 3.1 and all the lemmas from Section 3 extend to the semi-Pfaffian case without any difficulty.

Theorem 4.1. *Let $\mathcal{P}$ be a finite set of Pfaffian functions defined on the open cube $U := (-1,1)^{m+n} \subset \mathbb{R}^{m+n}$, with $\#\mathcal{P} = s$, and such that all functions in $\mathcal{P}$ have degree (α, β) and are derived from a common Pfaffian chain of order r. Then, there exists a finite set $A \subset \pi(U)$ with*

$$\#A \leqslant s^{O(nm)} 2^{O(n(m^2+nr^2))} (nm(\alpha+\beta))^{O(n(m+r))},$$

such that for every $\mathbf{y} \in \pi(U)$ there exists $\mathbf{z} \in A$ such that for every $\mathcal{P}$-semi-Pfaffian set $S \subset U$, the set $\pi_S^{-1}(\mathbf{y})$ is homotopy equivalent to $\pi_S^{-1}(\mathbf{z})$. In particular, for any fixed $\mathcal{P}$-semi-Pfaffian set S, the number of different homotopy types of fibres $\pi_S^{-1}(\mathbf{y})$ for various $\mathbf{y} \in \pi(S)$ is also bounded by

$$s^{O(nm)} 2^{O(n(m^2+nr^2))} (nm(\alpha+\beta))^{O(n(m+r))}.$$

Proof. We add to $\mathcal{P}$ the $2(m+n)$ functions, $X_i \pm (1-\delta)$, $Y_j \pm (1-\delta)$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, where $\delta \in \mathbb{R}$ is chosen sufficiently small and positive. We restrict attention to those $\mathcal{P}$-semi-Pfaffian sets which are contained in the cube defined by, $-1+\delta < X_i < 1-\delta$, $-1+\delta < Y_j < 1-\delta$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. For any $\mathcal{P}$-semi-Pfaffian set S, the intersection $S_0 := S \cap (-1+\delta, 1-\delta)^{n+m}$ is homotopy equivalent to S.

As in the proof of Theorem 2.1, let $B \subset \mathbb{R}^n$ be a fixed finite set of points such that for each $\mathcal{P}$-semi-Pfaffian set S and $\mathbf{y} \in \pi(S)$, there exists $\mathbf{z} \in B \cap \pi(S)$ (depending on S) such that $\pi_S^{-1}(\mathbf{y}) \simeq \pi_S^{-1}(\mathbf{z})$.

Then, for all sufficiently small $\delta > 0$, for each $\mathcal{P}$-semi-pfaffian set S, $\pi_{S_0}^{-1}(\mathbf{y}) \simeq \pi_S^{-1}(\mathbf{y})$ for each $\mathbf{y} \in B$. It follows that it is sufficient to bound from above the number of homotopy types of fibres of the projection of the restricted sets S_0.

The rest of the proof of Theorem 4.1 is identical to the corresponding part of the proof of Theorem 2.1. The only essential difference is the replacement of the reference to Proposition 3.10 by the reference to the following proposition. □

Proposition 4.2 [13]. *Let $V \subset U$ be a closed and bounded $\mathcal{P}$-semi-Pfaffian set defined by a Boolean formula such that $\#\mathcal{P} = s$, all functions from $\mathcal{P}$ are defined in U, and have degree (α, β), and a common Pfaffian chain is of order r. Then the kth Betti number of the projection is as follows:*

$$b_k(\pi(V)) \leqslant (ks)^{O(n+km)} 2^{O(kr)^2} ((n+km)(\alpha+\beta))^{O(n+km+kr)}.$$

In Section 5 we will need the following technical improvement of Theorem 4.1. We first introduce some notation.

Definition 8. Let $\tau \in \mathrm{Sign}(\{X_1, \dots, X_m\})$ be a sign condition for the family of coordinate functions. We denote the realization $\mathcal{R}(\tau)$ of τ by $\mathbb{R}^m_\tau$ and call it an *octant* of $\mathbb{R}^m$. In general, for any subset $S \subset \mathbb{R}^m$ and $\tau \in \mathrm{Sign}(\{X_1, \dots, X_m\})$ we introduce $S_\tau := S \cap \mathbb{R}^m_\tau$.

Theorem 4.3. *Let*

$$\mathcal{P} = \bigcup_{\tau \in \mathrm{Sign}(\{X_1,\dots,X_m\})} \mathcal{P}_\tau$$

be a finite set of Pfaffian functions such that for each $\tau \in \mathrm{Sign}(\{X_1, \dots, X_m\})$, functions in $\mathcal{P}_\tau$ are defined in the domain $U_\tau \times V$, where U is either $(-1,1)^m$ or $\mathbb{R}^m$, and V is either $(-1,1)^n$ or $\mathbb{R}^n$. Let $\#\mathcal{P} = s$, let all functions in $\mathcal{P}$ have degree (α, β), and for each τ let functions in $\mathcal{P}_\tau$ be derived from a common Pfaffian chain of order at most r. Then there exists a finite set $A \subset U$ with

$$\#A \leqslant s^{O(nm)} 2^{O(n(m^2+nr^2))} (nm(\alpha+\beta))^{O(n(m+r))},$$

such that for every $\mathbf{y} \in U$ there exists $\mathbf{z} \in A$ such that for every set $S = \bigcup_\tau S_\tau \subset (U \times V)$, where every S_τ is a $\mathcal{P}_\tau$-semi-Pffafian set, the set $\pi_S^{-1}(\mathbf{y})$ is homotopy equivalent to $\pi_S^{-1}(\mathbf{z})$. In particular, for any fixed set S, the number of different homotopy types of fibres $\pi_S^{-1}(\mathbf{y})$ for various $\mathbf{y} \in \pi(S)$ is bounded by

$$s^{O(nm)} 2^{O(n(m^2+nr^2))} (nm(\alpha+\beta))^{O(n(m+r))}.$$

Proof. The proof is a straightforward modification of the proof of Theorem 4.1. We express S as a union of realizations of sign conditions $\sigma \wedge \tau$ on the families $\mathcal{P}_\tau \cup \{X_1, \dots, X_m\}$, where $\sigma \in \mathrm{Sign}(\mathcal{P}_\tau)$ and $\tau \in \mathrm{Sign}(\{X_1, \dots, X_m\})$. Thus,

$$S = \bigcup_{\tau \in \mathrm{Sign}(\{X_1,\dots,X_m\}),\ \sigma \in \Sigma_{\tau,S}} \mathcal{R}(\sigma \wedge \tau) \tag{4.2}$$

with $\Sigma_{\tau,S} \subset \mathrm{Sign}(\mathcal{P}_\tau)$.

Given $\tau \in \mathrm{Sign}_\ell(\{X_1, \dots, X_m\})$, let $\tau_>$ and $\tau_<$ be sign conditions in $\mathrm{Sign}_{\ell-1}(\{X_1, \dots, X_m\})$ in which some equation $X_i = 0$ is replaced by $X_i > 0$ and $X_i < 0$, respectively. Since functions in $\mathcal{P}_\tau$ are defined in $U_\tau \times V$ and do not depend on X_i, they are also defined in the larger domain $(U_\tau \cup U_{\tau_>} \cup U_{\tau_<}) \times V$. It follows that for S given by (4.2), the set $S'(\bar{\varepsilon})$ is well defined and is homotopy equivalent to S.

The rest of the proof is the same as the proof of Theorem 4.1. □

5. *On a conjecture of Benedetti and Risler*

In their book [**4**], Benedetti and Risler introduced an axiomatic definition of the *complexity* of a polynomial, and consequently, of a semi-algebraic set. They conjectured that there is a finite number of homeomorphism types of semi-algebraic sets of a given complexity. Benedetti and Risler also formulated their conjecture for two particular cases: the number of monomials and *additive complexity*. In these cases the conjecture was proved independently by van den Dries [**10**] and Coste [**6**] using o-minimality theory, without producing any explicit upper bounds on the number of homeomorphism types as functions of complexity.

In this section we prove weaker, but effective, versions of these two particular cases of the conjecture. Namely, we give explicit, single exponential upper bounds on the number of possible *homotopy types* of semi-algebraic sets. Our proofs are strongly influenced by arguments of van den Dries in [**10**, § 3, Chapter 9].

5.1. *Homotopy types of sets defined by fewnomials*

We first prove a single exponential upper bound on the number of homotopy types of semi-algebraic subsets of $\mathbb{R}^m$ defined by a family of polynomials having in total at most r monomials.

More precisely, let $\mathcal{M}_{m,r}$ be the family of ordered lists $\mathcal{P} = (P_1, \ldots, P_s)$ of polynomials $P_i \in \mathbb{R}[X_1, \ldots, X_m]$, with the total number of monomials in all polynomials in $\mathcal{P}$ not exceeding r. Recall Definition 1, of the homotopy type of an ordered list of polynomials.

Theorem 5.1. *The number of different homotopy types of ordered lists in $\mathcal{M}_{m,r}$ does not exceed*

$$2^{O(mr)^4}. \tag{5.1}$$

In particular, the number of different homotopy types of semi-algebraic sets defined by a fixed formula $\phi_{\mathcal{P}}$, where $\mathcal{P}$ varies over $\mathcal{M}_{m,r}$, does not exceed (5.1).

Proof. Observe that the function

$$|X|^Y = e^{(Y \ln |X|)}$$

is Pfaffian of constant order and degree in the domain $\{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$.

We first prove the theorem for semi-algebraic sets defined in one fixed octant $\mathbb{R}^m_\tau$ for some sign condition $\tau \in \mathrm{Sign}(\{X_1, \ldots, X_m\})$ (see Definition 8).

Let $\{i_1, \ldots, i_t\} \subset \{1, \ldots, m\}$ be such that, $\tau(X_{i_j}) \in \{-1, 1\},\ 1 \leqslant j \leqslant t$, and $\tau(X_j) = 0,\ j \in \{1, \ldots, m\} \setminus \{i_1, \ldots, i_t\}$.

Consider the family of 2^r Pfaffian functions

$$\sum_{1 \leqslant j \leqslant r} \pm Z_j |X_{i_1}|^{Y_{ji_1}} \ldots |X_{i_t}|^{Y_{ji_t}} \tag{5.2}$$

in $\mathbb{R}^m_\tau \times \mathbb{R}^{tr+r}$. Observe that if

$$P := \sum_{1 \leqslant j \leqslant r} q_j X_{i_1}^{\alpha_{ji_1}} \ldots X_{i_t}^{\alpha_{ji_t}} \in \mathbb{R}[X_{i_1}, \ldots, X_{i_t}]$$

is a polynomial having r monomials defined over $\mathbb{R}^m_\tau$, then there is a function

$$F_\tau(X_{i_1}, \ldots, X_{i_t}, Y_{1i_1}, \ldots, Y_{ri_t}, Z_1, \ldots, Z_r)$$

in the family (5.2) such that

$$P = F_\tau(X_{i_1}, \ldots, X_{i_t}, \alpha_{1i_1}, \ldots, \alpha_{ri_t}, q_1, \ldots, q_r),$$

where the choice of the function is determined by parities (values mod 2) of integers in the sequence $\alpha_{ji_1}, \ldots, \alpha_{ji_t}$. For example, if $\tau(X_{i_c}) = -1$, and α_{ji_c} is odd, then the term $|X_{i_c}|^{Y_{ji_c}}$ contributes the factor (-1) to the coefficient of the monomial $Z_j |X_{i_1}|^{Y_{ji_1}} \ldots |X_{i_t}|^{Y_{ji_t}}$; likewise, if α_{ji_c} is even, the $|X_{i_c}|^{Y_{ji_c}}$ contributes the factor 1 to the coefficient.

It follows that all polynomials having r_0 monomials defined in $\mathbb{R}^m_\tau$ can be divided into 2^{r_0} disjoint classes so that each class is *represented* by a Pfaffian function in the family (5.2).

Consider an ordered list $\mathcal{P}_\tau = (P_1, \ldots, P_s)$ of polynomials $P_i \in \mathbb{R}[X_{i_1}, \ldots, X_{i_t}]$. Assume that the total number of monomials appearing in all polynomials $P_1, \ldots, P_s$ is r. Let $\mathcal{F}_\tau = (F_{\tau 1}, \ldots, F_{\tau s})$ be the list of Pfaffian functions representing the respective polynomials in $\mathcal{P}_\tau$. We say that $\mathcal{F}_\tau$ *represents* $\mathcal{P}_\tau$.

Let

$$\pi_\tau : \mathbb{R}^m_\tau \times \mathbb{R}^{tr+r} \longrightarrow \mathbb{R}^{tr+r}$$

be the projection map. According to Theorem 4.1, for each list $\mathcal{F}_\tau$ there exists a finite set $A_{\mathcal{F}_\tau} \subset \mathbb{R}^{tr+r}$ with cardinality not exceeding

$$2^{O(mr)^4}, \tag{5.3}$$

such that for every $\mathbf{y} \in \mathbb{R}^{tr+r}$ there exists $\mathbf{z} \in A_{\mathcal{F}_\tau}$ such that for every $\mathcal{F}_\tau$-semi-Pfaffian set S, the set $\pi_\tau^{-1}(\mathbf{y}) \cap S$ is homotopy equivalent to $\pi_\tau^{-1}(\mathbf{z}) \cap S$. In particular, (5.3) is an upper bound on the number of homotopy types of lists $\mathcal{P}_\tau$ which are represented by $\mathcal{F}_\tau$. Since there are 2^r different lists $\mathcal{F}_\tau$, the number of homotopy types of all lists $\mathcal{P}_\tau$ is at most (5.3) multiplied by 2^r.

Now we consider the general case of a semi-algebraic set defined in $\mathbb{R}^m$. The proof will follow the same pattern as the simpler case of a set defined in an octant $\mathbb{R}^m_\tau$ just described. Observe that

$$\mathbb{R}^m = \bigcup_{\tau \in \mathrm{Sign}(\{X_1,\dots,X_m\})} \mathbb{R}^m_\tau.$$

Each ordered list $\mathcal{P} = (P_1, \dots, P_s) \subset \mathbb{R}[X_1, \dots, X_m]$, having in total r monomials, and defined in $\mathbb{R}^m$, has its representative list $\mathcal{F}_\tau = (F_{\tau 1}, \dots, F_{\tau s})$ of Pfaffian functions (among 2^r lists) when restricted on the octant $\mathbb{R}^m_\tau$. Thus, each $\mathcal{P}$, defined in $\mathbb{R}^m$, is *represented* by a family of lists $\{\mathcal{F}_\tau \mid \tau \in \mathrm{Sign}(\{X_1, \dots, X_m\})\}$ of Pfaffian functions, corresponding to all 3^m octants. For a given $\mathcal{P}$, the corresponding family of lists $\{\mathcal{F}_\tau \mid \tau \in \mathrm{Sign}(\{X_1, \dots, X_m\})\}$ is determined by the parities (values mod 2) of integers in the total ordered sequence of powers of all r monomials appearing in the polynomials in $\mathcal{P}$. Since the number of different sequences of parities of r monomials in m variables is 2^{mr}, the total number of families that we need to consider is bounded by 2^{mr}.

For a Boolean formula ϕ with atoms $\{a_i, b_i, c_i \mid 1 \leqslant i \leqslant s\}$, denote by $\phi_{\mathcal{F}_\tau}$ the formula obtained from ϕ by replacing for each i, $1 \leqslant i \leqslant s$, the atom a_i, b_i and c_i by $F_{\tau i} = 0$, $F_{\tau i} > 0$ and $F_{\tau i} < 0$, respectively. Let $S(\phi_{\mathcal{F}_\tau})$ be the semi-Pfaffian set defined by $\phi_{\mathcal{F}_\tau}$.

Let

$$\pi : \mathbb{R}^m \times \mathbb{R}^{mr+r} \longrightarrow \mathbb{R}^{mr+r}$$

be the projection map. According to Theorem 4.3, for each fixed family of lists $\{\mathcal{F}_\tau \mid \tau \in \mathrm{Sign}(\{X_1, \dots, X_m\})\}$, there exists a finite set $A \subset \mathbb{R}^{mr+r}$ with cardinality not exceeding (5.3), such that for every $\mathbf{y} \in \mathbb{R}^{mr+r}$ there exists $\mathbf{z} \in A$ such that for every Boolean formula ϕ and

$$S := \bigcup_{\tau \in \mathrm{Sign}(\{X_1,\dots,X_m\})} S(\phi_{\mathcal{F}_\tau}),$$

the set $\pi_\tau^{-1}(\mathbf{y}) \cap S$ is homotopy equivalent to $\pi_\tau^{-1}(\mathbf{z}) \cap S$. Hence (5.3) is also an upper bound on the number of homotopy types of lists $\mathcal{P}$ represented by the fixed family $\{\mathcal{F}_\tau \mid \tau \in \mathrm{Sign}(\{X_1, \dots, X_m\})\}$ of Pfaffian functions. It follows that the number of all homotopy types of lists $\mathcal{P} \in \mathcal{M}_{m,r}$ is at most (5.3) multiplied by 2^{mr} since there are 2^{mr} different families of lists to consider. □

5.2. *Homotopy types of sets with bounded additive complexity*

DEFINITION 9. A polynomial $P \in \mathbb{R}[X_1, \dots, X_m]$ has *additive complexity at most a* if there are polynomials $Q_1, \dots, Q_a \in \mathbb{R}[X_1, \dots, X_m]$ such that:

(i) $Q_1 = a_1 X_1^{\alpha_{11}} \dots X_m^{\alpha_{1m}} + b_1 X_1^{\beta_{11}} \dots X_m^{\beta_{1m}}$,
where $a_1, b_1 \in \mathbb{R}$, and $\alpha_{11}, \dots, \alpha_{1m}, \beta_{11}, \dots, \beta_{1m} \in \mathbb{N}$;

(ii) $Q_j = a_j X_1^{\alpha_{j1}} \dots X_m^{\alpha_{jm}} \prod_{1 \leqslant i \leqslant j-1} Q_i^{\gamma_{ji}} + b_j X_1^{\beta_{j1}} \dots X_m^{\beta_{jm}} \prod_{1 \leqslant i \leqslant j-1} Q_i^{\delta_{ji}}$,
where $1 < j \leqslant a$, $a_j, b_1 j \in \mathbb{R}$, and $\alpha_{j1}, \dots, \alpha_{jm}, \beta_{j1}, \dots, \beta_{jm} \gamma_{ji}, \delta_{ji} \in \mathbb{N}$ for $1 \leqslant i < j$;

(iii) $P = c X_1^{\zeta_1} \dots X_m^{\zeta_m} \prod_{1 \leqslant j \leqslant a} Q_j^{\eta_j}$,
where $c \in \mathbb{R}$, and $\zeta_1, \dots, \zeta_m, \eta_1, \dots, \eta_a \in \mathbb{N}$.

In other words, P has additive complexity at most a if, starting with variables $X_1, \dots, X_m$ and constants in $\mathbb{R}$, and applying additions and multiplications, a formula representing P can be obtained using at most a additions (and an unlimited number of multiplications).

EXAMPLE 2. The polynomial $P := (X+1)^d \in \mathbb{R}[X]$ with $0 < d \in \mathbb{Z}$, has $d+1$ monomials when expanded but additive complexity at most 1.

Let $\mathcal{A}_{m,a}$ be the family of ordered lists $\mathcal{P} = (P_1, \dots, P_s)$ of polynomials $P_i \in \mathbb{R}[X_1, \dots, X_m]$, with the additive complexity of every P_k not exceeding a_k, and $a = \sum_{1 \leqslant k \leqslant s} a_k$.

THEOREM 5.2. *The number of different homotopy types of ordered lists in $\mathcal{A}_{m,a}$ does not exceed*

$$2^{O((m+a)a)^4}. \tag{5.4}$$

In particular, the number of different homotopy types of semi-algebraic sets defined by a fixed formula $\phi_{\mathcal{P}}$, where $\mathcal{P}$ varies over $\mathcal{A}_{m,a}$, does not exceed (5.4).

Proof. Fix an ordered list $\mathcal{P} \in \mathcal{A}_{m,a}$. For each polynomial $P_k \in \mathcal{P}$, $1 \leqslant k \leqslant s$, consider the sequence of polynomials $Q_{k1}, \dots, Q_{ka_k}$ as in Definition 9, so that

$$P_k := c_k X_1^{\zeta_{k1}} \dots X_m^{\zeta_{km}} \prod_{1 \leqslant j \leqslant a_k} Q_{kj}^{\eta_{kj}}.$$

Introduce a_k new variables $Y_{k1}, \dots, Y_{ka_k}$. Fix a semi-algebraic set $S \subset \mathbb{R}^m$, defined by a formula $\phi_{\mathcal{P}}$. Consider the semi-algebraic set $\widehat{S}$, defined by the conjunction of a 3-nomial equations obtained from the equalities in Definition 9 (i), (ii) by replacing Q_j by Y_{kj} for all $1 \leqslant k \leqslant s$, $1 \leqslant j \leqslant a_k$, and the formula $\phi_{\mathcal{P}}$ in which every occurrence of an atomic formula of the kind $P_k * 0$, where $* \in \{=, >, <\}$, is replaced by the formula

$$c_k X_1^{\zeta_{k1}} \dots X_m^{\zeta_{km}} \prod_{1 \leqslant j \leqslant a_k} Y_{kj}^{\eta_{kj}} * 0.$$

Note that $\widehat{S}$ is a semi-algebraic set in $\mathbb{R}^{m+a}$ defined by a formula $\widehat{\phi}_{\widehat{\mathcal{P}}}$, where $\widehat{\mathcal{P}} \in \mathcal{M}_{m+a,s+3a}$. According to Theorem 5.1, the number of different homotopy types of ordered lists in $\mathcal{M}_{m+a,s+3a}$ does not exceed (5.4).

Let $\rho : \mathbb{R}^{m+a} \to \mathbb{R}^m$ be the projection map on the subspace of coordinates $X_1, \dots, X_m$. It is clear that the restriction $\rho_{\widehat{S}} : \widehat{S} \to S$ is a homeomorphism. Therefore, if two lists $\mathcal{P}, \mathcal{Q} \in \mathcal{A}_{m,a}$ are *not* homotopy equivalent, then the corresponding lists $\widehat{\mathcal{P}}, \widehat{\mathcal{Q}} \in \mathcal{M}_{m+a,s+3a}$ are also not homotopy equivalent. It follows that the number of different homotopy types of lists in $\mathcal{A}_{m,a}$ does not exceed the number of different homotopy types of lists in $\mathcal{M}_{m+a,s+3a}$, and therefore does not exceed (5.4). $\square$

REMARK 2. A polynomial $P \in \mathbb{R}[X_1, \dots, X_m]$ is said to have *rational additive complexity at most a* if there are *rational functions* $Q_1, \dots, Q_a \in \mathbb{R}(X_1, \dots, X_m)$ satisfying conditions (i), (ii), and (iii) of Definition 9 with $\mathbb{N}$ replaced by $\mathbb{Z}$. For example, the polynomial $X^d + \dots + X + 1 = (X^{d+1} - 1)/(X - 1) \in \mathbb{R}[X]$ with $0 < d \in \mathbb{Z}$, has rational additive complexity at most 2. Define the family of ordered lists of polynomials $\mathcal{A}_{m,s,a}$ as above but interpreting a as the sum of rational additive complexities. According to [**10**] there is a finite number of different homeomorphism types of semi-algebraic sets defined by s polynomials in $\mathbb{R}[X_1, \dots, X_m]$, with the sum of additive complexities at most a. We conjecture that the number of different homotopy types of lists in $\mathcal{A}_{m,s,a}$ does not exceed

$$2^{(ma)^{O(1)}}.$$

6. *Metric upper bounds*

In this section we consider semi-algebraic sets defined by polynomials with integer coefficients, and use the technique from previous sections to obtain some 'metric' upper bounds related to homotopy types.

Let $V \subset R^m$ be a $\mathcal{P}$-semi-algebraic set, where $\mathcal{P} \subset \mathbb{Z}[X_1, \ldots, X_m]$. For each $P \in \mathcal{P}$, let $\deg(P) < d$, and the maximum of the absolute values of coefficients in P be less than some constant M, $0 < M \in \mathbb{Z}$. For $a > 0$ we denote by $B_m(0, a)$ the open ball of radius a in R^m centered at the origin.

The following proposition is well known (see, for instance, [**1**, Theorem 4.1.1]).

Proposition 6.1 [**1**, **15**, **17**, **19**]. *There exists a constant $c > 0$, such that for any $r > M^{d^{cm}}$, and for any connected component W of V the intersection $W \cap B_m(0, r)) \neq \emptyset$, and $W \subset B_m(0, r)$ if W is bounded.*

Notice that there is no dependence in the bound on the cardinality of the family $\mathcal{P}$. On the other hand, the dependence on the absolute values of integer coefficients is essential, as shown by an example of the semi-algebraic set defined by the system of equations

$$X = Y, \quad X = (1 - \varepsilon)Y + 1,$$

in which all coefficients are $\Theta(1)$, independently of any small real $\varepsilon > 0$.

We prove the following generalization of Proposition 6.1.

Theorem 6.2. *There exists a constant $c > 0$, such that for any $r_1 > r_2 > M^{d^{cm}}$ we have:*

(i) $V \cap B_m(0, r_1) \simeq V \cap B_m(0, r_2)$, *and*
(ii) $V \setminus B_m(0, r_1) \simeq V \setminus B_m(0, r_2)$.

In the proof of Theorem 6.2 it will be convenient to use *infinitesimals* instead of sufficiently small (or large) elements of the ground real closed field R. We do this by considering non-archimedean extensions of R. More precisely, denote by $R\langle\varepsilon\rangle$ the real closed field of algebraic Puiseux series in ε with coefficients in R (see, for instance, [**2**] for more details about Puiseux series). The sign of a Puiseux series in $R\langle\varepsilon\rangle$ agrees with the sign of the coefficient of the lowest degree term in ε. This induces a unique order on $R\langle\varepsilon\rangle$ which makes ε infinitesimal: ε is positive and smaller than any positive element of R. When $a \in R\langle\varepsilon\rangle$ is bounded from above by an element of R, the symbol $\lim_\varepsilon(a)$ denotes the constant term of a, obtained by substituting 0 for ε in a; clearly $\lim_\varepsilon(a) \in R$. We will also denote by $R\langle\bar{\varepsilon}\rangle$ the field $R\langle\varepsilon_1\rangle \ldots \langle\varepsilon_s\rangle$, where ε_{i+1} is infinitesimal with respect to $R\langle\varepsilon_1\rangle \ldots \langle\varepsilon_i\rangle$ for every $0 \leqslant i < s$.

Proof of Theorem 6.2. We will only prove the homotopy equivalence (i), the proof of (ii) being similar.

Let ϕ be the $\mathcal{P}$-formula defining V. Let $S \subset R^{m+1}$ be the set defined by the formula, $\phi \wedge (X_1^2 + \ldots + X_m^2 - Y^2 \leqslant 0)$, and let $\pi : R^{m+1} \to R$ be the projection on the Y coordinate. We will follow the notation introduced in the proof of Theorem 2.1, but in the definition of $S' = S'(\bar{\varepsilon})$ (cf. Definition 4) we let $1 \gg \frac{1}{\omega} \gg \bar{\varepsilon} > 0$ be infinitesimals and let R' be the field of Puiseux series, $R\langle\frac{1}{\omega}\rangle\langle\bar{\varepsilon}\rangle$. The set S' is then a semi-algebraic subset of R'^{m+1}.

Consider now the set $G(S, \bar{\varepsilon}) \subset R'$ (cf. Definition 5). It follows as a consequence of the complexity analysis of efficient quantifier elimination algorithms (see [**2**, **15**]), that $G(S, \bar{\varepsilon})$ is a finite subset of R' consisting of roots in R' of the univariate polynomials $h \in \mathbb{Z}[\omega, \bar{\varepsilon}][Y]$, the degrees of which do not exceed $d^{O(k)}$, and coefficients of which are integers with absolute values not exceeding $M^{d^{O(k)}}$. Now suppose that $\alpha \in R'$, $h(\alpha) = 0$ and α is bounded from above by an element of R. Then, $\lim_{\frac{1}{\omega}}(\alpha) \in R$ is a root of some polynomial in $\mathbb{Z}[Y]$ of degree not exceeding $d^{O(k)}$, with absolute values of coefficients at most $M^{d^{O(k)}}$. It follows that $|\alpha| \leqslant M^{d^{O(k)}}$.

Now let $G_b(S, \bar{\varepsilon}) \subset G(S, \bar{\varepsilon})$ be the set of all elements of G which are bounded from above by some element of R. Let $r = \max_{\alpha \in G_b} |\alpha|$, then $r \leqslant M^{d^{O(k)}}$. Moreover, if elements $r_1, r_2 \in R$ are both greater than r, and belong to the same connected component of $R' \setminus G(S, \bar{\varepsilon})$, then $\pi_{S'}^{-1}(r_1) \simeq \pi_{S'}^{-1}(r_2)$. Since $\pi_S^{-1}(y) \simeq V$ for all sufficiently large $y \in R$, and $\pi_{S'}^{-1}(y) \simeq \pi_S^{-1}(y)$ for all $y \in R$, it follows that

$$\pi_{S'}^{-1}(r_1) \simeq \pi_{S'}^{-1}(r_2) \simeq \pi_S^{-1}(r_1) \simeq \pi_S^{-1}(r_2) \simeq V.$$

This proves the theorem. □

It is well known that a semi-algebraic set has a *local conic structure* (see, for example, [7]). In particular, a semi-algebraic set is locally contractible. The following theorem gives a quantitative version of the latter statement.

THEOREM 6.3. *Let $V \subset R^m$ be a $\mathcal{P}$-semi-algebraic set, with $\mathcal{P} \subset \mathbb{Z}[X_1, \ldots, X_m]$ and $0 \in V$. Let $\deg(P) < d$ for each $P \in \mathcal{P}$, and let the maximum of absolute values of coefficients of $P \in \mathcal{P}$ be less than M, $0 < M \in \mathbb{Z}$. Then there exists a constant $c > 0$ such that for every $0 < r < M^{-d^{cm}}$ the set $V \cap B_m(0, r)$ is contractible.*

Proof. This proof is similar to the proof of Theorem 6.2. □

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