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RESEARCH ARTICLE

On Stone sublattices of the lattice of totally local Fitting classes

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Dedicated to Professor V. V. Kirichenko
on the occasion of his 65th birthday

Abstract. Totally local Fitting classes with a Stone lattice of totally local Fitting subclasses are described.

Introduction

All groups considered are finite. Recall that a formation $\mathfrak{F}$ is called local if there exists such a function $f : \mathbb{P} \to \{\text{formations of groups}\}$ that a group $G \in \mathfrak{F}$ if and only if $G/O_{p',p}(G) \in f(p)$ for every prime p dividing the order $|G|$ of G and in this case they write $\mathfrak{F} = LF(f)$. Analogously, a Fitting class $\mathfrak{F}$ is called local [1, 2] if there exists such a function f : $\mathbb{P} \to \{\text{Fitting classes}\}$ that $G \in \mathfrak{F}$ if and only if $O^{p,p'}(G) \in f(p)$ for every prime p dividing $|G|$ where $O^{p,p'}(G) = (G^{\mathfrak{G}_{p'}})^{\mathfrak{N}_p}$ (i.e. $O^{p,p'}(G)$ is the $\mathfrak{N}_p$-residual of the $\mathfrak{G}_{p'}$-residual $G^{\mathfrak{G}_{p'}}$ of G). In this case we write $\mathfrak{F} = LR(f)$ according to [3].

Reviewing the most famous concrete Fitting classes of groups one can see that a majority of them may be defined by functions all non-empty values of which are local Fitting classes themselves. This circumstance led us to the following natural construction [4]: by definition every Fitting class is 0-multiply local and for $n \geqslant 1$ a Fitting class $\mathfrak{F}$ is called n-multiply local if $\mathfrak{F} = LR(f)$ where all non-empty values of the function f

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are $(n-1)$-multiply local Fitting classes. A Fitting class is called totally local if it is n-multiply local for all natural n. Multiply local and totally local formations are defined analogously. It is not difficult to show that the class of all soluble totally local formations coincides with the class of all so-called primitive saturated formations introduced by Hawkes in [5] (see also the book by Doerk and Hawkes [6, Chapter VII]). Note also that the class of all totally local formations coincides with the class of all totally saturated formations introduced by Doerk in his paper [7]. Moreover the n-multiply local class concept in fact was originated in this paper. The theory of multiply local classes is interesting by itself and basides it is a very useful instrument in studying of many questions of the theory of classes and its various applications (for example see [8-12] and the books [13, 14]).

Being the extremal case, totally local classes have a series of specific properties. In particular we note that for every non-negative integer n the lattices of all n-multiply local formations, of all n-multiply local hereditary formations, of all n-multiply local normally hereditary formations etc. are modular but all of them are not distributive even in the class of all soluble groups $\mathfrak{S}$ (see [13, Chapter 2] and [14, Chapter 4]). Moreover as it was mentioned in [15] (see also [13, 14]) for every two non-negative integers n and m the systems of all laws of the lattices of all n-multiply local and all m-multiply local formations coincide. On the other hand the lattice of all soluble totally local formations is distributive [13] but we know nothing about it in the general case (see [14, Question 4.2.14] and [16, Question 14.80]).

In this connection the following natural questions arise:

Question 1 (see [16, Question 14.47]). Is the lattice of all Fitting classes modular (in the class of all soluble groups at least)?

Question 2 (see [16, Question 14.80], [13, Question 10.11] and [14, Question 4.2.14]). Is the lattice of all totally local formations distributive (or modular at least)?

Question 3. Do the systems of all laws of the lattices of all n-multiply local and all m-multiply local Fitting classes coincide for every non-negative integers n and m?

Question 4. Is the lattice of all totally local Fitting classes distributive (or modular at least)?

Let $\mathfrak{F}$ be an n-multiply local (totally local) Fitting class. Then the symbol $L^n(\mathfrak{F})$ ($L^\infty(\mathfrak{F})$) denotes the lattice of all its n-multiply local Fitting subclasses (the lattice of all its totally local Fitting subclasses).

In this paper we prove the following

Theorem A. *Let $\mathfrak{F}$ be an n-multiply local Fitting class. Then $L^n(\mathfrak{F})$ is a Stone lattice if and only if $\mathfrak{F} \subseteq \mathfrak{N}$.*

Theorem B. *Let $\mathfrak{F}$ be a totally local Fitting class. Then $L^\infty(\mathfrak{F})$ is a Stone lattice if and only if $\mathfrak{F} \subseteq \mathfrak{N}$.*

1. The proof of Theorem A

Recall some definitions [3] connected to local Fitting classes. If $\mathfrak{F} = LR(f)$ then f is called an H-function of $\mathfrak{F}$ (and in this case we say that $\mathfrak{F}$ has an H-function f). The symbol l^n (l^∞) denotes the lattice of all n-multiply local Fitting classes (the lattice of all totally local Fitting classes). An H-function f is called l^n-valued (l^∞-valued) if every its non-empty value belongs to l^n (l^∞). Let $\{f_i(p) \mid i \in I\}$ be an arbitrary set of l^n-valued (l^∞-valued) H-functions. For every $p \in \mathbb{P}$ we set $(\bigcap\limits_{i\in I} f_i)(p) = \bigcap\limits_{i\in I} f_i(p)$. The H-function $\bigcap\limits_{i\in I} f_i$ is called an intersection of H-functions f_i. If a Fitting class $\mathfrak{F}$ has at least one l^n-valued (l^∞-valued) H-function then the intersection of all such H-functions of $\mathfrak{F}$ is called a minimal l^n-valued (l^∞-valued) H-function of $\mathfrak{F}$.

Let $\mathfrak{X}$ be an arbitrary non-empty set of groups. The intersection of all n-multiply local Fitting classes (totally local Fitting classes) contained $\mathfrak{X}$ is denoted by $l^n\mathrm{fit}\mathfrak{X}$ ($l^\infty\mathrm{fit}\mathfrak{X}$) and it is called an n-multiply local Fitting class generated by $\mathfrak{X}$ (a totally local Fitting class generated by $\mathfrak{X}$). If $\mathfrak{X} = \{G\}$ then we write $l^n\mathrm{fit}\, G$ instead of $l^n\mathrm{fit}\{G\}$ ($l^\infty\mathrm{fit}\, G$ instead of $l^\infty\mathrm{fit}\{G\}$). Every Fitting class in this form is called a one-generated n-multiply (totally local) Fitting class.

We write $\mathcal{K}(G) = (\Sigma)$ where Σ is a set of all composition factors of G and $\mathcal{K}(\mathfrak{F})$ denotes the join of the sets $\mathcal{K}(G)$ for all groups G of $\mathfrak{F}$.

Lemma 1. *Let a simple group A belongs to $l^n\mathit{fit}\mathfrak{X}$ where $\mathfrak{X}$ is a class of groups. Then if $n = 0$, $A \simeq H/K$ where $H/K \in \mathcal{K}(G)$ for some group $G \in \mathfrak{X}$.*

If $n > 0$ the following statements satisfy

1) if A is non-abelian then $A \simeq H/K$ where $H/K \in \mathcal{K}(G)$ for some group $G \in \mathfrak{X}$;

2) if $A = Z_p$ is a group of an order p then $Z_p \simeq H \leqslant G$ for some subgroup H of $G \in \mathfrak{X}$.

Proof. Let $n = 0$. Evidently the class $l^n\mathrm{fit}\mathfrak{X}$ consists of all the groups which can be obtained as a result of finite application of the operations S_n and R to groups of $\mathfrak{X}$. Clearly if N is a normal subgroup of G then $\mathcal{K}(N) \subseteq \mathcal{K}(G)$. If $G = AB$ where A, B are normal subgroups of G then $\mathcal{K}(G) = \mathcal{K}(A) \cup \mathcal{K}(B)$. Consequently $\mathcal{K}(\mathfrak{X}) = \mathcal{K}(\mathrm{fit}\mathfrak{X})$.

Let $n > 0$. We suppose that for $n - 1$ the statement is true. Let $p \in \pi(A)$, $\mathfrak{F} = l^n\mathrm{fit}\mathfrak{X}$ and let f be a minimal l^{n-1}-valued H-function of the Fitting class $\mathfrak{F}$. If A is non-abelian then $O^{p,p'}(A) = A$. Hence

$$A = O^{p,p'}(A) \in f(p) = l^{n-1}\text{fit}(\mathfrak{X}(F^p)) = l^{n-1}\text{fit}(\text{fit}(O^{p,p'}(A) \mid A \in \mathfrak{X}),$$

by hypothethis $A \simeq H/K$ where $H/K \in \mathcal{K}(G)$ for some group $G \in \mathfrak{X}(F^p)$. As proved above $\mathcal{K}(\mathfrak{X}(F^p)) \subseteq \mathcal{K}(\text{fit}\mathfrak{X}) = \mathcal{K}(\mathfrak{X})$.

If A is a group of an order p then $p \in \pi(\mathfrak{F})$. Hence $A \simeq H \leqslant G$ for some subgroup H of $G \in \mathfrak{F}$.

□

Lemma 2. *Let $\mathfrak{F} = l^n fitG$ be a one-generated n-multiply local Fitting class. Then the lattice $L^n(\mathfrak{F})$ contains a finite number of atoms only.*

Proof. Let $\mathfrak{M}$ be an atom of the lattice $L^n(\mathfrak{F})$. Then $\mathfrak{M} = l^n\text{fit}A$ where A is a simple group in $\mathfrak{M}$. If A is non-abelian then by Lemma 1 we have $A \simeq H/K$ where $H/K \in \mathcal{K}(G)$. The group G is finite so there is a finite number of composition factors in G. Consequently the lattice $L^n(\mathfrak{F})$ contains a finite number of nonsoluble atoms.

Let $A = Z_p$ be a group of a prime order p. As p devides $|G|$ there exists only a finite number of soluble atoms in the lattice $L^n(\mathfrak{F})$.

□

The system $\{\mathfrak{F}_i \mid i \in I\}$ of non-empty classes of groups $\mathfrak{F}_i$ is called orthogonal [17] if:

1) $|I| > 1$;

2) $\mathfrak{F}_i \cap \mathfrak{F}_j = (1)$ for every two different $i, j \in I$.

According to [14] for every orthogonal system of classes $\{\mathfrak{F}_i \mid i \in I\}$ we denote by $\oplus_{i\in I}\mathfrak{F}_i$ (or else by $\mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_n$ in the case when $I = \{1, \ldots, n\}$) the set of all groups in the form $A_1 \times \ldots \times A_t$ where $A_1 \in \mathfrak{F}_{i_1}, \ldots, A_t \in \mathfrak{F}_{i_t}$ for some natural numbers n, t and $i_1, \ldots, i_t \in I$. Let us agree to write $\mathfrak{F} = \oplus_{i\in I}\mathfrak{F}_i$ if $|I| = 1$ and $\mathfrak{F}_i = \mathfrak{F}$. Any representation of a class of groups $\mathfrak{F}$ in the form $\mathfrak{F} = \oplus_{i\in I}\mathfrak{F}_i$ where $|I| > 1$ is called a direct decomposition of this class.

Lemma 3. *[17, Lemma 4] Let $\mathfrak{F}_1, \ldots, \mathfrak{F}_t$ be an orthogonal system of Fitting classes. Then $\mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_t$ is a Fitting class.*

Lemma 4. *[17, Theorem 1] Let $\mathfrak{F} = \oplus_{i\in I}\mathfrak{F}_i$ where $\mathfrak{F}_i$ is a Fitting class for all $i \in I$. Then the Fitting class $\mathfrak{F}$ is n-multiply local if and only if every Fitting class $\mathfrak{F}_i$ is n-multiply local.*

Lemma 5. *[17, Lemma 1] Let $\mathfrak{F} = \oplus_{i\in I}\mathfrak{F}_i$ and $\mathfrak{M}$ be a non-empty Fitting subclass in $\mathfrak{F}$. Then $\mathfrak{M} = \oplus_{i\in I}(\mathfrak{M} \cap \mathfrak{F}_i)$.*

Lemma 6. *Let $\mathfrak{F}$ be an n-multiply local Fitting class. Then if the class $\mathfrak{N}_p$ is complemented in the lattice $L^n(\mathfrak{F})$ for every $p \in \pi(\mathfrak{F})$, $\mathfrak{F} \subseteq \mathfrak{N}$.*

Proof. Let $\mathfrak{M}$ be an n-multiply local Fitting subclass in $\mathfrak{F}$. We show that if $\mathfrak{N}_p \subseteq \mathfrak{M}$ then the subclass $\mathfrak{N}_p$ is complemented in the lattice $L^n(\mathfrak{M})$. Really let $\mathfrak{H}$ be a complement to $\mathfrak{N}_p$ in $L^n(\mathfrak{F})$. Then by Lemmas 3 and 4

$$\mathfrak{F} = \mathfrak{N}_p \vee^n \mathfrak{H} = \mathfrak{N}_p \oplus \mathfrak{H},$$

and so by Lemma 5

$$\mathfrak{M} = \mathfrak{M} \cap \mathfrak{F} = \mathfrak{M} \cap (\mathfrak{N}_p \oplus \mathfrak{H}) =$$
$$(\mathfrak{M} \cap \mathfrak{N}_p) \oplus (\mathfrak{M} \cap \mathfrak{H}) = \mathfrak{N}_p \oplus (\mathfrak{M} \cap \mathfrak{H}) =$$
$$\mathfrak{N}_p \vee^n (\mathfrak{M} \cap \mathfrak{H}),$$

where $\mathfrak{N}_p \cap (\mathfrak{M} \cap \mathfrak{H}) = (1)$. Hence $\mathfrak{M} \cap \mathfrak{H}$ is a complement to $\mathfrak{N}_p$ in $L^n(\mathfrak{M})$. Thus it is sufficient to consider the case when $\mathfrak{F}$ is a one-generated n-multiply local Fitting class.

Note that in this case in view of Lemma 2 we can use the induction by the number of atoms in $L^n(\mathfrak{F})$.

Let $\mathfrak{N}_p \subseteq \mathfrak{F}$ and $\mathfrak{H}$ be a complement to $\mathfrak{N}_p$ in $L^n(\mathfrak{F})$. Then $\mathfrak{N}_p \not\subseteq \mathfrak{H}$ and so by induction $\mathfrak{H} \subseteq \mathfrak{N}$. Hence $\mathfrak{F} \subseteq \mathfrak{N}$. □

Let L be a lattice with 0. Then an element a^* is called a pseudocomplement to an element $a (\in L)$ if from $a \wedge a^* = 0$ and $a \wedge x = 0$ it follows that $x \leqslant a^*$. A lattice with 0 is called a lattice with pseudocomplements if every its element has a pseudocomplement. A distributive lattice with pseudocomplements satisfying an identity

$$a^* \vee (a^*)^* = 1$$

is called Stone lattice.

Proof of Theorem A. Let $\mathfrak{F}$ be an n-multiply local Fitting class. Assume that $L^n(\mathfrak{F})$ is a Stone lattice and let $\mathfrak{N}_p \subseteq \mathfrak{F}$. Note that for every n-multiply local Fitting subclass $\mathfrak{M}$ in $\mathfrak{F}$ the class $\mathfrak{F} \cap \mathfrak{G}_{\pi'}$ is a pseudocomplement to $\mathfrak{M}$ in $L^n(\mathfrak{F})$ where $\pi = \pi(\mathfrak{M})$. Really for an n-multiply local Fitting class $\mathfrak{H} \subseteq \mathfrak{F}$ we have $\mathfrak{M} \cap \mathfrak{H} = (1)$ if and only if $\pi(\mathfrak{M}) \cap \pi(\mathfrak{H}) = \varnothing$. Hence $\mathfrak{H} \subseteq \mathfrak{F} \cap \mathfrak{G}_{\pi'}$ and so $\mathfrak{F} \cap \mathfrak{G}_{\pi'}$ is a pseudocomplement to $\mathfrak{M}$ in $L^n(\mathfrak{F})$.

Thus $\mathfrak{F} \cap \mathfrak{G}_{p'}$ is a pseudocomplement to $\mathfrak{N}_p$ in $L^n(\mathfrak{F})$ and $\mathfrak{N}_p$ is a pseudocomplement to $\mathfrak{F} \cap \mathfrak{G}_{p'}$ in $L^n(\mathfrak{F})$. However by hypothesis $\mathfrak{F} = \mathfrak{N}_p \vee^n (\mathfrak{F} \cap \mathfrak{G}_{p'})$. Hence for every $p \in \pi(\mathfrak{F})$ the class $\mathfrak{N}_p$ is complemented in $L^n(\mathfrak{F})$ and so by Lemma 6 $\mathfrak{F} \subseteq \mathfrak{N}$.

Now let $\mathfrak{F} \subseteq \mathfrak{N}$. Let $\mathfrak{M} \in L^n(\mathfrak{F})$, $\pi_1 = \pi(\mathfrak{M})$ and $\pi_2 = \pi(\mathfrak{F}) \setminus \pi(\mathfrak{M})$. If $\pi_1 = \pi(\mathfrak{F})$, $\mathfrak{M} = \mathfrak{F}$ and (1) is a complement to $\mathfrak{M}$ in $L^n(\mathfrak{F})$. Otherwise $\mathfrak{N}_{\pi_2}$ is a complement to $\mathfrak{M}$ in $L^n(\mathfrak{F})$. So $L^n(\mathfrak{F})$ is a Boolean lattice. □

2. The proof of Theorem B

The following result is not hard to obtain.

Lemma 7. *Let $\mathfrak{F}$ be a totally local Fitting class. Then if $p \in \pi(\mathfrak{F})$, $\mathfrak{N}_p \subseteq \mathfrak{F}$.*

Lemma 8. *Let $\mathfrak{F}$ be a one-generated totally local Fitting class. Then the lattice $L^\infty(\mathfrak{F})$ contains a finite number of atoms only.*

Proof. Let $p \in \pi(G)$. We shall show that $\mathfrak{N}_p$ is an atom of $L^\infty(\mathfrak{F})$. Let $\mathfrak{M}$ be an atom of $L^\infty(\mathfrak{F})$. Then by Lemma 7 $\mathfrak{N}_p \subseteq \mathfrak{M}$. But $\mathfrak{M}$ is an atom. So we have $\mathfrak{N}_p = \mathfrak{M}$.

Since G is a finite group there exists a finite number of atoms in the lattice $L^\infty(\mathfrak{F})$. □

Lemma 9. *[17, Corollary 2] Let $\mathfrak{F} = \oplus_{i \in I}\mathfrak{F}_i$ where $\mathfrak{F}_i$ is a Fitting class for all $i \in I$. Then the Fitting class $\mathfrak{F}$ is totally local if and only if every Fitting class $\mathfrak{F}_i$ is totally local.*

Lemma 10. *Let $\mathfrak{F}$ be a totally local Fitting class. Then if the class $\mathfrak{N}_p$ is complemented in the lattice $L^\infty(\mathfrak{F})$ for every $p \in \pi(\mathfrak{F})$, $\mathfrak{F} \subseteq \mathfrak{N}$.*

Proof. Let $\mathfrak{M}$ be a totally local Fitting subclass in $\mathfrak{F}$. We will show that if $\mathfrak{N}_p \subseteq \mathfrak{M}$ then the subclass $\mathfrak{N}_p$ is complemented in the lattice $L^\infty(\mathfrak{M})$. Really, let $\mathfrak{H}$ be a complement to $\mathfrak{N}_p$ in $L^\infty(\mathfrak{F})$. Then by Lemmas 3 and 9

$$\mathfrak{F} = \mathfrak{N}_p \vee^\infty \mathfrak{H} = \mathfrak{N}_p \oplus \mathfrak{H},$$

and so by Lemma 5

$$\mathfrak{M} = \mathfrak{M} \cap \mathfrak{F} = \mathfrak{M} \cap (\mathfrak{N}_p \oplus \mathfrak{H}) =$$

$$(\mathfrak{M} \cap \mathfrak{N}_p) \oplus (\mathfrak{M} \cap \mathfrak{H}) = \mathfrak{N}_p \oplus (\mathfrak{M} \cap \mathfrak{H}) =$$

$$\mathfrak{N}_p \vee^\infty (\mathfrak{M} \cap \mathfrak{H}),$$

where $\mathfrak{N}_p \cap (\mathfrak{M} \cap \mathfrak{H}) = (1)$. Hence $\mathfrak{M} \cap \mathfrak{H}$ is a complement to $\mathfrak{N}_p$ in $L^\infty(\mathfrak{M})$. Thus it is sufficient to consider the case when $\mathfrak{F}$ is a one-generated totally local Fitting class.

Note that in this case in view of Lemma 8 we can use the induction by the number of atoms in $L^\infty(\mathfrak{F})$.

Let $\mathfrak{N}_p \subseteq \mathfrak{F}$ and $\mathfrak{H}$ be a complement to $\mathfrak{N}_p$ in $L^\infty(\mathfrak{F})$. Then $\mathfrak{N}_p \not\subseteq \mathfrak{H}$ and so by induction $\mathfrak{H} \subseteq \mathfrak{N}$. Hence $\mathfrak{F} \subseteq \mathfrak{N}$. □

Proof of Theorem B. Let $\mathfrak{F}$ be a totally local Fitting class. Assume that $L^\infty(\mathfrak{F})$ is a Stone lattice and let $\mathfrak{N}_p \subseteq \mathfrak{F}$. Note that for every totally local Fitting subclass $\mathfrak{M}$ in $\mathfrak{F}$ the class $\mathfrak{F} \cap \mathfrak{G}_{\pi'}$ is a pseudocomplement to $\mathfrak{M}$ in $L^\infty(\mathfrak{F})$ where $\pi = \pi(\mathfrak{M})$. Really for a totally local Fitting class $\mathfrak{H} \subseteq \mathfrak{F}$ we have $\mathfrak{M} \cap \mathfrak{H} = (1)$ if and only if $\pi(\mathfrak{M}) \cap \pi(\mathfrak{H}) = \varnothing$. Hence $\mathfrak{H} \subseteq \mathfrak{F} \cap \mathfrak{G}_{\pi'}$ and so $\mathfrak{F} \cap \mathfrak{G}_{\pi'}$ is a pseudocomplement to $\mathfrak{M}$ in $L^\infty(\mathfrak{F})$.

Thus $\mathfrak{F} \cap \mathfrak{G}_{p'}$ is a pseudocomplement to $\mathfrak{N}_p$ in $L^\infty(\mathfrak{F})$ and $\mathfrak{N}_p$ is a pseudocomplement to $\mathfrak{F} \cap \mathfrak{G}_{p'}$ in $L^\infty(\mathfrak{F})$. However by hypothesis $\mathfrak{F} = \mathfrak{N}_p \vee^\infty (\mathfrak{F} \cap \mathfrak{G}_{p'})$. Hence for every $p \in \pi(\mathfrak{F})$ the class $\mathfrak{N}_p$ is complemented in $L^\infty(\mathfrak{F})$ and so by Lemma 10, $\mathfrak{F} \subseteq \mathfrak{N}$.

Now let $\mathfrak{F} \subseteq \mathfrak{N}$. Let $\mathfrak{M} \in L^\infty(\mathfrak{F})$, $\pi_1 = \pi(\mathfrak{M})$ and $\pi_2 = \pi(\mathfrak{F}) \setminus \pi(\mathfrak{M})$. If $\pi_1 = \pi(\mathfrak{F})$, $\mathfrak{M} = \mathfrak{F}$ and (1) is a complement to $\mathfrak{M}$ in $L^\infty(\mathfrak{F})$. Otherwise $\mathfrak{N}_{\pi_2}$ is a complement to $\mathfrak{M}$ in $L^\infty(\mathfrak{F})$. Basides $L^\infty(\mathfrak{F})$ is distributive. Consequently $L^\infty(\mathfrak{F})$ is a Boolean lattice. Therefore $L^\infty(\mathfrak{F})$ is a Stone lattice. □

3. Some corollaries and open questions

Corollary 1. *The following statements are equivalent:*

1. *A Fitting class $\mathfrak{F}$ is local and $L^1(\mathfrak{F})$ is a Stone lattice.*
2. *A Fitting class $\mathfrak{F}$ is n-multiply local and $L^n(\mathfrak{F})$ is a Stone lattice.*
3. *A Fitting class $\mathfrak{F}$ is totally local and $L^\infty(\mathfrak{F})$ is a Stone lattice.*
4. *A Fitting class $\mathfrak{F}$ is local and $\mathfrak{F} \subseteq \mathfrak{N}$.*

Corollary 2. *The lattice of all soluble totally local Fitting classes is the lattice with pseudocomplements.*

Remind that a representation of an element a in the form $x_0 \vee \ldots \vee x_{n-1}$ is called cancellable [20] if

$$a = x_0 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_{n-1}$$

for some $0 \leqslant i < n$. Otherwise it is called uncancellable.

A totally local Fitting class $\mathfrak{F}$ is called l^∞-irreducible if it is not possible to represent in the form $\mathfrak{F} = \vee^\infty(\mathfrak{F}_i \mid i \in I)$ where $\{\mathfrak{F}_i \mid i \in I\}$ is the set of all proper totally local Fitting subclasses from $\mathfrak{F}$.

Using Lemma 8 and Theorem B we obtain

Corollary 3. *Let $\mathfrak{F}$ be a soluble one-generated totally local Fitting class. Then $\mathfrak{F}$ has the unique representation in the form of the uncancellable join $\mathfrak{F}_1 \vee^\infty \ldots \vee^\infty \mathfrak{F}_t$ of some its totally local l^∞-irreducible Fitting subclasses $\mathfrak{F}_1, \ldots, \mathfrak{F}_t$.*

For any two totally local Fitting classes $\mathfrak{M}$ and $\mathfrak{H}$ where $\mathfrak{M} \subseteq \mathfrak{H}$ the lattice of all totally local Fitting classes contained between $\mathfrak{M}$ and $\mathfrak{H}$ is denoted (see [14]) by $\mathfrak{H}/^{\infty}\mathfrak{M}$.

Corollary 4. *For any two soluble totally local Fitting classes $\mathfrak{M}$ and $\mathfrak{H}$ a lattice isomorphism*

$$\mathfrak{M} \vee^{\infty} \mathfrak{H}/^{\infty}\mathfrak{M} \simeq \mathfrak{H}/^{\infty}\mathfrak{H} \cap \mathfrak{M}$$

is valid.

Question 5. Is it possible to classify all l^{∞}-irreducible Fitting classes?
Question 6. Is the lattice $L^{\infty}(\mathfrak{F})$ coalgebraic for every $\mathfrak{F} \in l^{\infty}$?

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