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ON THE BOOLEAN LATTICES OF n TIMES LOCAL FITTING CLASSES

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All groups under consideration are assumed finite. We use the standard terminology of the books [1, 2].

We recall that if $\mathfrak{F}$ is an arbitrary nonempty formation then $G^{\mathfrak{F}}$ denotes the intersection of all kernels of the epimorphisms of a group G onto the groups of $\mathfrak{F}$ and $\mathfrak{M}\mathfrak{F}$ denotes the class of all groups G with $G^{\mathfrak{F}} \in \mathfrak{M}$. The symbols $\mathfrak{N}_p$ and $\mathfrak{S}_{p'}$ stand for the class of all p-groups and the class of all groups with a trivial Sylow p-subgroup.

Functions of the shape $f : \mathbb{P} \to \{Fitting\ classes\}$ are referred to as *Hartley* functions or, in short, *H-functions*. Given a Fitting class $\mathfrak{F}$, we write $\mathfrak{F} = LR(f)$ and say that f is an H-function of the class $\mathfrak{F}$, provided that

$$\mathfrak{F} = \mathfrak{S}_\pi \cap \Big(\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'}\Big),$$

where $\pi = \pi(\mathfrak{F})$ is the collection of all prime divisors of the orders of all groups in $\mathfrak{F}$. By analogy to [4], N. T. Vorob'ëv introduced and began studying n times local Fitting classes in the article [3]. Here every Fitting class is considered 0 times local and, for $n > 0$, a Fitting class $\mathfrak{F}$ is called n times local provided that $\mathfrak{F} = LR(f)$, where all nonempty values of the H-function f are $(n-1)$ times local Fitting classes.

It was shown in [5] that the lattice of (local) formations is modular. This property made it possible later to apply the methods of the general lattice theory to solving many open problems of formation theory (see [2, Chapter 4; 6, Chapters 4 and 5]). At the same time, no essential information is available about the lattice of Fitting classes. In particular, it is unknown by now whether or not the lattice of these classes is modular and which Fitting classes have the distributive lattice of Fitting subclasses.

In the present article, developing some ideas of [7], we describe the Boolean sublattices of the lattice of n times local Fitting classes. In contrast to [7], we use a series of new observations about direct decompositions of classes of groups which ascend conceptually to [8] (see details in [6, Chapter 4]).

A system $\{\mathfrak{F}_i \mid i \in I\}$ of nonempty classes $\mathfrak{F}_i$ of groups is called *orthogonal* if

(1) $|I| > 1$;

(2) $\mathfrak{F}_i \cap \mathfrak{F}_j = (1)$ for arbitrary two distinct $i, j \in I$.

Following [6], given an arbitrary orthogonal system of classes $\{\mathfrak{F}_i \mid i \in I\}$, we denote by $\bigoplus_{i\in I}\mathfrak{F}_i$ the collection of all groups of the shape $A_1 \times A_2 \times \dots \times A_t$, where $A_1 \in \mathfrak{F}_{i_1}, A_2 \in \mathfrak{F}_{i_2}, \dots, A_t \in \mathfrak{F}_{i_t}$ for some $i_1, i_2, \dots, i_t \in I$ (in particular, we write $\mathfrak{F} = \mathfrak{F}_1 \oplus \dots \oplus \mathfrak{F}_n$ whenever $I = \{1, \dots, n\}$). We also agree to write $\mathfrak{F} = \bigoplus_{i\in I}\mathfrak{F}_i$ in case $|I| = 1$ and $\mathfrak{F}_i = \mathfrak{F}$. Every representation of a class $\mathfrak{F}$ of groups in the shape $\mathfrak{F} = \bigoplus_{i\in I}\mathfrak{F}_i$ with $|I| > 1$ is referred to as a *direct decomposition* of this class.

Observe that if $\{\mathfrak{F}_i \mid i \in I\}$ is an orthogonal system of Fitting classes (formations) then this system is an orthogonal system of elements in the lattice of all Fitting classes (in the lattice of all formations) in the conventional sense (see, for instance, [9, p. 238]).

Lemma 1. *Assume that* $\mathfrak{F} = \bigoplus_{i\in I}\mathfrak{F}_i$ *and* $\mathfrak{M}$ *is a nonempty Fitting subclass of* $\mathfrak{F}$. *Then* $\mathfrak{M} = \bigoplus_{i\in I}(\mathfrak{M} \cap \mathfrak{F}_i)$.

PROOF. Take $G \in \mathfrak{M}$. Since $\mathfrak{M} \subseteq \mathfrak{F}$, there are $i_1, \dots, i_t \in I$ such that $G = A_1 \times \dots \times A_t$, where

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$A_1 \in \mathfrak{F}_{i_1}, \dots, A_t \in \mathfrak{F}_{i_t}$. Therefore,

$$G \in (\mathfrak{M} \cap \mathfrak{F}_{i_1}) \oplus \cdots \oplus (\mathfrak{M} \cap \mathfrak{F}_{i_t}) \subseteq \bigoplus_{i \in I} (\mathfrak{M} \cap \mathfrak{F}_i),$$

and $\mathfrak{M} \subseteq \bigoplus_{i\in I} (\mathfrak{M} \cap \mathfrak{F}_i)$.

Conversely, assume that $G \in \bigoplus_{i\in I}(\mathfrak{M} \cap \mathfrak{F}_i)$. Then there are $i_1, \dots, i_t \in I$ such that $G = A_1 \times \cdots \times A_t$, where $A_1 \in \mathfrak{M} \cap \mathfrak{F}_{i_1}, \dots, A_t \in \mathfrak{M} \cap \mathfrak{F}_{i_t}$. Since $\mathfrak{M}$ is a Fitting class, we infer that $G \in \mathfrak{M}$. Hence, $\bigoplus_{i\in I}(\mathfrak{M} \cap \mathfrak{F}_i) \subseteq \mathfrak{M}$. Thereby $\mathfrak{M} = \bigoplus_{i\in I}(\mathfrak{M} \cap \mathfrak{F}_i)$, which finishes the proof of the lemma.

The next two lemmas are easy to establish by straightforward verification.

Lemma 2. *Let $I = \bigcup_{j\in J} I_j$ be a partition of a set I (i.e., $I_{j_1} \cap I_{j_2} = \varnothing$ for arbitrary $j_1, j_2 \in J$ with $j_1 \neq j_2$). If $\mathfrak{F}_j = \bigoplus_{i\in I_j} \mathfrak{H}_i$, $j \in J$, and $\mathfrak{F} = \bigoplus_{j\in J} \mathfrak{F}_j$, then $\mathfrak{F} = \bigoplus_{i\in I} \mathfrak{H}_i$.*

Lemma 3. *If $\mathfrak{F} = \mathfrak{F}_1 \oplus \mathfrak{F}_2$, where $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are Fitting classes, and if A is a group of the shape $A = A_1 \times A_2$, where $A_1 \in \mathfrak{F}_1$ and $A_2 \in \mathfrak{F}_2$, then A_1 and A_2 are characteristic subgroups in A.*

Lemma 4. *Let $\mathfrak{F}_1, \dots, \mathfrak{F}_t$ be an orthogonal system of Fitting classes. Then $\mathfrak{F} = \mathfrak{F}_1 \oplus \cdots \oplus \mathfrak{F}_t$ is a Fitting class.*

PROOF. We proceed by induction on t. Assume that $t = 2$ and let H be a normal subgroup of a group $G \in \mathfrak{F} = \mathfrak{F}_1 \oplus \mathfrak{F}_2$. Then $G = A \times B$, where $A \in \mathfrak{F}_1$ and $B \in \mathfrak{F}_2$. Demonstrate that $H \in \mathfrak{F}$. If A_1 and B_1 are projections of H into A and B respectively, then A_1 is normal in A, B_1 is normal in B, and H is a subdirect summand of $A_1 \times B_1 \in \mathfrak{F}$. Hence, we may confine exposition to the case in which H is a subdirect summand of G.

Let $A \cap H = 1$. It is easy to see that $G = AH = BH$. Hence,

$$H \simeq AH/A = G/A \simeq B \in \mathfrak{F}_2 \subseteq \mathfrak{F}.$$

Therefore, we may assume that $A_1 = A \cap H \neq 1 \neq B_1 = B \cap H$. Since

$$H/A_1B_1 \cap AA_1B_1/A_1B_1 = (H \cap AB_1)/A_1B_1 = B_1(H \cap A)/A_1B_1 = B_1A_1/A_1B_1,$$

we have $H/A_1B_1 \subseteq C_{G/A_1B_1}(AB_1/A_1B_1)$. Analogously,

$$H/A_1B_1 \subseteq C_{G/A_1B_1}(BA_1/A_1B_1).$$

Hence,

$$H/A_1B_1 \subseteq Z(G/A_1B_1) = Z((AB_1/A_1B_1)(BA_1/A_1B_1)).$$

If $A_1B_1 = H$ then the fact that H is a subdirect summand of $A \times B$ and $H = (A \cap H)(B \cap H)$ implies that $H = A \times B \in \mathfrak{F}$. Assume that $A_1B_1 \subset H$. Since

$$HB/B \simeq H/B \cap H = H/B_1 \simeq A, \quad HA/A \simeq H/A \cap H = H/A_1 \simeq B$$

and H/A_1B_1 is a decomposable group, by [10, Chapter IX, Lemma 1.7] there is a prime number p such that $\mathfrak{N}_p \subseteq \mathfrak{F}_1 \cap \mathfrak{F}_2 = (1)$. This contradiction shows that $H \in \mathfrak{F}$.

Now, assume that $G = (A_1 \times B_1)(A_2 \times B_2)$, where $A_1, A_2 \in \mathfrak{F}_1$, $B_1, B_2 \in \mathfrak{F}_2$, and the subgroups $A_1 \times B_1$ and $A_2 \times B_2$ are normal in G. Then by Lemma 3 A_1, B_1, A_2, and B_2 are normal subgroups of G. Hence, $G = (A_1A_2)(B_1B_2)$, and since $A_1A_2 \cap B_1B_2 \in \mathfrak{F}_1 \cap \mathfrak{F}_2 = (1)$, it follows that $G = (A_1A_2) \times (B_1B_2) \in \mathfrak{F}$.

Assume that $t > 2$ and that the claim of the lemma has been proven for $t - 1$. It is clear that $\mathfrak{F}_2, \dots, \mathfrak{F}_t$ is an orthogonal system. Hence, $\mathfrak{F}_2 \oplus \cdots \oplus \mathfrak{F}_t$ is a Fitting class by our hypothesis. Let $\mathfrak{M} = \mathfrak{F}_1 \cap (\mathfrak{F}_2 \oplus \cdots \oplus \mathfrak{F}_t)$. Then by Lemma 1 $\mathfrak{M} = (\mathfrak{M} \cap \mathfrak{F}_2) \oplus \cdots \oplus (\mathfrak{M} \cap \mathfrak{F}_t)$. However, $\mathfrak{M} \subseteq \mathfrak{F}_1$ and $\mathfrak{F}_1 \cap \mathfrak{F}_i = (1)$ for all $i = 2, \dots, t$. Therefore, $\mathfrak{M} = (1) \oplus \cdots \oplus (1) = (1)$, and $\mathfrak{F}_1, \mathfrak{F}_2 \oplus \cdots \oplus \mathfrak{F}_t$ is an orthogonal system. Now, by the above and Lemma 2 $\mathfrak{F}_1 \oplus (\mathfrak{F}_2 \oplus \cdots \oplus \mathfrak{F}_t) = \mathfrak{F}_1 \oplus \mathfrak{F}_2 \oplus \cdots \oplus \mathfrak{F}_t$ is a Fitting class. The proof of the lemma is over.

Corollary 1. *If $\mathfrak{F} = \mathfrak{F}_1 \oplus \cdots \oplus \mathfrak{F}_t$, where $\mathfrak{F}_1, \dots, \mathfrak{F}_t$ are Fitting classes, and if A is a group of the shape $A = A_1 \times \cdots \times A_t$, where $A_1 \in \mathfrak{F}_1, \dots, A_t \in \mathfrak{F}_t$, then $A_1, \dots, A_t$ are characteristic subgroups in A.*

PROOF. For $t = 2$, the claim is valid by virtue of Lemma 3.

Assume that $t > 2$. By Lemma 1 $(\mathfrak{F}_1 \cap \mathfrak{F}_2 \oplus \cdots \oplus \mathfrak{F}_t) = (\mathfrak{F}_1 \cap \mathfrak{F}_2) \oplus \cdots \oplus (\mathfrak{F}_1 \cap \mathfrak{F}_t) = (1) \oplus \cdots \oplus (1) = (1)$. Thus, $\mathfrak{F}_1, \mathfrak{F}_2 \oplus \cdots \oplus \mathfrak{F}_t$ is an orthogonal system. By Lemma 3, A_1 is therefore a characteristic subgroup of the group $A = A_1 \times (A_2 \times \cdots \times A_t)$. This completes the proof of the corollary.

Observe that in the solvable case Lemma 4 for $t = 2$ ensues from [11, Lemma 4].

Given an arbitrary class $\mathfrak{X}$ of groups, we denote the intersection of all n times local Fitting classes including $\mathfrak{X}$ by $l^n \operatorname{fit} \mathfrak{X}$ [12].

Given n times local Fitting classes $\mathfrak{M}$ and $\mathfrak{H}$, we put

$$\mathfrak{M} \vee^n \mathfrak{H} = l^n \operatorname{fit}(\mathfrak{M} \cup \mathfrak{H}).$$

Theorem 1. *Suppose that $\mathfrak{F} = \bigoplus_{i \in I} \mathfrak{F}_i$ for some Fitting classes $\mathfrak{F}_i$. Then $\mathfrak{F}$ is an n times local Fitting class if and only if each of the Fitting classes $\mathfrak{F}_i$ is n times local.*

PROOF. Assume that $\mathfrak{F}_i$ is an n times local Fitting class for every $i \in I$. Demonstrate that $\mathfrak{F}$ is an n times local Fitting class.

First, we examine the case in which $n = 0$. Let H be a normal subgroup of a group $G \in \mathfrak{F}$. Then there are indices $i_1, \dots, i_t \in I$ such that $G = A_1 \times \cdots \times A_t$, where $A_1 \in \mathfrak{F}_{i_1}, \dots, A_t \in \mathfrak{F}_{i_t}$. Now, by Lemmas 2 and 4

$$H \in \mathfrak{F}_{i_1} \oplus (\mathfrak{F}_{i_2} \oplus \cdots \oplus \mathfrak{F}_{i_t}) = \mathfrak{F}_{i_1} \oplus \cdots \oplus \mathfrak{F}_{i_t} \subseteq \bigoplus_{i \in I} \mathfrak{F}_i = \mathfrak{F}.$$

Thus, the class $\mathfrak{F}$ is closed with respect to the taking of normal subgroups.

Assume that $G = AB$, where A and B are normal $\mathfrak{F}$-subgroups of a group G. Then there are indices $i_1, \dots, i_t, j_1, \dots, j_a \in I$ such that $A = A_1 \times \cdots \times A_t$ and $B = B_1 \times \cdots \times B_a$, where $A_1 \in \mathfrak{F}_{i_1}, \dots, B_a \in \mathfrak{F}_{j_a}$. By Corollary 1, the subgroups $A_1, \dots, A_t, B_1, \dots, B_a$ are normal in G. Hence, $G = C_1 \times \cdots \times C_b$, where each factor C_i satisfies one of the following conditions:

(1) C_i coincides with A_{i_l} for some index $i_l \notin \{j_1, \dots, j_a\}$;

(2) $C_i = B_{j_l}$ for some index $j_l \notin \{i_1, \dots, i_t\}$;

(3) $C_i = A_{i_l} B_{j_k}$ for some index $i_l = j_k$.

Therefore, $G \in \mathfrak{F}$. Assume that $n > 0$ and that f_i is a minimal l^{n-1}-valued H-function of the n times local Fitting class $\mathfrak{F}_i$ for every $i \in I$. If $i \neq j$ then by hypothesis $\mathfrak{F}_i \cap \mathfrak{F}_j = (1)$. Thereby $\pi(\mathfrak{F}_i) \cap \pi(\mathfrak{F}_j) = \varnothing$. Construct the H-function f so as to have $f(p) = f_i(p)$ whenever $p \in \pi(\mathfrak{F}_i)$ for some $i \in I$ and $f(p) = \varnothing$ for all $p \in \mathbb{P} \setminus \bigcup_{i \in I} \pi(\mathfrak{F}_i)$. Demonstrate that $\mathfrak{F} = LR(f)$. Let G be a group of minimal order in $LR(f) \setminus \mathfrak{F}$. Then G is a comonolithic group and its comonolith M equals $G_{\mathfrak{F}}$. Since $G \in LR(f)$, we have $F^p(G) \in f(p)$ for all $p \in \pi(G)$. Therefore, if $p \in \pi(G)$ then, by the construction of the H-function f, there is an $i \in I$ such that $f(p) = f_i(p) \neq \varnothing$. This means that $p \in \pi(\mathfrak{F}_i)$. Hence, $\pi(G) \subseteq \bigcup_{i \in I} \pi(\mathfrak{F}_i)$.

Assume that $p \in \pi(G/M) \subseteq \pi(G)$. Then $p \in \pi(\mathfrak{F}_i)$ for some $i \in I$. If G/M is a nonabelian group then $F^p(G) = G$. Therefore,

$$G = F^p(G) \in f(p) = f_i(p) \subseteq \mathfrak{F}_i \subseteq \mathfrak{F};$$

a contradiction.

Assume that G/M is a p-group. Then

$$F^p(G) = O^p(G) \in f(p) = f_i(p).$$

Hence, by [12, Lemma 23] $G \in \mathfrak{F}_i \subseteq \mathfrak{F}$; a contradiction. Thus, $LR(f) \subseteq \mathfrak{F}$.

Suppose that the reverse inclusion fails and let G be a group of minimal order in $\mathfrak{F} \setminus LR(f)$. Then G is a comonolithic group. Therefore, there is an $i \in I$ such that $G \in \mathfrak{F}_i = LR(f_i)$. Hence,

$F^p(G) \in f_i(p) = f(p)$ for all $p \in \pi(G)$. Consequently, $G \in LR(f)$ and $\mathfrak{F} \subseteq LR(f)$. Thus, $\mathfrak{F} = LR(f)$ is an n times local Fitting class.

Now, suppose that the Fitting class $\mathfrak{F}$ is n times local and let f be a minimal l^{n-1}-valued H-function of this class. Take $i \in I$ and let f_i be an H-function such that $f_i(p) = f(p)$ if $p \in \pi(\mathfrak{F}_i)$ and $f_i(p) = \varnothing$ if $p \in \mathbb{P} \setminus \pi(\mathfrak{F}_i)$. Demonstrate that $\mathfrak{F}_i = LR(f_i)$.

Assume that $\mathfrak{F}_i$ does not belong to $LR(f_i)$ and let G be a group of minimal order in $\mathfrak{F}_i \setminus LR(f_i)$. Then G is a comonolithic group with the comonolith $M = G_{LR(f_i)}$. Since $G \notin LR(f_i)$, by [12, Lemma 28] there is $p \in \pi(G/M)$ such that $F^p(G) \notin f_i(p)$. However, $G \in \mathfrak{F}_i \subseteq \mathfrak{F}$. Hence, for all $q \in \pi(G)$ we have

$$F^q(G) \in f(q) = f_i(q);$$

a contradiction. Thus, $\mathfrak{F}_i \subseteq LR(f_i)$.

Suppose that the reverse inclusion fails and let G be a group of minimal order in $LR(f_i) \setminus \mathfrak{F}_i$. Then G is a comonolithic group with the comonolith $M = G_{\mathfrak{F}_i}$.

Take $p \in \pi(G/M) \subseteq \pi(G)$. In this case, it follows from $G \in LR(f_i)$ that $F^p(G) \in f_i(p)$. Hence, $f_i(p) \neq \varnothing$, and by the construction of the H-function f_i we have $p \in \pi(\mathfrak{F}_i)$. Thus, $\pi(G/M) \subseteq \pi(\mathfrak{F}_i)$. Moreover, $f_i \leq f$ by the construction of the H-function f_i. In consequence, $G \in \mathfrak{F}$, and since G is a comonolithic group, there is $j \in I$ such that $G \in \mathfrak{F}_j$. Then $\pi(G) \subseteq \pi(\mathfrak{F}_j)$. Therefore,

$$\pi(G/M) \subseteq \pi(\mathfrak{F}_i) \cap \pi(\mathfrak{F}_j) = \varnothing.$$

Thereby $i = j$; i.e., $G \in \mathfrak{F}_i$; a contradiction. Hence, $LR(f_i) \subseteq \mathfrak{F}_i$. Thus, $\mathfrak{F}_i = LR(f_i)$ is an n-times local Fitting class, which finishes the proof of the theorem.

We recall that a Fitting class $\mathfrak{F}$ is called ***totally local*** [3] if it is n times local for every nonnegative integer n.

Corollary 2. *Suppose that $\mathfrak{F} = \bigoplus_{i\in I} \mathfrak{F}_i$ for some Fitting classes $\mathfrak{F}_i$. Then the Fitting class $\mathfrak{F}$ is totally local if and only if so is every Fitting class $\mathfrak{F}_i$.*

A nonempty Fitting class $\mathfrak{F} \neq (1)$ is called *directly indecomposable* if $\mathfrak{F}$ cannot be represented in the shape $\mathfrak{F} = \bigoplus_{i\in I} \mathfrak{F}_i$, where each Fitting class $\mathfrak{F}_i$ differs from $\mathfrak{F}$.

The next lemma is an analog of the Remak–Schmidt theorem for the Fitting classes.

Theorem 2. *Suppose that $\mathfrak{F} = \bigoplus_{i\in I} \mathfrak{F}_i = \bigoplus_{j\in J} \mathfrak{M}_j$, where the Fitting classes $\mathfrak{F}_i$ and $\mathfrak{M}_j$ are directly indecomposable for all $i \in I$ and $j \in J$. Then $|I| = |J|$ and, for some bijection $\varphi : I \to J$, the equality $\mathfrak{F}_i = \mathfrak{M}_{\varphi(i)}$ holds for all $i \in I$.*

PROOF. Take $i \in I$. Then by Lemma 1 we have the equality $\mathfrak{F}_i = \bigoplus_{j\in J}(\mathfrak{F}_i \cap \mathfrak{M}_j)$. By hypothesis, the Fitting class $\mathfrak{F}_i$ is directly indecomposable; therefore, $\mathfrak{F}_i = \mathfrak{F}_i \cap \mathfrak{M}_j$ for some $j \in J$; i.e., $\mathfrak{F}_i \subseteq \mathfrak{M}_j$. Furthermore, for arbitrary distinct j_1 and j_2, by hypothesis we have $\mathfrak{M}_{j_1} \cap \mathfrak{M}_{j_2} = (1)$. Hence, there is a unique index $j = j(i)$ such that $\mathfrak{F}_i \subseteq \mathfrak{M}_j$. Assume that $j \in J$. Then from Lemma 1 we infer $\mathfrak{M}_j = \bigoplus_{k\in I}(\mathfrak{F}_k \cap \mathfrak{M}_j)$. Arguing by analogy, we conclude that there is a unique index $k = k(j)$ such that $\mathfrak{M}_j \subseteq \mathfrak{F}_k$. Thus, $\mathfrak{F}_i \subseteq \mathfrak{M}_j \subseteq \mathfrak{F}_k$, but $\mathfrak{F}_i \cap \mathfrak{F}_k = (1)$. Therefore, $\mathfrak{M}_j = \mathfrak{F}_i = \mathfrak{F}_k$. It is clear that if $a, b \in I$ and $a \neq b$ then $j(a) \neq j(b)$. Hence, $|I| = |J|$, which completes the proof of the theorem.

Lemma 5. *Suppose that a simple group A belongs to $l^n \operatorname{fit} \mathfrak{X}$, where $\mathfrak{X}$ is a class of groups. If $n = 0$ then $A \simeq H/K$, where $H/K \in K(G)$ for some group $G \in \mathfrak{X}$. If $n > 0$ then the following assertions hold:*

(1) *if A is a nonabelian group then $A \simeq H/K$, where $H/K \in K(G)$ for some group $G \in \mathfrak{X}$;*

(2) *if $A = Z_p$ is a group of order p then $Z_p \simeq H \leq G$ for some subgroup H of a group $G \in \mathfrak{X}$.*

PROOF. Assume that $n = 0$. The class $l^0 \operatorname{fit} \mathfrak{X} = \operatorname{fit} \mathfrak{X}$ obviously consists of all groups that result from applying the operations S_n and R finitely many times to the groups in $\mathfrak{X}$. Clearly, if N is a normal subgroup of a group G then $K(N) \subseteq K(G)$. If $G = AB$, where A and B are normal subgroups of G, then $K(G) = K(A) \cup K(B)$. It follows that $K(\mathfrak{X}) = K(\operatorname{fit} \mathfrak{X})$.

Assume that $n > 0$ and that the claim of the lemma is true for $n-1$. Take $p \in \pi(A)$, $\mathfrak{F} = l^n \operatorname{fit} \mathfrak{X}$, and let f be a minimal l^{n-1}-valued H-function of the Fitting class $\mathfrak{F}$. If A is a nonabelian group then

$F^p(A) = A$. Hence,

$$A = F^p(A) \in f(p) = l^{n-1}\,\mathrm{fit}(\mathfrak{X}(F^p)) = l^{n-1}\,\mathrm{fit}[\mathrm{fit}(F^p(A) \mid A \in \mathfrak{X})],$$

and by induction we have $A \simeq H/K$, where $H/K \in K(G)$ for some group $G \in \mathfrak{X}(F^p)$. However, by the above $K(\mathfrak{X}(F^p)) \subseteq K(\mathrm{fit}\,\mathfrak{X}) = K(\mathfrak{X})$. Whence $A \in K(\mathfrak{X})$.

If A is a group of prime order p then $p \in \pi(\mathfrak{F})$. Hence, $A \simeq H \leq G$ for some subgroup H of a group $G \in \mathfrak{F}$, which completes the proof of the lemma.

Given an arbitrary n times local Fitting class $\mathfrak{F}$, we denote by $L^n(\mathfrak{F})$ the lattice of all n times local Fitting subclasses of $\mathfrak{F}$.

Lemma 6. *Let $\mathfrak{F} = l^n\,\mathrm{fit}\,G$ be a one-generated n times local Fitting class. Then the lattice $L^n(\mathfrak{F})$ has only finitely many atoms.*

PROOF. Let $\mathfrak{M}$ be an atom of the lattice $L^n(\mathfrak{F})$. Then $\mathfrak{M} = l^n\,\mathrm{fit}\,A$, where A is a simple group in $\mathfrak{M}$. If A is a nonabelian group then by Lemma 5 we have $A \simeq H/K$, where $H/K \in K(G)$. Since the group G is finite, it has finitely many composition factors. Therefore, the lattice $L^n(\mathfrak{F})$ has finitely many nonsolvable atoms.

Suppose that $A = Z_p$ is a group of prime order p. Since p divides $|G|$, the lattice $L^n(\mathfrak{F})$ has only finitely many solvable atoms. The proof of the lemma is over.

Lemma 7. *Let $\{\mathfrak{M}_i \mid i \in I\}$ be a collection of atoms of the lattice l^n and $\mathfrak{F} = \bigoplus_{i\in I}\mathfrak{M}_i$. Then the Fitting class $\mathfrak{F}$ belongs to the lattice l^n and if $\mathfrak{M} \neq \varnothing$ is an arbitrary nonidentity n times local Fitting subclass of $\mathfrak{F}$ then the set $\{\mathfrak{M}_i \mid i \in I\}$ has a subset $\{\mathfrak{M}_j \mid j \in J\}$ such that $\mathfrak{M} = \bigoplus_{j\in J}\mathfrak{M}_j$.*

PROOF. The fist assertion of the lemma follows from Theorem 1.

According to Lemma 1, $\mathfrak{M} = \bigoplus_{i\in I}(\mathfrak{M}_i \cap \mathfrak{M})$. Since $\mathfrak{M}_i$ is an atom of the lattice l^n, we obtain $\mathfrak{M}_i \cap \mathfrak{M} = \{(1), \mathfrak{M}_i\}$. Let J be a subset of I such that $j \in J$ if and only if $\mathfrak{M}_j \cap \mathfrak{M} = \mathfrak{M}_j$. Obviously, $\mathfrak{M} = \bigoplus_{j\in J}\mathfrak{M}_j$, which completes the proof of the lemma.

Theorem 3. *Let $\mathfrak{F}$ be a nonidentity n times local Fitting class. Then the following conditions are equivalent:*

(1) *$L^n(\mathfrak{F})$ is a Boolean lattice;*

(2) *$\mathfrak{F} = \bigoplus_{i\in I}\mathfrak{F}_i$, where $\{\mathfrak{F}_i \mid i \in I\}$ is the collection of all atoms of the lattice $L^n(\mathfrak{F})$;*

(3) *each Fitting subclass which is an atom of the lattice $L^n(\mathfrak{F})$ is complemented in $\mathfrak{F}$.*

PROOF. Prove that condition (3) implies condition (2). First, observe that condition (3) holds for every nonidentity n times local subclass $\mathfrak{F}_1$ of $\mathfrak{F}$. Indeed, if $\mathfrak{M}$ is an arbitrary atom of the lattice $L^n(\mathfrak{F}_1)$ then by condition (3) there is a Fitting subclass $\mathfrak{H}$ in $\mathfrak{F}$ such that $\mathfrak{F} = \mathfrak{M} \vee^0 \mathfrak{H}$ and $\mathfrak{M} \cap \mathfrak{H} = (1)$. Hence, $\mathfrak{F} = \mathfrak{M} \oplus \mathfrak{H}$ by Lemma 4. Therefore, in view of Lemma 1 we have $\mathfrak{F}_1 = \mathfrak{M} \oplus (\mathfrak{H} \cap \mathfrak{F}_1)$; i.e., $\mathfrak{H} \cap \mathfrak{F}_1$ is a complement to $\mathfrak{M}$ in $\mathfrak{F}_1$.

We now validate condition (2) for an arbitrary nonidentity n times local subclass $\mathfrak{F}_1$ of $\mathfrak{F}$ with a finite number m of atoms of the lattice $L^n(\mathfrak{F}_1)$. We induct on m. Let $\mathfrak{L}$ be an atom of the lattice $L^n(\mathfrak{F}_1)$. Then, as mentioned above, $\mathfrak{F}_1 = \mathfrak{L} \oplus \mathfrak{K}$ for some Fitting subclass $\mathfrak{K}$ of $\mathfrak{F}_1$. By Theorem 1, $\mathfrak{K}$ is an n times local Fitting class. If $m = 1$ then $\mathfrak{K} = (1)$. Hence, $\mathfrak{F}_1 = \mathfrak{L}$ and validity of condition (2) for $\mathfrak{F}_1$ is trivial. Assume that $m > 1$ and suppose that condition (2) holds for every nonidentity n times local Fitting class $\mathfrak{N} \subseteq \mathfrak{F}$ for which the number of atoms of the lattice $L^n(\mathfrak{N})$ does not exceed $m - 1$. Observe that from $m > 1$ it follows that $\mathfrak{K} \neq (1)$. Clearly, the number of atoms in the lattice $L^n(\mathfrak{K})$ for the n times local Fitting class $\mathfrak{K}$ is less than m. Our hypothesis implies that $\mathfrak{K} = \mathfrak{M}_1 \oplus \cdots \oplus \mathfrak{M}_t$, where $\{\mathfrak{M}_i \mid i \in I\}$ is the collection of all atoms of the lattice $L^n(\mathfrak{K})$. Using Lemmas 1 and 2, we now conclude that $\mathfrak{F}_1 = \mathfrak{L} \oplus \mathfrak{M}_1 \oplus \cdots \oplus \mathfrak{M}_t$, where $\{\mathfrak{L}, \mathfrak{M}_1, \dots, \mathfrak{M}_t\}$ is the collection of all atoms of the lattice $L^n(\mathfrak{F}_1)$.

Let $\{\mathfrak{F}_i \mid i \in I\}$ be the collection of all atoms of the lattice $L^n(\mathfrak{F})$ and let $\mathfrak{H} = \bigoplus_{i\in I}\mathfrak{F}_i$. Then by Lemma 7 $\mathfrak{H}$ is an n times local Fitting class. Demonstrate that $\mathfrak{F} = \mathfrak{H}$. Assume $\mathfrak{F} \neq \mathfrak{H}$ and $G \in \mathfrak{F} \setminus \mathfrak{H}$. According to Lemma 6, there are only finitely many subclasses in $\mathfrak{F}_2 = l^n\,\mathrm{fit}\,G$ which are atoms of the lattice $L^n(\mathfrak{F}_2)$. By the above, $\mathfrak{F}_2 = \mathfrak{M}_1 \oplus \cdots \oplus \mathfrak{M}_a$, where $\mathfrak{M}_1, \dots, \mathfrak{M}_a$ is the collection of all atoms of the lattice $L^n(\mathfrak{F}_2)$, and $G \in \mathfrak{F}_2 \subseteq \mathfrak{H}$; a contradiction. Thus, $\mathfrak{F} = \mathfrak{H}$.

Now, suppose that condition (2) holds and validate condition (1). First, we establish that $L^n(\mathfrak{F})$ is a complemented lattice. Let $\mathfrak{M}$ be an arbitrary n times local Fitting subclass of $\mathfrak{F}$. Then by Lemma 7 there is a collection $\{\mathfrak{F}_i \mid i \in I_1\}$ of Fitting subclasses in $\mathfrak{M}$ which are atoms of the lattice $L^n(\mathfrak{F})$ and for which $\mathfrak{M} = \bigoplus_{i\in I_1} \mathfrak{F}_i$. Put $I_2 = I \setminus I_1$ and $\mathfrak{H} = \bigoplus_{i\in I_2} \mathfrak{F}_i$. Demonstrate that $\mathfrak{H}$ is a complement to $\mathfrak{M}$ in the lattice $L^n(\mathfrak{F})$. Obviously, $\mathfrak{M} \vee^n \mathfrak{H} = \mathfrak{F}$. Assume that $\mathfrak{M} \cap \mathfrak{H} \neq (1)$ and let $\mathfrak{R}$ be a Fitting subclass in $\mathfrak{M} \cap \mathfrak{H}$ which is an atom of the lattice $L^n(\mathfrak{F})$. Then there is $i \in I$ such that $\mathfrak{R} = \mathfrak{F}_i$. However, $I_1 \cap I_2 = \varnothing$. Therefore, by Lemma 7 $\mathfrak{R}$ is in one of the Fitting classes $\mathfrak{M}$ and $\mathfrak{H}$; a contradiction. Hence $L^n(\mathfrak{F})$ is a complemented lattice.

Demonstrate that $L^n(\mathfrak{F})$ is a distributive lattice. Let $\mathfrak{M}$, $\mathfrak{H}$, and $\mathfrak{R}$ be arbitrary n times local Fitting classes in $\mathfrak{F}$. The inclusion $(\mathfrak{M} \cap \mathfrak{R}) \vee^n (\mathfrak{M} \cap \mathfrak{H}) \subseteq \mathfrak{M} \cap (\mathfrak{R} \vee^n \mathfrak{H})$ is obvious.

Suppose that the reverse inclusion fails and let A be a group of minimal order in

$$\mathfrak{M} \cap (\mathfrak{R} \vee^n \mathfrak{H}) \setminus (\mathfrak{M} \cap \mathfrak{R}) \vee^n (\mathfrak{M} \cap \mathfrak{H}).$$

Then A is a comonolithic group. Hence, there is $i \in I$ such that $A \in \mathfrak{F}_i$. Since $\mathfrak{F}_i$ is an atom of the lattice $L^n(\mathfrak{F})$, we have $\mathfrak{F}_i = l^n \operatorname{fit} A \subseteq \mathfrak{R} \vee^n \mathfrak{H}$. Whence, in view of condition (2) and Lemma 7, we infer that either $\mathfrak{F}_i \subseteq \mathfrak{R}$ or $\mathfrak{F}_i \subseteq \mathfrak{H}$. In both cases we arrive at the fact that the lattice $L^n(\mathfrak{F})$ is distributive. Thus, $L^n(\mathfrak{F})$ is a Boolean lattice.

Suppose that condition (1) holds and validate condition (3). This is obvious for $n = 0$. Assume that $n \geq 1$. First, demonstrate that the Fitting class $\mathfrak{F}$ is nilpotent. Suppose that $\mathfrak{F} \not\subseteq \mathfrak{N}$ and let G be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{N}$. Then G is a comonolithic group with the comonolith $R = F(G)$. Let p divide $|G|$. Since $\mathfrak{F}$ is a local class, we have $\mathfrak{N}_p \subseteq \mathfrak{F}$. Since $L^n(\mathfrak{F})$ is a complemented lattice, there is a Fitting subclass $\mathfrak{H}$ in $\mathfrak{F}$ such that $\mathfrak{F} = \mathfrak{N}_p \oplus \mathfrak{H}$. Since G is a comonolithic group, from here we infer that either $G \in \mathfrak{N}_p$ or $G \in \mathfrak{H}$. In the first case, the group G is nilpotent, which contradicts its choice. Assume that $G \in \mathfrak{H}$. From the fact that $\mathfrak{H}$ is a local class and that p divides $|G|$ we obtain $\mathfrak{N}_p \subseteq \mathfrak{H}$. However, this contradicts the fact that $\mathfrak{N}_p \cap \mathfrak{H} = (1)$. Hence, $\mathfrak{F} \subseteq \mathfrak{N}$.

Take $p \in \pi(\mathfrak{F}), \pi = \pi(\mathfrak{F}) \setminus \{p\}$. If $\pi = \varnothing$ then $\mathfrak{F} = \mathfrak{N}_p$ and (1) is a complement to $\mathfrak{N}_p$ in $\mathfrak{F}$. Suppose that $\pi \neq \varnothing$. Demonstrate that $\mathfrak{N}_\pi$ is a complement to $\mathfrak{N}_p$ in $\mathfrak{F}$. It is clear that $\mathfrak{N}_p \cap \mathfrak{N}_\pi = (1)$. Let $\mathfrak{H} = \operatorname{fit}(\mathfrak{N}_p \cup \mathfrak{N}_\pi) \neq \mathfrak{F}$ and let G be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{H}$. Then G is a comonolithic group. Furthermore, G is nilpotent. Hence, G is a primal group. Therefore, either $G \in \mathfrak{N}_p$ or $G \in \mathfrak{N}_\pi$. In both cases we arrive at a contradiction which completes the proof of the theorem.

Corollary 3. *Let $\mathfrak{F}$ be an n times local Fitting class with $n \geq 1$. Then the following conditions are equivalent:*

(1) *$L^n(\mathfrak{F})$ is a Boolean lattice;*

(2) $\mathfrak{F} = \mathfrak{N}_{\pi(\mathfrak{F})}$;

(3) *each Fitting subclass of the shape $\mathfrak{N}_p$ is complemented in $\mathfrak{F}$.*

Corollary 4. *Let $\mathfrak{M}$ and $\mathfrak{H}$ be n times local Fitting classes with $L^n(\mathfrak{M}) \simeq L^n(\mathfrak{H})$, where $n \geq 1$. If the Fitting class $\mathfrak{M}$ is nilpotent then so is the Fitting class $\mathfrak{H}$.*

PROOF. According to Corollary 3, $L^n(\mathfrak{M})$ is a Boolean lattice. Hence, so is the lattice $L^n(\mathfrak{H})$. The Fitting class $\mathfrak{H}$ is nilpotent by Corollary 3, which completes the proof of the corollary.

Corollary 5. *Assume that $\mathfrak{F} = l^n \operatorname{fit} G$. Then the group G is nilpotent if and only if $L^n(\mathfrak{F})$ is a Boolean lattice.*

References

1. L. A. Shemetkov, Formations of Finite Groups [in Russian], Nauka, Moscow (1978).
2. L. A. Shemetkov and A. N. Skiba, Formations of Algebraic Systems [in Russian], Nauka, Moscow (1989).
3. N. T. Vorob′ëv, "On Hawkes's conjecture for radical classes," Sibirsk. Mat. Zh., **37**, No. 6, 1296–1302 (1996).

4. A. N. Skiba, "Characterization of finite solvable groups of given nilpotent length," in: Problems of Algebra [in Russian], Universitetskoe, Minsk, 1987, No. 3, pp. 21–31.
5. A. N. Skiba, "On local formations of length 5," in: Arithmetical and Subgroup Construction of Finite Groups [in Russian], Nauka i Tekhnika, Minsk, 1986, pp. 135–149.
6. A. N. Skiba, Algebra of Formations [in Russian], Belaruskaya Navuka, Minsk (1997).
7. A. N. Skiba, "On local formations with complemented local subformations," Izv. Vyssh. Uchebn. Zaved. Mat., No. 10, 75–80 (1994).
8. A. N. Skiba, "On complemented subformations," in: Problems of Algebra [in Russian], Gomel′sk. Univ., Gomel′, 1996, No. 9, pp. 55–62.
9. V. A. Artamonov, V. N. Saliĭ, L. A. Skornyakov, L. N. Shevrin, and E. G. Shul′geĭfer, General Algebra. Vol. 2 [in Russian], Nauka, Moscow (1991).
10. K. Doerk and T. Hawkes, Finite Soluble Groups, Walter de Gruyter, Berlin and New York (1992).
11. V. A. Vedernikov, "On local formations of finite groups," Mat. Zametki, **46**, No. 6, 32–37 (1989).
12. A. N. Skiba and L. A. Shemetkov, Multiply ω-Local Formations and Fitting Classes of Finite Groups [Preprint, No. 63], Gomel′sk. Univ., Gomel′ (1997).