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# SELF-SIMILAR LIE ALGEBRA $\mathcal{E}(2)\oplus\mathcal{R}$

Let the Euclidean or Lorentz scalar product be given in Lie algebra $\mathcal{G}$. A linear transformation $f\colon \mathcal{G}\to\mathcal{G}$ is called an *autosimilarity* if it is both an automorphism of the Lie algebra and a similarity with respect to a given scalar product. We call a Lie algebra self-similar if it admits a one-parameter autosimilarity group that is not an isometry group.

The problem of constructing self-similar homogeneous manifolds of a Lie group $G$ endowed with a left-invariant Lorentzian metric $g$, initially involves solving the problem of finding such an inner product on a given Lie algebra for which it is self-similar [1].

In this paper, we consider the direct sum $\mathcal{G}_4=\mathcal{E}(2)\oplus\mathcal{R}$, where $\mathcal{E}(2)$ – is the three-dimensional Lie algebra, of the group $E(2)$ of motions of the Euclidean plane. It was proved in [2] that the Lie algebra $\mathcal{E}(2)$ does not admit autosimilarity for any way of specifying the Lorentz scalar product on it.

In a suitable basis $(E_1, E_2, E_3, E_4)$ in $\mathcal{G}_4$ the bracket operation is given by the equalities $[E_1, E_2]=E_3$, $[E_1, E_3]=-E_2$, and the other brackets are equal to the zero vector. Such a basis will be called canonical. The Lie algebra $\mathcal{G}_4$ contains the three-dimensional commutative ideal $\mathcal{H}=\langle E_2, E_3, E_4\rangle$, the one-dimensional center $\boldsymbol{R}E_4$, and the derived Lie algebra is two-dimensional: $\mathcal{G}_4^{(2)}=\mathcal{L}=\langle E_2, E_3\rangle$. The vector $E_1$ acts on $\mathcal{H}$ via the transformation $\mathrm{ad}(E_1)$, and the kernel of this transformation is $\boldsymbol{R}E_4$.

In [3], a complete group of automorphisms of the considered Lie algebra was found and it was proved that it cannot be self-similar for any way of specifying a Euclidean scalar product on it. The purpose of this paper is to show that there is one and only one (up to isometry) way of specifying a Lorentz inner product on $\mathcal{G}_4$ such that this Lie algebra is self-similar.

***Theorem.*** *The Lie algebra $\mathcal{G}_4 = \mathcal{E}(2)\oplus\mathcal{R}$ is self-similar if and only if there exists a canonical basis with respect to which the Lorentz inner product is given by the Gram matrix:*

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \qquad (1)$$

*In this case, the given Lie algebra admits a one-parameter similarity group whose action in the canonical basis is given by the matrix*

$$F(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\nu t}\cos\varepsilon t & -e^{\nu t}\sin\varepsilon t & 0 \\ 0 & e^{\nu t}\sin\varepsilon t & e^{\nu t}\cos\varepsilon t & 0 \\ 0 & 0 & 0 & e^{2\nu t} \end{pmatrix}, \ \nu > 0\,, t \in \boldsymbol{R}, \qquad (2)$$

*where $\varepsilon$ can take one of two values:* 1 *or* 0.

***Proof.*** Assume that a Lorentz scalar product is given in the Lie algebra $\mathcal{G}_4$ such, that a nondegenerate scalar product is induced on $\mathcal{H}$. Under the autosimilarity $F: \mathcal{G}_4 \to \mathcal{G}_4$ the ideal $\mathcal{H}$ and its orthogonal complement must remain invariant. We can choose $E_1 \in \mathcal{H}^{\perp}$, so that the bracket operation won't change. If the similarity is not isometry, then $E_1' = F(E_1) = \alpha E_1$, $|\alpha| \neq 1$, and $\mathrm{ad}(E_1) = |\alpha|\cdot\mathrm{ad}(E_1)$. This means that the given transformation cannot be an automorphism.

Suppose now that a degenerate scalar product is induced on $\mathcal{H}$. The only isotropic direction in $\mathcal{H}$ must remain invariant under any similarity. We can prove that the vector $E_4$ must belong to this direction. Thus a positive-definite scalar product is induced on the ideal $\mathcal{L}$. Denote $\mathcal{P} = \mathcal{H}^{\perp}$, $\dim\mathcal{P} = 2$. The Lorentz scalar product is induced in $\mathcal{P}$. It contains two isotropic directions, one of which is $\boldsymbol{R}E_4$ (figure 1).

Without changing the bracket operation, we can choose $E_1$ belonging to the second direction, and then multiply this vector by such a number that the new vector (we keep the same notation for it) will have the property $E_4 \cdot E_1 = 1$. The vectors $E_2, E_3$ can be chosen in the ideal $\mathcal{L}$ to be unit and orthogonal. As a result, we obtain the Gram matrix (1).

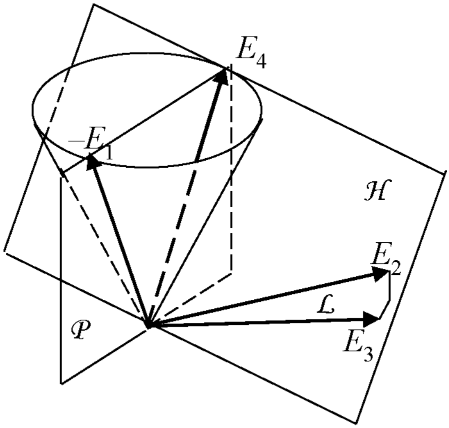


Figure 1

It is easy to see that the transformations that act according to formulas (2) are similarities and form a one-parameter group. ■

Having formulas (1) and (2), we can construct a homogeneous self-similar manifold of the Lie group $E(2)\times\boldsymbol{R}^{+}$ equipped with a left-invariant Lorentzian metric.

**References**


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