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ON THE DISTRIBUTIVITY AND MODULARITY SIGNS OF A FAMILY OF FITTING SETS OF A FINITE GROUP

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Abstract: We find conditions for the Fitting sets of a group to satisfy the distributive and modular laws.

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1. Introduction

Unless otherwise specified, all groups in this article are finite. We keep up with [1] as regards terminology and notation. A set $\mathscr{F}$ of subgroups in a group G (see [2, 3]) is a *Fitting set* in G whenever the following hold:

(1) if $T \trianglelefteq S \in \mathscr{F}$ then $T \in \mathscr{F}$;

(2) if $S, T \in \mathscr{F}$ and $S, T \trianglelefteq ST$ then $ST \in \mathscr{F}$;

(3) if $S \in \mathscr{F}$ and $x \in G$ then $S^x \in \mathscr{F}$.

It is easy to see that the $\subseteq$-ordered family of all Fitting sets of G constitutes a lattice in which $\mathscr{F} \cap \mathscr{H}$ is the meet and $\mathscr{F} \vee \mathscr{H} = \mathrm{Fitset}(\mathscr{F} \cup \mathscr{H})$ is the join for a pair of Fitting sets $\mathscr{F}$ and $\mathscr{H}$ in G. Here $\mathrm{Fitset}(\mathscr{F} \cup \mathscr{H})$ stands for the Fitting set G generated by $\mathscr{F} \cup \mathscr{H}$; i.e., $\mathrm{Fitset}(\mathscr{F} \cup \mathscr{H})$ is the intersection of all those Fitting sets G that include $\mathscr{F} \cup \mathscr{H}$.

Note that, the theory of group classes, describing the properties of lattices of classes, uses the concept of multiple localization. The latter was first proposed in formation theory by Skiba and found applications in classifying groups and their classes and finding the new families of lattices of group formations (see [4, Chapter 4]). This concept was dualized in the study of local Fitting classes in [5]. The idea of such a localization acquired further development in the study of generalized local Fitting classes in [6].

By now, the properties of the lattice of all Fitting classes remain little studied: It is still unknown whether the lattice of such classes is modular even in the soluble case [7, Question 14.47]; though Skiba and Vorob'ev [8, Theorem] and, independently, Reifferscheid [9, Theorem 2.2.14(b)] proved that the lattice of all totally local Fitting classes (primitive saturated functions) is distributive. Also known in the theory of Fitting classes are the results by Lausch [10] and Vorob'ev and Martsinkevich [11] stating that the lattice of all soluble normal Fitting classes and the lattice of all locally normal Fitting classes are modular. However, as we proved in [12], such lattices and the lattice of all Fitting classes are not distributive.

The study of lattices of formation Fitting sets is the contents of Skiba's work [13]. Nevertheless, until recently, there were no results on the properties of lattices of arbitrary Fitting sets of a group. In particular, the questions remain open whether *the lattice of all Fitting sets of a group is modular and the lattice of all totally local Fitting sets of a soluble group is distributive.*

The search for solving these questions leads to *the problem of describing the families of Fitting sets of a group for which the distributive and modular equalities hold.* Solving this problem is the contents of the present article. We define the families of Fitting sets and families of σ-local (in particular, local) Fitting sets that satisfy the distributive and modular equalities (see Theorems 4.2 and 4.5).

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2. Preliminaries

Recall that a *class of groups* is a collection of groups containing all groups isomorphic to G together with each particular member G. A group belonging to a class of groups $\mathfrak{X}$ is called a $\mathfrak{X}$*-group*.

A class of groups $\mathfrak{F}$ is called a *Fitting class* if $\mathfrak{F}$ is closed under normal subgroups and products of normal $\mathfrak{F}$-subgroups.

Let $\mathscr{F}$ be a Fitting set of a group G and let $\mathfrak{H}$ be a Fitting class. Then the set $\{S \leq G : S/S_{\mathscr{F}} \in \mathfrak{H}\}$ of subgroups in G is the *product* of $\mathscr{F}$ and $\mathfrak{H}$ denoted by $\mathscr{F} \odot \mathfrak{H}$. If $\mathscr{F} = \varnothing$ or $\mathfrak{H} = \varnothing$ then we assume that $\mathscr{F} \odot \mathfrak{H} = \varnothing$.

Lemma 2.1 [14, Proposition 3.1]. *If $\mathscr{F}$ is a Fitting set of G and $\mathfrak{H}$ is a Fitting class then the product $\mathscr{F} \odot \mathfrak{H}$ is a Fitting set in G.*

Lemma 2.2 [14, Proposition 3.4(3)]. *If $\mathscr{F}$ is a Fitting set of G, while $\mathfrak{H}$ and $\mathfrak{M}$ are Fitting classes; then $\mathscr{F} \odot (\mathfrak{H} \cap \mathfrak{M}) = (\mathscr{F} \odot \mathfrak{H}) \cap (\mathscr{F} \odot \mathfrak{M})$.*

Lemma 2.3 [14, Property 3.2(1)]. *If $\mathscr{F}$ is a Fitting set of G and $\mathfrak{H}$ is a nonempty Fitting class then $\mathscr{F} \subseteq \mathscr{F} \odot \mathfrak{H}$.*

A class of groups $\mathfrak{F}$ is a *homomorph* if $G \in \mathfrak{F}$ implies $G/N \in \mathfrak{F}$ for every normal subgroup N in G. If a class of groups $\mathfrak{F}$ is simultaneously a homomorph and a Fitting class then $\mathfrak{F}$ is a *radical homomorph.*

Lemma 2.4 [14, Proposition 3.4(1)]. *If $\mathscr{F}$ and $\mathscr{H}$ are Fitting groups in a group G, with $\mathscr{F} \subseteq \mathscr{H}$, and $\mathfrak{M}$ is a nonempty radical homomorph then $\mathscr{F} \odot \mathfrak{M} \subseteq \mathscr{H} \odot \mathfrak{M}$.*

A class of groups is a *formation* if it is closed under homomorphic images and subdirect products. If $\mathfrak{F}$ is a nonempty formation then each group G has a least normal subgroup the quotient group by which belongs to $\mathfrak{F}$. The latter is the $\mathfrak{F}$*-residual* of G, denoted by $G^{\mathfrak{F}}$.

Lemma 2.5 [6, Lemma 2.1]. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be nonempty formations. If $\mathfrak{F} \subseteq \mathfrak{H}$ then $G^{\mathfrak{H}} \leq G^{\mathfrak{F}}$ for all G.*

Recall that the *product* $\mathfrak{F}\mathfrak{H}$ *of two classes of groups* is the class $(G : \exists N \trianglelefteq G,\ N \in \mathfrak{F}, \text{ and } G/N \in \mathfrak{H})$; and the *product* $\mathfrak{F} \circ \mathfrak{H}$ *of formations* $\mathfrak{F}$ and $\mathfrak{H}$ is the class $(G : G^{\mathfrak{F}} \in \mathfrak{H})$. If $\mathfrak{F}$ or $\mathfrak{H}$ is an empty formation then we put $\mathfrak{F} \circ \mathfrak{H} = \varnothing$. It is well known that the product of every two formations is a formation and the operation $\circ$ is associative on the set of all formations (see [1, Theorem IV.1.8(a),(c)]). If a formation $\mathfrak{F}$ is closed under normal subgroups then $\mathfrak{F}\mathfrak{H} = \mathfrak{F} \circ \mathfrak{H}$ (see [1, p. 338]).

Lemma 2.6 [1, Theorem IV.1.8(b)]. *If $\mathfrak{F}$ and $\mathfrak{H}$ are nonempty formations then $G^{\mathfrak{F} \circ \mathfrak{H}} = (G^{\mathfrak{H}})^{\mathfrak{F}}$ for all groups G.*

For every nonempty Fitting set $\mathscr{F}$ of a group G, each subgroup H in G has a greatest normal $\mathscr{F}$-subgroup called the $\mathscr{F}$*-radical* of H and denoted by $H_{\mathscr{F}}$.

Lemma 2.7 [1, Proposition VIII.2.4(d)]. *Let $\mathscr{F}$ be a nonempty Fitting set of a group G. If $N \trianglelefteq G$ then $N_{\mathscr{F}} = N \cap G_{\mathscr{F}}$.*

3. σ-Local Fitting Sets of a Group and Their Properties

In the theory of Fitting sets, we develop the σ-method of studying the structure of groups and their classes which was proposed by Skiba (see [13, 15, 16]).

Let $\mathbb{P}$ be the set of all primes, $\pi \subseteq \mathbb{P}$, and $\pi' = \mathbb{P} \setminus \pi$. Denote by $\pi(n)$ the set of all prime divisors of an integer n and designate as $\pi(G) = \pi(|G|)$ the set of all prime divisors of a group G. Let σ be a partition of $\mathbb{P}$; i.e., $\sigma = \{\sigma_i : i \in I\}$, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$, and $\sigma_i \cap \sigma_j = \varnothing$ for all $i \neq j$; $\sigma(n) = \{\sigma_i : \sigma_i \cap \pi(n) \neq \varnothing\}$ and $\sigma(G) = \sigma(|G|)$.

Following [6], call each mapping of the form $f : \sigma \to \{\text{a Fitting set of } G\}$ a *Hartley σ-function* of G or simply an *H_σ-function* of G. If f is an H_σ-function then $\mathrm{Supp}(f)$ is the support of f, i.e., the set of all σ_i such that $f(\sigma_i) \neq \varnothing$.

Suppose that $LFS_\sigma(f) = \{S \le G : S = 1 \text{ or } S \ne 1 \text{ and } S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in f(\sigma_i) \text{ for all } \sigma_i \in \sigma(S)\}$, where $\mathfrak{E}_{\sigma_i}$ and $\mathfrak{E}_{\sigma_i'}$ are the classes of σ_i-groups and σ_i'-groups.

DEFINITION 3.1. Call a Fitting set of a group G *σ-local* if $\mathscr{F} = LFS_\sigma(f)$ for some H_σ-function f. In particular, if $\sigma = \sigma^1 = \{\{2\}, \{3\}, \dots\}$ then we refer to $\mathscr{F}$ as a *local* Fitting set of G.

Observe that to each Fitting class of $\mathfrak{F}$ there corresponds the set of $\mathfrak{F}$-subgroups in G; i.e., $\{S \le G : S \in \mathfrak{F}\}$, which is denoted by $\mathrm{Tr}_\mathfrak{F}(G)$ and called the *trace* of the Fitting class $\mathfrak{F}$ in G. Obviously, $\mathrm{Tr}_\mathfrak{F}(G)$ is a Fitting set in G, though the converse fails in general (see [1, Example VII.2.2(c)]).

EXAMPLES 3.2. (1) The Fitting set consisting only of the identity subgroup $\{1\}$ in G is local; i.e., $\{1\} = LFS_\sigma(f)$ for the H_σ-function f such that $f(\sigma_i) = \varnothing$ for all $i \in I$.

(2) Let $\mathscr{F} = \mathrm{Tr}_{\mathfrak{E}_{\sigma_i}}(G)$ be the Fitting set of all σ_i-subgroups in a group G. If $S \le G$ and $S \in \mathscr{F}$ then $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in \mathscr{F}$ and $\mathscr{F} = LFS_\sigma(f)$ for the H_σ-function such that $f(\sigma_i) = \mathscr{F}$ and $f(\sigma_j) = \varnothing$ for all $i \ne j$.

(3) Recall that a group G is *σ-primary* if G is a σ_i-group for some $i \in I$. Also, G is *σ-nilpotent* (see [15]) if $G = G_1 \times \dots \times G_t$ for some σ-primary groups $G_1, \dots, G_t$. Suppose that $\mathscr{X}$ is a Fitting set of a group G, while $\mathfrak{N}_\sigma$ is the class of all σ-nilpotent groups, $\mathscr{X} \odot \mathfrak{N}_\sigma$ is the product of $\mathscr{X}$ and the fitting class $\mathfrak{N}_\sigma$, and f is an H_σ-function such that $f(\sigma_i) = \mathscr{X}$ for all i. Then, by Lemma 2.2,

$$LFS_\sigma(f) = \bigcap_{\sigma_i} \mathscr{X} \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'} = \mathscr{X} \odot \Big(\bigcap_{\sigma_i} \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}\Big) = \mathscr{X} \odot \mathfrak{N}_\sigma$$

and $\mathscr{X} \odot \mathfrak{N}_\sigma$ is a σ-local Fitting set in G.

(4) Put $\mathfrak{N}_\sigma^k = \underbrace{\mathfrak{N}_\sigma \dots \mathfrak{N}_\sigma}_{k}$ $(k \in \mathbb{N})$ and let $\mathfrak{N}_\sigma^0$ be the class of groups of order 1. Let $\mathrm{Tr}_{\mathfrak{N}_\sigma^k}(G)$ be the trace of the Fitting class $\mathfrak{N}_\sigma^k$ in a group G. Then $\mathrm{Tr}_{\mathfrak{N}_\sigma^k}(G)$ is a σ-local Fitting set in G with H_σ-function f such that $f(\sigma_i) = \mathrm{Tr}_{\mathfrak{N}_\sigma^{k-1}}(G)$ for all σ_i. In particular, $\mathrm{Tr}_{\mathfrak{N}_\sigma}(G)$ is a σ-local Fitting set in G with H_σ-function f such that $f(\sigma_i) = \{1\}$ for all σ_i.

Lemma 3.3. *Let* $\mathscr{F} = LFS_\sigma(f)$ *and* $\Pi = \mathrm{Supp}(f)$. *Then the following hold:*

(1) $\Pi = \sigma(\mathscr{F})$;

(2) $S \le G$ *and* $S \in \mathscr{F}$ *if and only if* $S \in f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$ *for all* $\sigma_i \in \sigma(S)$, *i.e.*, $\mathscr{F} = \{S \le G : S \in f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}\}$;

(3) $\mathscr{F} = \mathrm{Tr}_{\mathfrak{E}_\Pi}(G) \cap \big(\bigcap_{\sigma_i \in \Pi} f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}\big)$.

PROOF. (1): If $\sigma_i \in \Pi$ then $\{1\} \in f(\sigma_i)$; and, for every σ_i-subgroup S in G such that $S \ne 1$, we have $\sigma(S) = \{\sigma_i\}$. Since $\{1\} \in f(\sigma_i)$, by Lemma 2.3, $S \in \{1\} \odot \mathfrak{E}_{\sigma_i} \subseteq \{1\} \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'} \subseteq f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$ for all $\sigma_i \in \Pi$. By the definitions of a Fitting set of G and a Fitting class, $S/S_{f(\sigma_i)} \in \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$, and hence $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \le S_{f(\sigma_i)}$. By the definition of $f(\sigma_i)$-radical, $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \trianglelefteq S_{f(\sigma_i)}$. Consequently, $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in f(\sigma_i)$ and $S \in \mathscr{F}$. Hence, $\Pi \subseteq \sigma(\mathscr{F})$.

If $\sigma_i \in \sigma(\mathscr{F})$; then, for some subgroup $S \in \mathscr{F}$ in G, we have $\sigma_i \in \sigma(S)$ and $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in f(\sigma_i)$. Hence, $\sigma_i \in \Pi$ and $\sigma(\mathscr{F}) \subseteq \Pi$. Thus, $\Pi = \sigma(\mathscr{F})$.

(2): If $\sigma_i \in \sigma(S)$ and S is a $\mathscr{F}$-subgroup in G then $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in f(\sigma_i)$. Since $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \le S_{f(\sigma_i)}$ and $\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$ is a formation, we infer

$$(S/S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}})/(S_{f(\sigma_i)}/S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}}) \cong S/S_{f(\sigma_i)} \in \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}.$$

Consequently, $S \in f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$ for all $\sigma_i \in \sigma(S)$.

Conversely, if for all $\sigma_i \in \sigma(S)$ we have $S \in f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$ then $S/S_{f(\sigma_i)} \in \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$. Then $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \le S_{f(\sigma_i)}$. Since $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \trianglelefteq S_{f(\sigma_i)}$; therefore, $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in f(\sigma_i)$. Thus, $S \in \mathscr{F}$.

(3) Let S be a $\mathscr{F}$-subgroup in G. Then $|S|$ is a $\sigma(\mathscr{F})$-number. By item (1), $\sigma(\mathscr{F}) = \Pi$, $|S|$ is a Π-number, and S is a Π-subgroup in G. Consequently, $S \in \mathfrak{G}_\Pi$ and $S \in \mathrm{Tr}_{\mathfrak{E}_\Pi}(G)$. Moreover, by

item (2), $S \in \mathscr{F}$ implies that $S \in f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$ for all $\sigma_i \in \sigma(S)$. If $\sigma_i \in \Pi \setminus \sigma(S)$ then $S \in \mathfrak{E}_{\sigma_i'} \subseteq \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'} \subseteq f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$. Thus, $S \in \mathrm{Tr}_{\mathfrak{E}_\Pi}(G) \cap \left(\bigcap_{\sigma_i \in \Pi} f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}\right)$, and $\mathscr{F} \subseteq \mathrm{Tr}_{\mathfrak{E}_\Pi}(G) \cap \left(\bigcap_{\sigma_i \in \Pi} f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}\right)$. Conversely, if

$$S \in \mathrm{Tr}_{\mathfrak{E}_\Pi}(G) \cap \left(\bigcap_{\sigma_i \in \Pi} f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}\right)$$

for all $\sigma_i \in \sigma(S)$ then $S \in f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$. Therefore, by item (2) of the lemma, $S \in \mathscr{F}$ and $\mathrm{Tr}_{\mathfrak{E}_\Pi}(G) \cap \left(\bigcap_{\sigma_i \in \Pi} f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}\right) \subseteq \mathscr{F}$.

The lemma is proved.

DEFINITION 3.4. Let $\mathscr{F}$ be a σ-local Fitting set of a group G. Call an H_σ-function f of $\mathscr{F}$:
(1) *inner* if $f(\sigma_i) \subseteq \mathscr{F}$ for all $\sigma_i \in \Pi$;
(2) *complete* if $f(\sigma_i) = f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}$ for all $\sigma_i \in \Pi$;
(3) *complete inner* if f is a complete and inner H_σ-function simultaneously.

Lemma 3.5. *Let $\mathscr{F}$ be a σ-local Fitting set in a group G. Then the following hold:*
(1) $\mathscr{F}$ is defined by an inner H_σ-function;
(2) $\mathscr{F}$ is defined by a complete inner H_σ-function.

PROOF. (1): Since the Fitting set $\mathscr{F}$ is σ-local by Lemma 3.3(3), there exists an H_σ-function f such that

$$\mathscr{F} = \mathrm{Tr}_{\mathfrak{E}_\Pi}(G) \cap \left(\bigcap_{\sigma_i \in \Pi} f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}\right),$$

where $\Pi = \mathrm{Supp}(f) = \sigma(\mathscr{F})$. Consider the H_σ-function φ such that $\varphi(\sigma_i) = f(\sigma_i) \cap \mathscr{F}$ for all $i \in I$. Then $\varphi(\sigma_i) \subseteq f(\sigma_i)$ for every $\sigma_i \in \sigma(\mathscr{F})$. Consequently, by Lemma 2.4, $\varphi(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'} \subseteq f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$; and, hence, $LFS_\sigma(\varphi) \subseteq \mathscr{F}$.

Conversely, let $S \in \mathscr{F}$. Then $S \in f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$. Hence, $S/S_{f(\sigma_i)} \in \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$. Thus, $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \leq S_{f(\sigma_i)}$ and $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in f(\sigma_i)$. Since $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \trianglelefteq S$, we have $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in \mathscr{F}$. Consequently, $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in f(\sigma_i) \cap \mathscr{F} = \varphi(\sigma_i)$. This yields $S \in \varphi(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$ for all $\sigma_i \in \sigma(\mathscr{F})$ and

$$S \in \mathrm{Tr}_{\mathfrak{E}_\Pi(G)} \cap \left(\bigcap_{\sigma_i \in \Pi} \varphi(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}\right) = LFS_\sigma(\varphi).$$

Hence, $\mathscr{F} \subseteq LFS_\sigma(\varphi)$ and $\mathscr{F} = LFS_\sigma(\varphi)$. Item (1) is proved.

(2): Let $\mathscr{F} = LFS_\sigma(\varphi)$ for some inner H_σ-function. Consider the H_σ-function ψ such that $\psi(\sigma_i) = \varphi(\sigma_i) \odot \mathfrak{E}_{\sigma_i}$ for every $i \in I$. Obviously, ψ is a complete H_σ-function. Show that ψ is an inner H_σ-function for $\mathscr{F}$.

Let $S \in \psi(\sigma_i)$. Then $S/S_{\varphi(\sigma_i)} \in \mathfrak{E}_{\sigma_i} \subseteq \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$. Consequently, $S \in \varphi(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$. Let $\sigma_j \neq \sigma_i$. Then $\mathfrak{E}_{\sigma_i} \subseteq \mathfrak{E}_{\sigma_j'}$. By Lemma 2.5, $S^{\mathfrak{E}_{\sigma_j'}} \leq S^{\mathfrak{E}_{\sigma_i}}$. Hence, $(S^{\mathfrak{E}_{\sigma_j'}})^{\mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma_j'}} \in \varphi(\sigma_j)$. Using Lemma 2.6, we infer

$$(S^{\mathfrak{E}_{\sigma_j'}})^{\mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma_j'}} = S^{\mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma_j'}\mathfrak{E}_{\sigma_j'}} = S^{\mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma_j'}}, \quad S^{\mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma_j'}} \in \varphi(\sigma_j).$$

Thus,

$$(S/S^{\mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma_j'}})/(S_{\varphi(\sigma_j)}/S^{\mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma_j'}}) \cong S/S_{\varphi(\sigma_j)} \in \mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma_j'}.$$

Then $S \in \varphi(\sigma_j) \odot \mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma_j'}$ for all $j \neq i$. We obtain $S \in \varphi(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$ for all $\sigma_i \in \Pi$. Consequently, by Lemma 3.3(2), $S \in \mathscr{F}$.

The lemma is proved.

If $\mathscr{X}$ is a family of subgroups in a group G then denote by $\mathrm{Fitset}(\mathscr{X})$ the intersection of all Fitting sets in G including $\mathscr{X}$. Obviously, $\mathrm{Fitset}(\mathscr{X})$ is a Fitting set in G.

Let Ω be the family of all H_σ-functions of a Fitting set $\mathscr{F}$ in G. Endow Ω with the order $\leq$ as follows: If $f, \varphi \in \Omega$ then $f \leq \varphi$ if and only if $f(\sigma_i) \subseteq \varphi(\sigma_i)$ for all $\sigma_i \in \Pi$. Refer to a minimal element in Ω as a *minimal H_σ-function* of the Fitting set $\mathscr{F}$ of G.

Lemma 3.6. *Let $\mathscr{F}$ be a σ-local Fitting set of a group G. Then*

(1) $\mathscr{F}$ is defined by the only minimal H_σ-function f such that $f(\sigma_i) = \mathrm{Fitset}\{S \leq G : S$ is conjugate to $X^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}}$, $X \in \mathscr{F}\}$ for all $\sigma_i \in \mathrm{Supp}(f)$;

(2) $\mathscr{F}$ is defined by the only minimal complete inner H_σ-function f such that $f(\sigma_i) = \mathrm{Fitset}\{S \leq G : S^{\mathfrak{E}_{\sigma_i}}$ is conjugate to $X^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}}$, $X \in \mathscr{F}\} \odot \mathfrak{E}_{\sigma_i}$ for all $\sigma_i \in \mathrm{Supp}(f)$.

PROOF. (1): Let $\mathscr{F} = LFS_\sigma(\varphi)$ be an inner H_σ-function φ. Define the set of subgroups of G as follows: $f_1(\sigma_i) = \{S \leq G : S$ is conjugate to $X^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}}, X \in \mathscr{F}\}$. Put $f(\sigma_i) = \mathrm{Fitset}\, f_1(\sigma_i)$. In view of $X \in \mathscr{F}$, we obtain $X^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in \mathscr{F}$. By Lemma 3.3(2), $X \in \varphi(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$. Consequently, $X^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in \varphi(\sigma_i)$. Since S is conjugate to $X^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in \varphi(\sigma_i)$, we have $S \in \varphi(\sigma_i)$. Hence, $f_1(\sigma_i) \subseteq \varphi(\sigma_i)$ and $\mathrm{Fitset}\, f_1(\sigma_i) = f(\sigma_i) \subseteq \mathrm{Fitset}\, \varphi(\sigma_i) = \varphi(\sigma_i)$. Therefore, $LFS_\sigma(f) \subseteq \mathscr{F}$.

Prove the reverse inclusion. Assume that $S \leq G$ and $S \in \mathscr{F}$. Since the $\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$-residual of $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}}$ is conjugate to $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}}$; by the definition of f, we have $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in \{S \leq G : S$ is conjugate to $X^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}}\} \subseteq f(\sigma_i)$. Consequently, $S \in f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$ for all $\sigma_i \in \mathrm{Supp}(f)$. By Lemma 3.3(2), $S \in LFS_\sigma(f)$. Hence, $\mathscr{F} \subseteq LFS_\sigma(f)$ and $\mathscr{F} = LFS_\sigma(f)$. Assertion (1) is proved.

(2): By Lemma 3.5(2), $\mathscr{F} = LFS_\sigma(\varphi)$ for a complete inner H_σ-function φ of the Fitting set $\mathscr{F}$ in G, i.e.,

$$\mathscr{F} = \mathrm{Tr}_{\mathfrak{E}_\Pi}(G) \cap \Big(\bigcap_{\sigma_i \in \Pi} \varphi(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'} \Big).$$

Suppose that $\varphi_1(\sigma_i) = \{S \leq G : S^{\mathfrak{E}_{\sigma_i}}$ is conjugate to $X^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}}$, $X \in \mathscr{F}\}$. If $S \in \varphi_1(\sigma_i)$ then $S^{\mathfrak{E}_{\sigma_i}}$ is conjugate to $X^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}}$ for some subgroup $X \in \mathscr{F}$. Since $X \in \mathscr{F}$, by Lemma 3.3(2), we have $X \in \varphi(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$. Then $X/X_{\varphi(\sigma_i)} \in \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$. Hence, $X^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in \varphi(\sigma_i)$. Owing to the conjugacy of $S^{\mathfrak{E}_{\sigma_i}}$ and $X^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}}$, we have $S^{\mathfrak{E}_{\sigma_i}} \in \varphi(\sigma_i)$. Therefore, $S \in \varphi(\sigma_i) \odot \mathfrak{E}_{\sigma_i} = \varphi(\sigma_i)$ and $\varphi_1(\sigma_i) \subseteq \varphi(\sigma_i)$ for all $\sigma_i \in \mathrm{Supp}(\varphi)$. Hence, $\mathrm{Fitset}\, \varphi_1(\sigma_i) \odot \mathfrak{E}_{\sigma_i} \subseteq \mathrm{Fitset}\, \varphi(\sigma_i) \odot \mathfrak{E}_{\sigma_i}$. Thus, $f \leq \varphi$ for every complete H_σ-function φ of the Fitting set $\mathscr{F}$ of G. In particular, $LFS_\sigma(f) \subseteq \mathscr{F}$.

Conversely, let S be a $\mathscr{F}$-subgroup in G. In view of Lemma 2.6, we have $(S^{\mathfrak{E}_{\sigma_i'}})^{\mathfrak{E}_{\sigma_i}} = S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}}$. Consequently, $S^{\mathfrak{E}_{\sigma_i'}} \in \varphi_1(\sigma_i) \subseteq f(\sigma_i)$. By Lemma 2.5, $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \leq S^{\mathfrak{E}_{\sigma_i}}$. Then $S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in f(\sigma_i)$ and $S \in f(\sigma_i) \odot \mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}$ for all $\sigma_i \in \sigma(\mathscr{F})$. By Lemma 3.3(2), $S \in LFS_\sigma(f)$. Thus, $\mathscr{F} \subseteq LFS_\sigma(f)$ and $\mathscr{F} = LFS_\sigma(f)$.

Let $\{f_i : i \in I\}$ be the set of all complete H_σ-functions. Then $\psi = \bigcap_{i \in I} f_i$ is a minimal complete H_σ-function. Since by Lemma 3.5(2) each σ-local Fitting set in G is defined by a complete inner H_σ-function, in the set of all complete H_σ-functions of $\mathscr{F}$, there exists at least one complete inner H_σ-function. Consequently, $\psi = \bigcap_{i \in I} f_i$ is a minimal complete inner H_σ-function of $\mathscr{F}$ and $\psi = f$. Claim (2) is proved.

The proof of the lemma is complete.

The following is straightforward from Lemma 3.6.

Corollary 3.7. *Assume that $\mathscr{F} = LFS_\sigma(\underline{f})$ and $\mathscr{H} = LFS_\sigma(\underline{h})$, where $\underline{f}$ and $\underline{h}$ are minimal H_σ-functions of Fitting sets $\mathscr{F}$ and $\mathscr{H}$ in a group G. Then $\underline{f} \leq \underline{h}$ if and only if $\mathscr{F} \subseteq \mathscr{H}$.*

If f and h are H_σ-functions then we denote by $f \vee_\sigma h$ the H_σ-function such that $(f \vee_\sigma h)(\sigma_i) = f(\sigma_i) \vee_\sigma h(\sigma_i)$ for each i and designate as $f \wedge h$ the H_σ-function such that $(f \cap h)(\sigma_i) = f(\sigma_i) \cap h(\sigma_i)$ for each i.

Note that the inclusion-ordered set of all σ-local Fitting sets of a group G forms a lattice with the operations $\wedge_\sigma$ and $\vee_\sigma$ defined for every two σ-local Fitting sets $\mathcal{F}$ and $\mathcal{H}$ of G as follows: $\mathcal{F} \wedge_\sigma \mathcal{H} = \mathcal{F} \cap \mathcal{H}$ and $\mathcal{F} \vee_\sigma \mathcal{H} = \mathrm{Fitset}(\mathcal{F} \cup \mathcal{H})$.

Lemma 3.8. *Suppose that $\mathcal{F} = LFS_\sigma(f)$ is a σ-local Fitting set of a group G, while $\Pi = \sigma(\mathcal{F})$ and m is the H_σ-function such that*

$$m(\sigma_i) = \mathrm{Fitset}\{S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} : S \leq G,\ S \in \mathcal{F}\}$$

for all $\sigma_i \in \Pi$ and $m(\sigma_i) = \varnothing$ for all $\sigma_i \in \Pi'$. Then

(1) $\mathcal{F} = LFS_\sigma(m)$;

(2) $m(\sigma_i) \subseteq h(\sigma_i) \cap \mathcal{F}$ for every H_σ-function h of the Fitting set $\mathcal{F}$ of G and each σ_i.

PROOF. Suppose that

$$\mathcal{F}(\sigma_i) = \{S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} : S \leq G,\ S \in \mathcal{F}\}$$

for all $\sigma_i \in \Pi$ and $\mathcal{M} = LFS_\sigma(m)$. Then $\mathcal{F} \subseteq \mathcal{M}$. On the other hand, $\mathcal{F}(\sigma_i) \subseteq f(\sigma_i)$, and hence $m(\sigma_i) \subseteq f(\sigma_i)$ for all $\sigma_i \in \Pi$. Moreover, $m(\sigma_i) = \varnothing \subseteq f(\sigma_i)$ for all $\sigma_i \in \Pi'$. Therefore, $\mathcal{M} \subseteq \mathcal{F}$, and hence $\mathcal{M} = \mathcal{F}$.

The lemma is proved.

4. On Distributivity and Modularity Signs

In this section, we define the families of Fitting sets of a group G which satisfy the distributive and modular equalities.

Lemma 4.1. *If $\mathcal{F}$ and $\mathcal{H}$ are Fitting sets of G, while $\mathfrak{X}$ and $\mathfrak{Y}$ are radical homomorphs such that $\mathfrak{X} \cap \mathfrak{Y} = (1)$, $\mathcal{F} \subseteq \mathcal{H} \odot \mathfrak{X}$, and $\mathcal{H} \subseteq \mathcal{F} \odot \mathfrak{Y}$; then $\mathcal{F} \vee \mathcal{H} = \{S \leq G : S = S_{\mathcal{F}} S_{\mathcal{H}}\}$.*

PROOF. By Lemma 2.3, $\mathcal{F} \subseteq \mathcal{F} \odot \mathfrak{Y}$ and, by hypothesis, $\mathcal{F} \subseteq \mathcal{H} \odot \mathfrak{X}$. Consequently, $\mathcal{F} \subseteq \mathcal{H} \odot \mathfrak{X} \cap \mathcal{F} \odot \mathfrak{Y}$. Likewise, $\mathcal{H} \subseteq \mathcal{H} \odot \mathfrak{X}$ and $\mathcal{H} \subseteq \mathcal{F} \odot \mathfrak{Y}$. Therefore, $\mathcal{H} \subseteq \mathcal{H} \odot \mathfrak{X} \cap \mathcal{F} \odot \mathfrak{Y}$.

Thus, $\mathcal{F}$ and $\mathcal{H}$ are Fitting sets of G from $\mathcal{H} \odot \mathfrak{X} \cap \mathcal{F} \odot \mathfrak{Y}$. Since, by the definition of a lattice join, $\mathcal{F} \vee \mathcal{H}$ is the least Fitting set in G that includes $\mathcal{F}$ and $\mathcal{H}$, we have

$$\mathcal{F} \vee \mathcal{H} \subseteq \mathcal{H} \odot \mathfrak{X} \cap \mathcal{F} \odot \mathfrak{Y}. \tag{4.1}$$

Prove the reverse inclusion: $\mathcal{H} \odot \mathfrak{X} \cap \mathcal{F} \odot \mathfrak{Y} \subseteq \mathcal{F} \vee \mathcal{H}$. Let $\mathcal{M} = \{S \leq G : S_{\mathcal{F}} S_{\mathcal{H}}\}$ and $\mathcal{N} = \mathcal{H} \odot \mathfrak{X} \cap \mathcal{F} \odot \mathfrak{Y}$. Show that

$$\mathcal{M} = \mathcal{N}. \tag{4.2}$$

Let $S \in \mathcal{N}$. Then $S \in \mathcal{H} \odot \mathfrak{X}$ and $S \in \mathcal{F} \odot \mathfrak{Y}$. Consequently, $S/S_{\mathcal{F}} \in \mathfrak{Y}$. Since $\mathfrak{Y}$ is a homomorph, $S/S_{\mathcal{F}}/S_{\mathcal{F}}S_{\mathcal{H}}/S_{\mathcal{F}} \in \mathfrak{Y}$. Owing to the isomorphism $S/S_{\mathcal{F}}/S_{\mathcal{F}}S_{\mathcal{H}}/S_{\mathcal{F}} \cong S/S_{\mathcal{F}}S_{\mathcal{H}}$ and, also, $S/S_{\mathcal{F}}S_{\mathcal{H}} \in \mathfrak{Y}$. Similarly, from $S \in \mathcal{H} \odot \mathfrak{X}$ we obtain $S/S_{\mathcal{H}} \in \mathfrak{X}$, $S/S_{\mathcal{H}}/S_{\mathcal{F}}S_{\mathcal{H}}/S_{\mathcal{H}} \in \mathfrak{X}$, and $S/S_{\mathcal{F}}S_{\mathcal{H}} \in \mathfrak{X}$. Hence, $S/S_{\mathcal{F}}S_{\mathcal{H}} \in \mathfrak{X} \cap \mathfrak{Y}$. Since $S/S_{\mathcal{F}}S_{\mathcal{H}}$ is the identity group, $S = S_{\mathcal{F}}S_{\mathcal{H}}$. Hence, $S \in \mathcal{M}$ and $\mathcal{N} \subseteq \mathcal{M}$. Conversely, let $S \in \mathcal{M}$. Then $S = S_{\mathcal{F}}S_{\mathcal{H}}$. By hypothesis, $\mathcal{F} \subseteq \mathcal{H} \odot \mathfrak{X}$, and hence $S_{\mathcal{F}} \subseteq S_{\mathcal{H} \odot \mathfrak{X}}$. Therefore, $S \subseteq S_{\mathcal{H} \odot \mathfrak{X}} S_{\mathcal{H}} = S_{\mathcal{H} \odot \mathfrak{X}}$. Thud, $S = S_{\mathcal{H} \odot \mathfrak{X}}$, i.e., $S \in \mathcal{H} \odot \mathfrak{X}$. Similarly, $S \in \mathcal{F} \odot \mathfrak{Y}$ and $S \in \mathcal{H} \odot \mathfrak{X} \cap \mathcal{F} \odot \mathfrak{Y}$, and (4.2) is proved.

Now, show that $\mathcal{M} \subseteq \mathcal{F} \vee \mathcal{H}$. Take $S \in \mathcal{M}$. Then $S = S_{\mathcal{F}}S_{\mathcal{H}}$. By the definition of $\mathcal{F}$-radical, $S_{\mathcal{F}} \in \mathcal{F} \subseteq \mathcal{F} \vee \mathcal{H}$. Likewise, $S_{\mathcal{H}} \in \mathcal{H} \subseteq \mathcal{F} \vee \mathcal{H}$. By the definition of Fitting set, $S_{\mathcal{F}}S_{\mathcal{H}} \in \mathcal{F} \vee \mathcal{H}$. Since $S = S_{\mathcal{F}}S_{\mathcal{H}}$, we have $S \in \mathcal{F} \vee \mathcal{H}$. Thus, $\mathcal{M} \subseteq \mathcal{F} \vee \mathcal{H}$. Considering (4.2), we obtain

$$\mathcal{H} \odot \mathfrak{X} \cap \mathcal{F} \odot \mathfrak{Y} \subseteq \mathcal{F} \vee \mathcal{H}. \tag{4.3}$$

Now, in view of (4.1), (4.3), and (4.2), $\mathcal{H} \odot \mathfrak{X} \cap \mathcal{F} \odot \mathfrak{Y} = \mathcal{F} \vee \mathcal{H}$ and $\mathcal{F} \vee \mathcal{H} = \{S \leq G : S = S_{\mathcal{F}}S_{\mathcal{H}}\}$.

The lemma is proved.

Define some family of Fitting sets in G that satisfy the distributive equality.

Theorem 4.2. *If $\mathscr{F}$, $\mathscr{H}$, and $\mathscr{M}$ are Fitting sets of a group G and $\mathscr{F} \subseteq \mathscr{H} \odot \mathfrak{E}_\pi$, $\mathscr{H} \subseteq \mathscr{F} \odot \mathfrak{E}_{\pi'}$ for some set of primes π; then*

$$\mathscr{M} \cap (\mathscr{F} \vee \mathscr{H}) = (\mathscr{M} \cap \mathscr{F}) \vee (\mathscr{M} \cap \mathscr{H}).$$

PROOF. Obviously, $(\mathscr{M} \cap \mathscr{F}) \vee (\mathscr{M} \cap \mathscr{H}) \subseteq \mathscr{M} \cap (\mathscr{F} \vee \mathscr{H})$.

Show that $\mathscr{M} \cap (\mathscr{F} \vee \mathscr{H}) \subseteq (\mathscr{M} \cap \mathscr{F}) \vee (\mathscr{M} \cap \mathscr{H})$. Let $S \leq G$ and $S \in \mathscr{M} \cap (\mathscr{F} \vee \mathscr{H})$. Consequently, $S \in \mathscr{M}$ and $S \in \mathscr{F} \vee \mathscr{H}$. Then, by Lemma 4.1, in view of $\mathscr{F} \subseteq \mathscr{H} \odot \mathfrak{E}_\pi$ and $\mathscr{H} \subseteq \mathscr{F} \odot \mathfrak{E}_{\pi'}$, we obtain $S = S_\mathscr{F} S_\mathscr{H}$. From $S \in \mathscr{M}$ and $S = S_\mathscr{F} S_\mathscr{H}$ it follows that $S_\mathscr{F} \in \mathscr{M}$ and $S_\mathscr{H} \in \mathscr{M}$. Then $S_{\mathscr{F}\cap\mathscr{M}} \in \mathscr{M}$ and $S_{\mathscr{H}\cap\mathscr{M}} \in \mathscr{M}$. Hence, $S = S_{\mathscr{F}\cap\mathscr{M}} S_{\mathscr{H}\cap\mathscr{M}}$. Therefore, $S \in (\mathscr{M} \cap \mathscr{F}) \vee (\mathscr{M} \cap \mathscr{H})$. Thus, we have the inclusion $\mathscr{M} \cap (\mathscr{F} \vee \mathscr{H}) \subseteq (\mathscr{M} \cap \mathscr{F}) \vee (\mathscr{M} \cap \mathscr{H})$. Consequently, $\mathscr{M} \cap (\mathscr{F} \vee \mathscr{H}) = (\mathscr{M} \cap \mathscr{F}) \vee (\mathscr{M} \cap \mathscr{H})$.

The theorem is proved.

Lemma 4.3. *Suppose that $\{\mathscr{F}_i = LFS_\sigma(f_i),\ i \in I\}$ is a set of σ-local Fitting sets of a group G and $\mathscr{F} = \bigcap_{i\in I} \mathscr{F}_i$ for all $i \in I$. Then $\mathscr{F} = LFS_\sigma\big(\bigcap_{i\in I} f_i\big)$ is a σ-local Fitting set of G.*

PROOF. Put $\Pi_i = \sigma(\mathscr{F}_i)$ for all $i \in I$. Then

$$\bigcap_{i\in I} \Pi_i = \bigcap_{i\in I} \sigma(\mathscr{F}_i) = \sigma\Big(\bigcap_{i\in I} \mathscr{F}_i\Big) = \sigma(\mathscr{F}).$$

Take $S \in \mathscr{F}$. Then

$$S \in \mathscr{F}_i = \mathrm{Tr}_{\mathfrak{E}_{\Pi_i}}(G) \cap \Big(\bigcap\nolimits_{\sigma_j \in \Pi_i} f_i(\sigma_j) \odot \mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma'_j}\Big)$$

for all $j \in J$ and $i \in I$. Hence, $S \in \mathscr{F}_i = \mathrm{Tr}_{\mathfrak{E}_{\Pi_i}}(G)$ and $S \in f_i(\sigma_j) \odot \mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma'_j}$ for all $j \in J$ and $i \in I$. From $S \in \mathscr{F}_i = \mathrm{Tr}_{\mathfrak{E}_{\Pi_i}}(G)$ it follows that $S \in \mathscr{F}_i = \bigcap_{i\in I} \mathrm{Tr}_{\mathfrak{F}_i}(G)$. Then

$$S \in \mathrm{Tr}_{\bigcap_{i\in I} \mathfrak{E}_{\Pi_i}}(G) = \mathrm{Tr}_{\mathfrak{E}_\Pi}(G).$$

Since $S \in f_i(\sigma_j) \odot \mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma'_j}$, we have $S/S_{f_i(\sigma_j)} \in \mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma'_j}$. Since $\mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma'_j}$ is a formation, $S/\bigcap_{i\in I} S_{f_i(\sigma_j)} \in \mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma'_j}$. Then $S/S_{\bigcap_{i\in I} f_i(\sigma_j)} \in \mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma'_j}$. Consequently, $S \in \bigcap_{i\in I} f_i(\sigma_j) \odot \mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma'_j}$. Thus,

$$S \in \mathrm{Tr}_{\mathfrak{E}_\Pi}(G) \cap \Big(\bigcap_{j\in J}\bigcap_{i\in I} f_i(\sigma_j) \odot \mathfrak{E}_{\sigma_j}\mathfrak{E}_{\sigma'_j}\Big).$$

Hence, $S \in LFS_\sigma\big(\bigcap_{i\in I} f_i\big)$ and $\mathscr{F} \subseteq LFS_\sigma\big(\bigcap_{i\in I} f_i\big)$ is a σ-local Fitting set of G. The reverse inclusion is obvious.

The lemma is proved.

Recall that if $\mathscr{X}$ is a collection of subgroups in G then $\mathscr{X}(\sigma_i)$ is $\{S \leq G : S^{\mathfrak{E}_{\sigma_i}\mathfrak{E}_{\sigma_i'}} \in \mathscr{X}\}$.

Lemma 4.4. *Suppose that $\mathscr{F}_1 = LFS_\sigma(f_1)$ and $\mathscr{F}_2 = LFS_\sigma(f_2)$ are σ-local Fitting sets of a group G, where f_1 and f_2 are inner H_σ-functions and $\mathscr{F} = \mathscr{F}_1 \vee_\sigma \mathscr{F}_2$. Then $\mathscr{F} = LFS_\sigma(f_1 \vee_\sigma f_2)$ is a σ-local Fitting set in G.*

PROOF. Let h_j be a minimal H_σ-function of the Fitting set $\mathscr{F}_j$ of G and let p_j be a minimal complete inner H_σ-function of the Fitting set $\mathscr{F}_j$ of G for $j \in \{1, 2\}$. For any $i \in I$, in view of Lemma 3.8, we have $h_j(\sigma_i) \subseteq f_j(\sigma_i) \subseteq p_j(\sigma_i)$. Moreover,

$$\begin{aligned} h(\sigma_i) &= \mathrm{Fitset}((\mathscr{F}_1 \cup \mathscr{F}_2)(\sigma_i)) = \mathrm{Fitset}(\mathscr{F}_1(\sigma_i) \cup \mathscr{F}_2(\sigma_i)) = \mathrm{Fitset}(h_1(\sigma_i) \cup h_2(\sigma_i)) \\ &\subseteq f(\sigma_i) \subseteq \mathrm{Fitset}(h_1(\sigma_i) \cup h_2(\sigma_i)) \odot \mathfrak{E}_{\sigma_i} \subseteq h(\sigma_i) \odot \mathfrak{E}_{\sigma_i} = p(\sigma_i). \end{aligned}$$

Hence, $h(\sigma_i) \subseteq f(\sigma_i) \subseteq p(\sigma_i)$ for all $i \in I$ and $\mathscr{F} = LFS_\sigma(f)$.

The lemma is proved.

If $\mathscr{X}$ is a family of subgroups of G then the closure operation S_n on $\mathscr{X}$ is defined as follows: $S_n(\mathscr{X}) = \{S \leq G : S \trianglelefteq\trianglelefteq H$ for some subgroup $H \in \mathscr{X}\}$ (see [17, p. 171]).

The following theorem defines some family of σ-local Fitting sets in a group G that satisfy the modular equality:

Theorem 4.5. *Let $\mathscr{F} = LFS_\sigma(f)$, $\mathscr{H} = LFS_\sigma(h)$, and $\mathscr{M} = LFS_\sigma(m)$ be σ-local Fitting sets of a group G and let f, h, and m be minimal H_σ-functions of the Fitting sets $\mathscr{F}$, $\mathscr{H}$, and $\mathscr{M}$ of G, where $f \leq m$. If the H_σ-functions f and h are such that $f(\sigma_i) \vee h(\sigma_i) = S_n\{S \leq G : S = S_{f(\sigma_i)}S_{h(\sigma_i)}\}$ for all σ_i, with $f(\sigma_i)$ and $h(\sigma_i)$ nonempty Fitting sets in G, then $(\mathscr{F} \vee_\sigma \mathscr{H}) \cap \mathscr{M} = \mathscr{F} \vee_\sigma (\mathscr{H} \cap \mathscr{M})$.*

PROOF. Let us first show that the minimal H_σ-functions f, h, and m satisfy $(f(\sigma_i) \vee_\sigma h(\sigma_i)) \cap m(\sigma_i) = f(\sigma_i) \vee_\sigma (h(\sigma_i) \cap m(\sigma_i))$ for all σ_i.

Since $f \leq m$ and $f \leq f \vee_\sigma h$, we have $f \leq (f \vee_\sigma h) \cap m$. Moreover, $h \cap m \leq f \vee_\sigma h$ and $h \cap m \leq m$. Consequently, $h \cap m \leq (f \vee_\sigma h) \cap m$ and

$$f \vee_\sigma (h \cap m) \leq (f \vee_\sigma h) \cap m. \tag{4.4}$$

Check the validity of the reverse embedding of H_σ-functions:

$$(f \vee_\sigma h) \cap m \leq f \vee_\sigma (h \cap f). \tag{4.5}$$

If $m(\sigma_i) = \varnothing$ then (4.5) is trivial. Separately consider the cases when $f(\sigma_i)$ or $h(\sigma_i)$ is empty. Assume that $f(\sigma_i) = \varnothing$. In this case, obviously, $(f(\sigma_i) \vee_\sigma h(\sigma_i)) \cap m(\sigma_i) = h(\sigma_i) \cap m(\sigma_i)$ and the H_σ-functions f, h, and m satisfy $(f(\sigma_i) \vee_\sigma h(\sigma_i)) \cap m(\sigma_i) = f(\sigma_i) \vee_\sigma (h(\sigma_i) \cap m(\sigma_i))$.

Suppose that $h(\sigma_i) = \varnothing$. Then, in view of the embedding $h \leq m$, we have

$$(f(\sigma_i) \vee_\sigma h(\sigma_i)) \cap m(\sigma_i) = f(\sigma_i) \vee_\sigma (h(\sigma_i) \cap m(\sigma_i)) = f(\sigma_i).$$

Thus, it remains to consider the case that neither of the Fitting sets $f(\sigma_i)$, $h(\sigma_i)$, or $m(\sigma_i)$ is empty. Let K be a subgroup in $(f(\sigma_i) \vee_\sigma h(\sigma_i)) \cap m(\sigma_i)$. Since $K \in f(\sigma_i) \vee_\sigma h(\sigma_i)$; by hypothesis, there exist a subgroup $S = S_{m(\sigma_i)}S_{h(\sigma_i)}$ and an isomorphism ψ such that $K \cong \psi(K) \trianglelefteq\trianglelefteq S$. Identifying K with its isomorphic image, we infer that

$$K \trianglelefteq\trianglelefteq S_{m(\sigma_i)} = S \cap S_{m(\sigma_i)} = S_{f(\sigma_i)}S_{h(\sigma_i)} \cap S_{m(\sigma_i)} = S_{f(\sigma_i)}(S_{h(\sigma_i)} \cap S_{m(\sigma_i)}) = S_{f(\sigma_i)}S_{h(\sigma_i)m(\sigma_i)}.$$

Therefore, $K \in f(\sigma_i) \vee_\sigma (h(\sigma_i) \cap m(\sigma_i))$ and the embedding (4.5) of H_σ-functions is proved. From (4.4) and (4.5) we obtain

$$(f \vee_\sigma h) \cap m = f \vee_\sigma (h \cap m). \tag{4.6}$$

Let us now prove the inequality

$$(\mathscr{F} \vee_\sigma \mathscr{H}) \cap \mathscr{M} = \mathscr{F} \vee_\sigma (\mathscr{H} \cap \mathscr{M}).$$

Since the H_σ-functions f and m of the Fitting sets $\mathscr{F}$ and $\mathscr{M}$ are inner; by Corollary 3.7, the embedding $f \leq m$ implies that $\mathscr{F} \subseteq \mathscr{M}$.

By Lemma 4.4, $f \vee_\sigma h$ is an H_σ-function of $\mathscr{F} \vee_\sigma \mathscr{H}$. Consequently, by Lemma 4.3, $(f \vee_\sigma h) \cap m$ is an H_σ-function of $(\mathscr{F} \vee_\sigma \mathscr{H}) \cap \mathscr{M}$.

Likewise, by Lemma 4.3, $h \cap m$ is an H_σ-function of $\mathscr{H} \cap \mathscr{M}$; and then, by Lemma 4.4, the H_σ-function $f \vee_\sigma (h \cap f)$ defines the Fitting set $\mathscr{F} \vee_\sigma (\mathscr{H} \cap \mathscr{M})$. Therefore, by (4.6),

$$(\mathscr{F} \vee_\sigma \mathscr{H}) \cap \mathscr{M} = \mathscr{F} \vee_\sigma (\mathscr{H} \cap \mathscr{M}).$$

The theorem is proved.

REMARK 4.6. Following Skiba's concept of multiple localization (see [16]), it is easy to obtain some generalization of the result on a test for the modularity of families of σ-local Fitting sets of a group G. Call each of the Fitting sets of G under consideration 0-*multiply* σ-*local*, and if $n \geq 1$ then call a Fitting set $\mathscr{F}$ of G n-*multiply* σ-*local* if either $\mathscr{F} = \{1\}$ or $\mathscr{F}$ has an H_σ-function f such that its every nonempty value is $(n-1)$-multiply σ-local.

Given two Fitting sets $\mathscr{F}$ and $\mathscr{H}$ of a group G, we put $\mathscr{F} \vee_\sigma^n \mathscr{H} = l_\sigma^n \operatorname{Fitset}(\mathscr{F} \cup \mathscr{H})$, where $l_\sigma^n \operatorname{Fitset}(\mathscr{F} \cup \mathscr{H})$ stands for the n-multiply σ-local Fitting set of G generated by $\mathscr{F} \cup \mathscr{H}$. If f and h are H_σ-functions, then $f \vee_\sigma^n h$ is an n-multiply local H_σ-function such that $(f \vee_\sigma^n h)(\sigma_i) = f(\sigma_i) \vee_\sigma^n h(\sigma_i)$ for all i and $f \cap h$ is an n-multiply local H_σ-function such that $(f \cap h)(\sigma_i) = f(\sigma_i) \cap h(\sigma_i)$ for all σ_i.

It is easy to see by following the proof of Theorem 4.5 and using induction on n that the theorem holds for n-multiply σ-local Fitting sets of a group G.

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