Данная работа является продолжением работы по созданию полномасштабного электронного конспекта лекций, начатой выпускницами ФМиИТ Черных В.В., Гаджиевой Ф.С. и Ивановой Е.А. Полученные результаты, вместе с изучением автоподобий и автоизометрий различных четырехмерных алгебр Ли (см. работы [2] – [5]) были представлены на Республиканские конкурсы научных работ студентов 2019–2021 годов, и им были присвоены две первых и одна вторая категории.

Заключение. В данной работе мы рассказали про продолжение создания электронного конспекта лекций по теории групп и алгебр Ли, который играет важную роль в организации индивидуального изучения теории студентами выпускных курсов и студентов второй ступени высшего образования при работе над курсовыми и конкурсными работами, а также дипломными проектами. Работа по созданию электронного конспекта будет продолжена в текущем учебном году.

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MAXIMAL GROUP OF ISOMETRIES OF THE LORENTZIAN LIE GROUP $A(1)\times A(1)$

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Ключевые слова. Группа Ли, левоинвариантная лоренцева метрика, однопараметрическая группа изометрий, самоподобное многообразие.

Keywords. Lie group, left-invariant Lorentzian metric, one-parameter isometry group, self-similar manifold.

Any Lie group G endowed with a left-invariant metric is a homogeneous manifold on which the Lie group itself acts simply and transitively. This means that the stationary subgroup G_e is trivial. Therefore, the question is of interest: does there exist a left-invariant metric on the Lie group such that the resulting homogeneous manifold has a group of motions whose dimension is greater than that of the Lie group itself, and this is equivalent to the fact that G_e contains a one-parameter subgroup.

The purpose of this paper is to indicate the metric tensor on the Lie group $G_{\text{IV}}=A^{+}(1)\times A^{+}(1)$, under which it admits a one-parameter group of motions that leave the identity element of the Lie group fixed, and write out the action of this one-parameter group in the coordinates associated with the matrix representation of this Lie group. For all other left-invariant Lorentzian metrics, the connected component of the group of motions of the resulting manifold is isomorphic to the Lie group G_{IV} itself.

Material and methods. We consider the 4-dimensional Lie group $G_{\text{IV}}=A^{+}(1)\times A^{+}(1)$ and its Lie algebra. We use methods of differential geometry.

Results and its discussion. Let the Euclidean or Lorentz scalar product be introduced in the Lie algebra $\mathcal{G}$. A linear transformation $F:\mathcal{G}\to\mathcal{G}$ is called an autoisometry if it is both an isometry with respect to the scalar product and an automorphism of the Lie algebra. In order to construct a one-parameter group of motions of a Lie group that leaves the identity element fixed, one must first find the one-parameter autoisometry group of its Lie algebra. Lie algebra $\mathcal{G}_{IV} = \mathcal{A}(1) \oplus \mathcal{A}(1)$ of the Lie group G_{IV} belongs to type IV according to the Bianchi classification. As was proved in [1], this Lie algebra does not admit autosimilarity for any way of specifying a Lorentzian scalar product on it, and there is a unique way of specifying a Lorentzian scalar product in this Lie algebra in which it admits a one-parameter autoisometry group. This proves that there is a unique class of left-invariant Lorentzian metrics such that the total isometry group of the resulting manifold is five-dimensional.

Matrix representation. In an appropriate basis (E_1, E_2, E_3, E_4) the commutation relations of the Lie algebra $\mathcal{G}_{IV}$ are given by two equalities: $[E_1, E_2] = E_2$, $[E_3, E_4] = E_4$, and the remaining brackets are equal to the zero vector. We will call such a basis canonical. The linear span of the vectors E_2 and E_4 is the derived Lie algebra $[\mathcal{G}_{IV}, \mathcal{G}_{IV}]$. It is a two-dimensional commutative ideal, which we denote by $\mathcal{H}$. The linear spans of the vectors E_1, E_2 and E_3, E_4 will be denoted by $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. These subspaces are two-dimensional non-commutative ideals.

The Lie algebra $\mathcal{G}_{IV}$ can be represented as consisting of matrices of the form

$$\begin{pmatrix} u_1 & u_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_3 & u_4 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the canonical basis is formed by the matrices

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ E_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(it's not the only one). The connected simply connected Lie group G_{IV} corresponding to it can be represented as a group of matrices of the form

$$\begin{pmatrix} x_1 & x_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x_3 & x_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ x_1 > 0, \ x_3 > 0,$$

with the operation of matrix multiplication.

Let's introduce coordinates on $\mathcal{G}_{IV}$ and G_{IV} by comparing the above matrices with the coordinates (u_1, u_2, u_3, u_4) and (x_1, x_2, x_3, x_4) respectively. The unit element of the group corresponds to the coordinates $(1,0,1,0)$. Then the group operation is given by the formulas

$$(x_1, x_2, x_3, x_4) \cdot (y_1, y_2, y_3, y_4) = (x_1 y_1, x_1 y_2 + x_2, \ x_3 y_3, x_3 y_4 + x_4),$$

and the inverse element is found like this: $(x_1, x_2, x_3, x_4)^{-1} = (x_1^{-1}, -x_2 x_1^{-1}, x_3^{-1}, -x_4 x_3^{-1})$.

By direct calculation, we find that the exponential mapping $\exp: \mathcal{G}_{IV} \to G_{IV}$ is given by the formulas

$$x_1 = e^{u_1}, \ x_2 = \frac{u_2}{u_1}(e^{u_1} - 1), \ x_3 = e^{u_3}, \ x_2 = \frac{u_4}{u_3}(e^{u_3} - 1). \qquad (1)$$

To eliminate indeterminancy, it is necessary to clarify that $\exp(0, u_2, 0, u_4) = (1, u_2, 1, u_4)$. From here we obtain the inverse mapping formulas $\exp^{-1}: G_{IV} \to \mathcal{G}_{IV}$:

$$u_1 = \ln x_1, \ u_2 = \frac{x_2}{x_1 - 1} \ln x_1, \ u_3 = \ln x_3, \ u_4 = \frac{x_4}{x_3 - 1} \ln x_3 \qquad (2)$$

Main result. Let the Lorentz inner product be introduced on the Lie algebra $\mathcal{G}_{IV}$. The main result proved in [1] is the following theorem (we have changed the order of the basis vectors).

***Theorem* 1.** 1) *There is only one way to specify the Lorentz inner product in the Lie algebra* $\mathcal{G}_{IV} = \mathcal{A}(1) \oplus \mathcal{A}(1)$, *in which case it admits a non-trivial one-parameter autoisometry group* F_t. *The action of this group in the canonical basis is given by the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\nu t} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-\nu t} \end{pmatrix}, \nu > 0, t \in \boldsymbol{R}, \qquad (3)$$

in this case, the Gram matrix has the form

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \qquad (4)$$

2) *The Lie algebra under consideration does not allow autosimilarity for any way of specifying the Lorentz scalar product in it.*

The location of the basis vectors relative to the cone of isotropic vectors, in the case when the Gram matrix has the form (4), is shown in Figure 1. On the ideal $\mathcal{H}$, the Lorentz scalar product is induced, the vectors E_2 and E_4 are isotropic, and the ideals L_1 are L_2 are the orthogonal complements of the vectors E_2 and E_4 respectively.

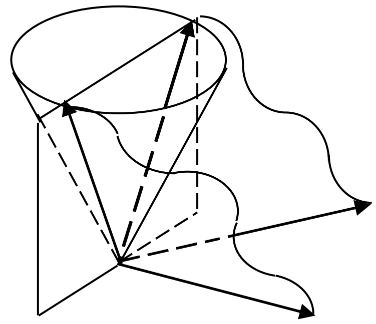

Figure 1

Theorem 1 allows us to prove the following main result of this paper.

***Theorem* 2.** 1) *There is only one, up to isometry, left-invariant Lorentzian metric g on the Lie group $G_{\mathrm{IV}} = A^{+}(1)\times A^{+}(1)$, for which the homogeneous manifold (G_{IV}, g) admits a one-parameter isometry group $f_t : G_{\mathrm{IV}} \to G_{\mathrm{IV}}$, leaving the identity element of the group Lie invariant. In the coordinates described above, the metric tensor is given by the matrix*

$$(g_{ij}(x)) = \begin{pmatrix} x_1^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1^{-1}x_3^{-1} \\ 0 & 0 & x_3^{-2} & 0 \\ 0 & x_1^{-1}x_3^{-1} & 0 & x_3^{-2} \end{pmatrix}, \qquad (5)$$

and the action of the group f_t is given by the formulas

$$x_1' = x_1,\ x_2' = e^{\nu t} x_2,\ x_3' = x_3,\ x_4' = e^{-\nu t} x_4,\ \nu > 0,\ t \in \boldsymbol{R}. \qquad (6)$$

2) *In the coordinates described above, the action of the complete isometry group of the constructed homogeneous manifold is described by the formulas*

$$x_1' = g_1x_1,\ x_2' = \pm e^{t}g_1x_2 + g_2,\ x_3' = g_3x_3,\ x_4' = \pm e^{-t}g_3x_4 + g_4,\ g_1 > 0,\ g_3 > 0 \qquad (7)$$

moreover, the signs "+" or "–" can be chosen independently in two cases. The connected component of the full group of isometries containing the identical transformation is given by formulas (7) with the choice of the "+" sign in both cases.

The technology for constructing a one-parameter isometry group is described in [2].

Conclusion. In this paper, we proved that there exists a unique, up to isometry, left-invariant Lorentzian metric on the four-dimensional Lie group $A^{+}(1)\times A^{+}(1)$, for which the resulting manifold admits a one-parameter isometry group that leaves the identity of the Lie group fixed. Only in this unique case is the full isometry group of the Lorentzian manifold of the group under consideration five-dimensional.

In [3], all autosimilarity of the Lie algebra $\mathcal{A}(1)\oplus\mathcal{R}^2$. This will allow us to construct self-similar manifolds of the Lie group $A^{+}(1)\times(\boldsymbol{R}^{+})^2$, equipped with a left-invariant Lorentzian metric, and this is the immediate goal of the further research.

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