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On σ-local Fitting classes

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A B S T R A C T

Let σ be a partition of the set of all primes $\mathbb{P}$. If G is a finite group and $\mathfrak{F}$ is a Fitting class of finite groups, the symbol $\sigma(G)$ denotes the set $\{\sigma_i|\sigma_i \cap \pi(|G|) \neq \emptyset\}$ and $\sigma(\mathfrak{F}) = \cup_{\sigma\in\mathfrak{F}}\sigma(G)$. We call any function f of the form $f : \sigma \longrightarrow \{\text{Fitting classes}\}$ a *Hartley σ-function* (or simply *H_σ-function*), and we put $LR_\sigma(f) = (G|G = 1$ or $G \neq 1$ and $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in f(\sigma_i)$ for all $\sigma_i \in \sigma(G))$. If there is an H_σ-function f such that $\mathfrak{F} = LR_\sigma(f)$, then we say that $\mathfrak{F}$ is σ-local and f is a σ-local definition of $\mathfrak{F}$. In this paper, we describe some properties of σ-local Fitting classes and prove that: 1) every σ-local Fitting class can be defined by a unique H_σ-function F such that $F(\sigma_i) = F(\sigma_i)\mathfrak{G}_{\sigma_i} \subseteq \mathfrak{F}$ and $F(\sigma_i)$ is a Lockett class for all $\sigma_i \in \sigma(\mathfrak{F})$; 2) the product of two σ-local

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Fitting classes is also a σ-local Fitting class. Moreover, we also discuss the n-multiply σ-local Fitting classes.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G. Following Shemetkov [1], σ is some partition of $\mathbb{P}$, that is, $\sigma = \{\sigma_i | i \in I\}$, where $\mathbb{P} = \bigcup_{i \in I} \sigma_i$, $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$ [2]; $\sigma(G) = \sigma(|G|)$. For any collection of groups $\mathfrak{X}$, the symbol $(\mathfrak{X})$ denotes the class of all groups G such that $G \cong H$ for some $H \in \mathfrak{X}$. Following [2–6], a group G is called: *σ-primary* if G is a σ_i-group for some $i \in I$; *σ-soluble* if every chief factor of G is σ-primary; *σ-nilpotent* if $G = G_1 \times \cdots \times G_n$ for some σ-primary groups $G_1, \cdots, G_n$.

Note that in the classical case when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \cdots\}$ (we use the notation in [6,7]), G is σ-soluble (respectively, σ-nilpotent) if and only if it is soluble (respectively, nilpotent).

Recall that a class $\mathfrak{F}$ of groups is said: (i) a *formation* if it is closed under taking homomorphic images and subdirect products; (ii) a *Fitting class* if it is closed under taking normal subgroups and products of normal $\mathfrak{F}$-subgroups.

Clearly, for a nonempty formation $\mathfrak{F}$, every group G has a smallest normal subgroup $G^{\mathfrak{F}}$ such that $G/G^{\mathfrak{F}} \in \mathfrak{F}$; for a nonempty Fitting class $\mathfrak{F}$, every group G has a largest normal $\mathfrak{F}$-subgroup $G_{\mathfrak{F}}$. The subgroups $G^{\mathfrak{F}}$ and $G_{\mathfrak{F}}$ are called the $\mathfrak{F}$-residual and the $\mathfrak{F}$-radical of G, respectively.

Recall that the product $\mathfrak{F}\mathfrak{H}$ of two classes of groups $\mathfrak{F}$ and $\mathfrak{H}$ is the class $(G|G$ has a normal subgroup N with $G/N \in \mathfrak{H})$; the product $\mathfrak{F} \circ \mathfrak{H}$ of two formations $\mathfrak{F}$ and $\mathfrak{H}$ is the class $(G|G^{\mathfrak{H}} \in \mathfrak{F})$ and the product $\mathfrak{F} \diamond \mathfrak{H}$ of two Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ is the class $(G|G/G_{\mathfrak{F}} \in \mathfrak{H})$. It is well known that if $\mathfrak{F}$ is closed under taking normal subgroups and $\mathfrak{H}$ is closed under taking homomorphic images, then $\mathfrak{F} \circ \mathfrak{H} = \mathfrak{F}\mathfrak{H} = \mathfrak{F} \diamond \mathfrak{H}$ (see [8, p. 388 and 566]). Also it is known that the product of two formations (resp. the product of two Fitting classes) is a formation (resp. a Fitting class) and the multiplication of formations (resp. Fitting classes) satisfies the associative law (see [8, IV, Theorem 1.8(a)(c) and IX, Theorem (1.12)(a)(c)]).

In [9], the author introduced the concept of σ-local formations: a function f of the form $f : \sigma \longrightarrow \{\text{formations of groups}\}$ is called a *formation σ-function*. Let $LF_\sigma(f) = (G|G = 1$ or $G \neq 1$ and $G/O_{\sigma_i', \sigma_i}(G) \in f(\sigma_i)$ for all $\sigma_i \in \sigma(G))$. A formation is called *σ-local* if $\mathfrak{F} = LF_\sigma(f)$ for some formation σ-function f. Note that if $\sigma = \sigma^1$, then a formation σ-function and a σ-local formation are just a local function and a local formation in the usual sense, respectively (see [8, IV, Definition (3.1)] or [10, p. 2]).

If $\Pi \subseteq \sigma$, then following [2], we write $\mathfrak{G}_\Pi$ to denote the class of all Π-groups. In particular, $\mathfrak{G}_{\sigma_i}$ is the class of all σ_i-groups and $\mathfrak{G}_{\sigma_i'}$ is the class of all σ_i'–groups. Clearly, the classes $\mathfrak{G}_{\sigma_i}$ and $\mathfrak{G}_{\sigma_i'}$ are both formations and Fitting classes.

A formation σ-function f of a formation $\mathfrak{F}$ is said: *integrated* if $f(\sigma_i) \subseteq \mathfrak{F}$ for all $\sigma_i \in \sigma(\mathfrak{F})$; *full* if $\mathfrak{G}_{\sigma_i} f(\sigma_i) = f(\sigma_i)$ for all $\sigma_i \in \sigma(f)$; *full integrated* if f is full and integrated.

In [11,12], the authors proved the following basic results for σ-local formations:

Theorem A. *[11, Proposition 2.5]. Every σ-local formation $\mathfrak{F}$ can be defined by a unique full integrated formation σ-function F such that $F(\sigma_i) = \mathfrak{G}_{\sigma_i} F(\sigma_i) \subseteq \mathfrak{F}$ for all $\sigma_i \in \sigma(\mathfrak{F})$.*

Theorem B. *[12, Theorem 1.14]. The product $\mathfrak{F} \circ \mathfrak{H}$ of two σ-local formations $\mathfrak{F}$ and $\mathfrak{H}$ is a σ-local formation.*

Note that in the case that $\sigma = \sigma^1$, Theorems A and B are the well-known basic results of Carter-Hawkes-Schmidt [13,14] and Gaschütz-Shemetkov [15,16] (see, also [8, IV, Proposition 3.8 and IV, Theorem 3.13]).

It is well known that Fitting classes may be regarded as dual to formations (see Hartley [17]). Therefore, in connection with the above theory of σ-local formations, it is natural to ask:

Question 1. Can we establish the theory of σ-local Fitting classes?

The main purpose of this article is to solve the question.

We call any function f of the form

$$f : \sigma \longrightarrow \{\textit{Fitting classes}\}$$

a *Hartley σ-function* (or simply *H_σ-function*), and we put

$$LR_\sigma(f) = (G | G = 1 \text{ or } G \neq 1 \text{ and } G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in f(\sigma_i) \text{ for all } \sigma_i \in \sigma(G)).$$

Definition 1.1. Let $\mathfrak{F}$ be a Fitting class. If there is an H_σ-function f such that $\mathfrak{F} = LR_\sigma(f)$, then we say that $\mathfrak{F}$ is σ-local and f is a σ-local definition of $\mathfrak{F}$.

In the particular case when $\sigma = \sigma^1$, we use symbol $LR(f)$ instead of $LR_\sigma(f)$, which was used in [18,19] (see also [10, p. 310]).

If f is an H_σ-function, then the symbol $Supp(f)$ denotes the support of f, that is, the set of all σ_i such that $f(\sigma_i) \neq \emptyset$. Let $\sigma(\mathfrak{F}) = \bigcup\{\sigma(G) | G \in \mathfrak{F}\}$.

Before continuing, consider some examples.

Examples 1.2. (i) For the Fitting class of all identity groups (1) we have $(1) = LR_\sigma(f)$, where $f(\sigma_i) = \emptyset$ for all i.

(ii) Let $\mathfrak{F} = \mathfrak{G}_{\sigma_i}$ be the class of all σ_i-groups. Then $\mathfrak{F} = LR_\sigma(f)$, where $f(\sigma_i) = \mathfrak{F}$ and $f(\sigma_j) = \emptyset$ for all $j \neq i$.

(iii) Let $\mathfrak{X}$ be a nonempty Fitting classes and $\mathfrak{N}_\sigma$ be the class of all σ-nilpotent groups. Then $\mathfrak{X}\mathfrak{N}_\sigma$ is a σ-local Fitting class. In fact, let f be the H_σ-function such that $f(\sigma_i) = \mathfrak{X}$ for all i. Clearly, every σ_i-group belongs to $\mathfrak{X}\mathfrak{N}_\sigma$, so $\Pi = Supp(f) = \bigcup \sigma_i = \mathbb{P}$. Then by Lemma 3.1 below, $LR_\sigma(f) = \bigcap_{\sigma_i} \mathfrak{X}\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'} = \mathfrak{X}(\bigcap_i \mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}) = \mathfrak{F}\mathfrak{N}_\sigma$. Hence $\mathfrak{X}\mathfrak{N}_\sigma$ is σ-local.

(iv) We use $\mathfrak{N}_\sigma^k$ to denote the product $\mathfrak{N}_\sigma \cdots \mathfrak{N}_\sigma$ of k copies of $\mathfrak{N}_\sigma$ ($k \in \mathbb{N}$), and $\mathfrak{N}_\sigma^0$ is the class of groups of order 1. Clearly, $\mathfrak{N}_\sigma^k$ is a σ-local Fitting class with the H_σ-function f such that $f(\sigma_i) = \mathfrak{N}_{\sigma_i}^{k-1}$ for all σ_i by (iii). In particular, $\mathfrak{N}_\sigma$ is the σ-local Fitting class with an H_σ-function f such that $f(\sigma_i) = (1)$ for all σ_i.

(v) Let $\mathfrak{X}$ be a nonempty Fitting class and $\mathfrak{S}_\sigma$ be the class of all σ-soluble groups. Let f be the H_σ-function such that $f(\sigma_i) = \mathfrak{X}\mathfrak{S}_\sigma$ for all σ_i. Then by Lemma 3.1 below, $LR_\sigma(f) = \bigcap_{\sigma_i} \mathfrak{X}\mathfrak{S}_\sigma\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'} = \mathfrak{X}\mathfrak{S}_\sigma(\bigcap_\sigma \mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}) = \mathfrak{X}\mathfrak{S}_\sigma\mathfrak{N}_\sigma = \mathfrak{X}\mathfrak{S}_\sigma$ and so $\mathfrak{X}\mathfrak{S}_\sigma$ is a σ-local Fitting class. In particular, the class $\mathfrak{S}_\sigma$ is σ-local.

It is well known that the operations * and $_*$ defined by Lockett [20] play important roles in investigations of the structure of classes of groups and canonical subgroups (see [8, Chapters IX-X] and [10, Chapter 5, Section 5.9]). In fact, every nonempty Fitting class can be connected with a Fitting class $\mathfrak{F}^*$, where $\mathfrak{F}^*$ is the smallest Fitting class containing $\mathfrak{F}$ such that $(G \times H)_{\mathfrak{F}^*} = G_{\mathfrak{F}^*} \times H_{\mathfrak{F}^*}$ for all groups G and H. Moreover, $\mathfrak{F}_* = \bigcap\{\mathfrak{X} | \mathfrak{X}$ is a Fitting class such that $\mathfrak{X}^* = \mathfrak{F}^*\}$. A Fitting class $\mathfrak{F}$ is called a *Lockett class* if $\mathfrak{F} = \mathfrak{F}^*$.

Definition 1.3. Let $\mathfrak{F} = LR_\sigma(f)$ for some H_σ-function f. Then we say that

(a) f is integrated if $f(\sigma_i) \subseteq \mathfrak{F}$ for all i;
(b) f is full if $f(\sigma_i)\mathfrak{G}_{\sigma_i} = f(\sigma_i)$ for all i;
(c) full integrated if f is full and integrated;
(d) Lockett H_σ-function if $f(\sigma_i)$ is a Lockett class for all i.

In connection with Theorems A and B, the following dual questions for σ-local Fitting classes naturally arises:

Question 2. Is it true that every σ-local Fitting $\mathfrak{F}$ can be defined by a unique full integrated H_σ-function F such that $F(\sigma_i) = F(\sigma_i)\mathfrak{G}_{\sigma_i}$ for all $\sigma_i \in \sigma(\mathfrak{F})$?

Question 3. Is it true that the product $\mathfrak{F} \diamond \mathfrak{H}$ of two σ-local Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ is a σ-local Fitting class?

The following Theorems 1.1 and 1.2 resolve Questions 2 and 3.

Theorem 1.1. *Every σ-local Fitting class can be defined by a unique full integrated H_σ-function F such that $F(\sigma_i) = F(\sigma_i)\mathfrak{G}_{\sigma_i} \subseteq \mathfrak{F}$ and $F(\sigma_i)$ is a Lockett class for all $\sigma_i \in \sigma(\mathfrak{F})$.*

Theorem 1.2. *The product $\mathfrak{M} = \mathfrak{F} \diamond \mathfrak{H}$ of two σ-local Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ is a σ-local Fitting class. Moreover, if $\Pi = \sigma(\mathfrak{H})$, $\mathfrak{F} = LR_\sigma(f)$ for integrated H_σ-function f and $\mathfrak{H} = LR_\sigma(h)$ for integrated H_σ-function h, then $\mathfrak{M}$ can be defined by an H_σ-function m such that*

$$m(\sigma_i) = \begin{cases} \mathfrak{F} \diamond h(\sigma_i) & \text{if } \sigma_i \in \Pi; \\ f(\sigma_i) & \text{if } \sigma_i \in \sigma \setminus \Pi. \end{cases}$$

In the case when $\sigma = \sigma^1$, we get from Theorems 1.1 and 1.2 the following well-known results.

Corollary 1.3. *(Vorob'ev [21]). Every local Fitting class $\mathfrak{F}$ can be defined by a unique full integrated H-function F such that $F(p) = F(p)\mathfrak{N}_p \subseteq \mathfrak{F}$ and $F(p)$ is a Lockett class for all $p \in \pi(\mathfrak{F})$.*

Corollary 1.4. *(Vorob'ev [22]). The product of two every local Fitting classes is also a local Fitting class.*

2. Preliminaries

Lemma 2.1. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be nonempty formations. If $\mathfrak{F} \subseteq \mathfrak{H}$, then $G^{\mathfrak{H}} \leq G^{\mathfrak{F}}$ for all groups G.*

Proof. It is clear. □

Lemma 2.2. *[8, IV, Theorem (1.8)(b)]. If $\mathfrak{F}$ and $\mathfrak{H}$ are nonempty formations, then $G^{\mathfrak{F}\mathfrak{H}} = (G^{\mathfrak{H}})^{\mathfrak{F}}$ for all groups G.*

Lemma 2.3. *[8, IX, Lemma (1.1)(a)]. Let $\mathfrak{F}$ be a nonempty Fitting class and G a group. If N is a subnormal subgroup of G, then $N_{\mathfrak{F}} = N \cap G_{\mathfrak{F}}$.*

Recall that a Fitting class $\mathfrak{F}$ is called a *Fischer class* [17] if $H \in \mathfrak{F}$ whenever $K \leq H \leq G \in \mathfrak{F}$, $K \trianglelefteq G$ and H/K is a p-group for some $p \in \mathbb{P}$.

Lemma 2.4. *[8, X, Proposition (1.25)]. If $\mathfrak{F}$ is a Fischer class, then $\mathfrak{F}$ is a Lockett class.*

Lemma 2.5. *Let $\mathfrak{F}$ be a nonempty Fitting class. Then*

(a) $(\mathfrak{F}_*)_* = \mathfrak{F}_* = (\mathfrak{F}^*)_* \subseteq \mathfrak{F} \subseteq \mathfrak{F}^* = (\mathfrak{F}^*)^* = (\mathfrak{F}_*)^*$. *[8, X, Theorems (1.8) and (1.15)]*

(b) *If $\mathfrak{H}$ is also nonempty Fitting class and $\mathfrak{F} \subseteq \mathfrak{H}$, then $\mathfrak{F}^* \subseteq \mathfrak{H}^*$. [8, X, Theorem (1.8)]*
(c) *Let $\mathfrak{F}$ be a nonempty Fitting class and $\mathfrak{H}$ be both a nonempty local formation and a Fitting class, then $(\mathfrak{F}\mathfrak{H})^* = \mathfrak{F}^*\mathfrak{H}$. [23, Lemma 3]*
(d) *If $\mathfrak{F}$ and $\mathfrak{H}$ are nonempty Fitting classes, then $(\mathfrak{F}\cap\mathfrak{H})^* = \mathfrak{F}^* \cap \mathfrak{H}^*$. [8, X, Proposition (1.13)]*

Let G and H be groups. Then $G \wr H$ denotes the regular wreath product of G with H. If $K \leq G$, we denote by $K^\natural$ the subgroup of the base group of $K \wr H$ which is isomorphic to the product of $|H|$ copies of K. In particular, $G^\natural$ denotes the base group of $G \wr H$.

Lemma 2.6. *[8, X, Proposition (2.1)(a)]. Let $\mathfrak{F}$ be a Lockett class. If $G \notin \mathfrak{F}$, then $(G \wr H)_{\mathfrak{F}} = (G_{\mathfrak{F}})^\natural$ for any group H.*

Lemma 2.7. *[8, A, Lemma (18.2)(d)]. Let $W = G \wr H$. If $K \trianglelefteq G$ and $K^\natural$ is the base group of $K \wr H$, then $K^\natural \trianglelefteq W$ and $W/K^\natural \cong (G/K) \wr H$.*

Lemma 2.8. *[1, Lemma 1.1] (see also [24, Chapter 2, Lemma 2.1.3]). If $\mathfrak{F}$ is a nonempty formation and $N \trianglelefteq G$, then $(G/K)^{\mathfrak{F}} = G^{\mathfrak{F}}K/K$.*

We use the following modification of [25, Lemma 1.1(2)].

Lemma 2.9. *Let N be a normal subgroup of G and $\pi \subseteq \mathbb{P}$. If G/N is a π'-group, then $G^{\mathfrak{G}_\pi\mathfrak{G}_{\pi'}} = N^{\mathfrak{G}_\pi\mathfrak{G}_{\pi'}}$.*

3. Properties of σ-local Fitting classes

In order to prove our theorems, we need discuss some properties of σ-local Fitting classes.

Lemma 3.1. *Let $\mathfrak{F} = LR_\sigma(f)$ and $\Pi = Supp(f)$. Then*

(a) $\Pi = \sigma(\mathfrak{F})$;
(b) *$G \in \mathfrak{F}$ if and only if $G \in f(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ for all $\sigma_i \in \sigma(G)$;*
(c) $\mathfrak{F} = \mathfrak{G}_\Pi \cap (\bigcap_{\sigma_i\in\Pi} f(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'})$;
(d) *if every group in $\mathfrak{F}$ is σ-soluble, then $\mathfrak{F} = \mathfrak{G}_\Pi \cap (\bigcap_{\sigma_i\in\Pi} f(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{S}_{\sigma_i'})$.*

Proof. (a) If $\sigma_i \in \Pi$, then $1 \in f(\sigma_i)$ and for every σ_i-group $G \neq 1$ we have $\sigma(G) = \{\sigma_i\}$. Hence $G \in \mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ and so $1 = G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in f(\sigma_i)$. Therefore $G \in LR_\sigma(f) = \mathfrak{F}$, and consequently $\Pi \subseteq \sigma(\mathfrak{F})$. On the other hand, if $\sigma_i \in \sigma(\mathfrak{F})$, then for some group $G \in \mathfrak{F}$ we have $\sigma_i \in \sigma(G)$ and $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in f(\sigma_i)$. It follows that $\sigma_i \in \Pi$. Thus $\Pi = \sigma(\mathfrak{F})$.

(b) If $G \in \mathfrak{F}$ and $\sigma_i \in \sigma(G)$, then $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in f(\sigma_i)$, and so $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \leq G_{f(\sigma_i)}$. Since $\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ is a formation,

$$(G/G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}})/(G_{f(\sigma_i)}/G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}}) \cong G/G_{f(\sigma_i)} \in \mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}.$$

It follows that $G \in f(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$. Conversely, if $G \in f(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ for any $\sigma_i \in \sigma(G)$, then $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in f(\sigma_i)$, and so $G \in \mathfrak{F}$.

(c) If $G \in \mathfrak{F}$, then $\sigma(G) \subseteq \sigma(\mathfrak{F}) = \Pi$ by (a). Hence $G \in \mathfrak{G}_\Pi$. Moreover, for every $\sigma_i \in \sigma(G)$ we have $G \in f(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ by (b). If $\sigma_i \in \Pi \setminus \sigma(G)$, then $G \in \mathfrak{G}_{\sigma_i'} \subseteq \mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'} \subseteq f(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$. Therefore $\mathfrak{F} \subseteq \mathfrak{G}_\Pi \cap (\bigcap_{\sigma_i\in\Pi} f(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'})$. The anti-inclusion relationship is obvious by (b).

(d) See the proof (c). □

Lemma 3.2. *Every σ-local Fitting class can be determined by a full integrated H_σ-function.*

Proof. Assume that $\mathfrak{F}$ is a σ-local Fitting class. Then there exists an H_σ-function f such that $\mathfrak{F} = LR_\sigma(f)$, and so $\mathfrak{F} = \mathfrak{G}_\Pi \cap (\bigcap_{\sigma_i\in\Pi} f(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'})$, where $\Pi = \sigma(\mathfrak{F}) = Supp(f)$, by Lemma 3.1. We define an H_σ-function φ as follows: $\varphi(\sigma_i) = f(\sigma_i) \cap \mathfrak{F}$ for every $\sigma_i \in \Pi$. It is clear that $\mathfrak{F} = LR_\sigma(\varphi)$.

Now we define the H_σ-function ψ such that $\psi(\sigma_i) = \varphi(\sigma_i)\mathfrak{G}_{\sigma_i}$ for every $\sigma_i \in \Pi$. It is clear that ψ is a full H_σ-function of $\mathfrak{F}$. We show that ψ is an integrated H_σ-function of $\mathfrak{F}$. Let $G \in \psi(\sigma_i)$. Then $G/G_{\varphi(\sigma_i)} \in \mathfrak{G}_{\sigma_i} \subseteq \mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ and so $G \in \varphi(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$. Assume that $\sigma_j \neq \sigma_i$, then $\mathfrak{G}_{\sigma_i} \subseteq \mathfrak{G}_{\sigma_j'}$ and therefore $G^{\mathfrak{G}_{\sigma_j'}} \leq G^{\mathfrak{G}_{\sigma_i}} \in \mathfrak{F}$ by Lemma 2.1. It follows that $(G^{\mathfrak{G}_{\sigma_j'}})^{\mathfrak{G}_{\sigma_j}\mathfrak{G}_{\sigma_j'}} \in \varphi(\sigma_j)$. Then using Lemma 2.2, $(G^{\mathfrak{G}_{\sigma_j'}})^{\mathfrak{G}_{\sigma_j}\mathfrak{G}_{\sigma_j'}} = G^{\mathfrak{G}_{\sigma_j}\mathfrak{G}_{\sigma_j'}\mathfrak{G}_{\sigma_j'}} = G^{\mathfrak{G}_{\sigma_j}\mathfrak{G}_{\sigma_j'}}$, so $G^{\mathfrak{G}_{\sigma_j}\mathfrak{G}_{\sigma_j'}} \in \varphi(\sigma_j)$. Therefore

$$G/G_{\varphi(\sigma_j)} \cong (G/G^{\mathfrak{G}_{\sigma_j}\mathfrak{G}_{\sigma_j'}})/(G_{\varphi(\sigma_j)}/G^{\mathfrak{G}_{\sigma_j}\mathfrak{G}_{\sigma_j'}}) \in \mathfrak{G}_{\sigma_j}\mathfrak{G}_{\sigma_j'}.$$

Then $G \in \varphi(\sigma_j)\mathfrak{G}_{\sigma_j}\mathfrak{G}_{\sigma_j'}$ for all $j \neq i$. This shows that $G \in \varphi(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ for all $i \in \Pi$. Hence $G \in \mathfrak{F}$ by Lemma 3.1(b). The lemma is proved. □

Lemma 3.3. (1) *Let $\mathfrak{F}$ be a nonempty Fitting class. Then the class $\mathfrak{F}\mathfrak{G}_\pi\mathfrak{G}_{\pi'}$ is a Fischer class, for every $\pi \subseteq \mathbb{P}$.*

(2) *Every σ-local Fitting class is a Fischer class.*

Proof. (1) Let $G \in \mathfrak{F}\mathfrak{G}_\pi\mathfrak{G}_{\pi'}$, $N \trianglelefteq G$, $N \leq H \leq G$ and H/N is a q-subgroup of G/N. Then $H = H_qN$, where H_q is some Sylow q-subgroup of H.

Suppose that $q \notin \pi$. Then $H/N \in \mathfrak{G}_{\pi'}$. Since $N \trianglelefteq G \in \mathfrak{F}\mathfrak{G}_\pi\mathfrak{G}_{\pi'}$, $N \in \mathfrak{F}\mathfrak{G}_\pi\mathfrak{G}_{\pi'}$ and so $H \in \mathfrak{F}\mathfrak{G}_\pi\mathfrak{G}_{\pi'}$. Now assume that $q \in \pi$. Since $|G : G_{\mathfrak{F}\mathfrak{G}_\pi}|$ is a π'-number, every q-subgroup of G is contained in $G_{\mathfrak{F}\mathfrak{G}_\pi}$. Hence $H_q \leq G_{\mathfrak{F}\mathfrak{G}_\pi}$. Then $[N, H_q] \leq [N, G_{\mathfrak{F}\mathfrak{G}_\pi}] \leq N \cap G_{\mathfrak{F}\mathfrak{G}_\pi} = N_{\mathfrak{F}\mathfrak{G}_\pi}$ by Lemma 2.3. Let $L = N_{\mathfrak{F}\mathfrak{G}_\pi}$. Then $LH_q \trianglelefteq NH_q$, $LH_q \in \mathfrak{F}\mathfrak{G}_\pi\mathfrak{G}_\pi = \mathfrak{F}\mathfrak{G}_\pi$ and $(NH_q)/(LH_q) \cong N/(L(N \cap H_q)) \in \mathfrak{G}_{\pi'}$. Thus $H = NH_q \in \mathfrak{F}\mathfrak{G}_\pi\mathfrak{G}_{\pi'}$. Note that if $\pi = \emptyset$ or $\pi' = \emptyset$, then $\mathfrak{F}\mathfrak{G}_\pi\mathfrak{G}_{\pi'} = \mathfrak{G}$ and the statement is obvious.

(2) Let $\mathfrak{F} = LR_\sigma(f)$ for some H_σ-function f. Then by Lemma 3.1(c), $\mathfrak{F} = \mathfrak{G}_\Pi \cap (\bigcap_{\sigma_i\in\Pi} f(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'})$, where $\Pi = Supp(f)$. Obviously, $\mathfrak{G}_\Pi$ is a Fischer class since $\mathfrak{G}_\Pi$ is

closed under taking subgroups. Moreover, $f(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ is a Fischer class for every $\sigma_i \in \Pi$ by the statement (1) of the lemma. It is clear that the intersection of Fischer classes is a Fischer class. Hence $\mathfrak{F}$ is a Fischer class.

The lemma is proved. □

Corollary 3.4. *Every σ-local Fitting class is a Lockett class.*

Proof. It follows from Lemma 3.3(2) and Lemma 2.4. □

Lemma 3.5. *Every σ-local Fitting class $\mathfrak{F}$ can be defined by a Lockett H_σ-function.*

Proof. Let φ be an H_σ-function of the Fitting class $\mathfrak{F}$. Then $\mathfrak{F} = \mathfrak{G}_\Pi \cap (\bigcap_{\sigma_i \in \Pi} \varphi(\sigma_i) \mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'})$, where $\Pi = Supp(\varphi)$, by Lemma 3.1(c). It is easy to see that for every $\sigma_i \in \Pi$ the Fitting class $\varphi(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ is a Fischer class by Lemma 3.3(1). Hence $\varphi(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ is a Lockett class by Lemma 2.4, that is, $(\varphi(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'})^* = \varphi(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$. Then by Lemma 2.5(c), $(\varphi(p))^*\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'} = \varphi(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ for all $\sigma_i \in \Pi$. We now construct an H_σ-function f as follows: $f(\sigma_i) = (\varphi(\sigma_i))^*$ for every $\sigma_i \in \Pi$. By Lemma 2.5(a), $f(\sigma_i)$ is a Lockett class and so f is a Lockett H_σ-function locally defining $\mathfrak{F}$. The lemma is proved. □

Lemma 3.6. *Every σ-local Fitting class $\mathfrak{F}$ can be defined by a full integrated Lockett H_σ-function.*

Proof. By Lemma 3.5, every σ-local Fitting class $\mathfrak{F}$ can be defined by a Lockett H_σ-function f. Let $\psi(\sigma_i) = (f(\sigma_i) \cap \mathfrak{F})\mathfrak{G}_{\sigma_i}$ for all $\sigma_i \in \Pi = Supp(f)$. Then from the proof of Lemma 3.1 we see that ψ is a full integrated H_σ-function of $\mathfrak{F}$. We now show that ψ is a Lockett H_σ-function. In fact, $\psi^*(\sigma_i) = (f(\sigma_i) \cap \mathfrak{F})^*\mathfrak{G}_{\sigma_i}$ by Lemma 2.5(c). Since $\mathfrak{F}$ is a Lockett class by Corollary 3.4, $(f(\sigma_i) \cap \mathfrak{F})^* = f(\sigma_i)^* \cap \mathfrak{F}^* = f(\sigma_i) \cap \mathfrak{F}$ by Lemma 2.5(d) and so $\psi^*(\sigma_i) = \psi(\sigma_i)$ for all $\sigma_i \in \Pi$. □

4. On minimal H_σ-functions of σ-local Fitting classes

For any two H_σ-functions f and φ of $\mathfrak{F}$, following Shemetkov [1], we define a partial as follows: $f \leqslant \varphi$ if $f(\sigma_i) \subseteq \varphi(\sigma_i)$ for all $\sigma_i \in \Pi$, where $\Pi = Supp(f)$.

If $\mathfrak{X}$ is a set of groups, then we use $Fit(\mathfrak{X})$ to denote the Fitting class generated by $\mathfrak{X}$, that is, the interection of all Fitting classes containing $\mathfrak{X}$.

Proposition 4.1. *Let $\mathfrak{F}$ be a σ-local Fitting class. Then*

(a) *$\mathfrak{F}$ can be defined by a unique minimal H_σ-function $\underline{f}$ such that $\underline{f}(\sigma_i) = Fit(G \mid G \cong X^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}}, X \in \mathfrak{F})$ for all $\sigma_i \in Supp(f)$.*

(b) *$\mathfrak{F}$ can be defined by a unique minimal full H_σ-function $\underline{\underline{f}}$ such that $\underline{\underline{f}}(\sigma_i) = Fit(G \mid G^{\mathfrak{G}_{\sigma_i}} \cong X^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}}, X \in \mathfrak{F})\mathfrak{G}_{\sigma_i}$ for all $\sigma_i \in Supp(\underline{\underline{f}})$.*

Proof. (a) Let $\mathfrak{F} = LR_\sigma(f)$ for some H_σ-function f. Then by definition of $\underline{f}$, we have $\underline{f} \leqslant f$. Hence $LR_\sigma(\underline{f}) \subseteq \mathfrak{F}$. Conversely, let $G \in \mathfrak{F}$. Then $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in \underline{f}(\sigma_i)$ by the definition of $\underline{f}$, and so $G \in \underline{f}(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ for all $\sigma_i \in Supp(f)$. By Lemma 3.1(b), $G \in LR_\sigma(\underline{f})$. Hence $\mathfrak{F} \subseteq LR_\sigma(\underline{f})$. Therefore (a) holds.

(b) By Lemma 3.2, $\mathfrak{F} = LR_\sigma(\varphi)$ for some integrated full H_σ-function φ of $\mathfrak{F}$. Let $\varphi_1(\sigma_i) = (G|G^{\mathfrak{G}_{\sigma_i}} \cong X^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}}, X \in \mathfrak{F})$. If $G \in \varphi_1(\sigma_i)$, then $G^{\mathfrak{G}_{\sigma_i}} \cong X^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}}$ for some group $X \in \mathfrak{F}$. Since $X \in \mathfrak{F}$, $X/X_{\varphi(p)} \in \mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ by Lemma 3.1(b) and the definition of the product of two Fitting classes, and so $G^{\mathfrak{G}_{\sigma_i}} \cong X^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in \varphi(\sigma_i)$. Hence $G \in \varphi(\sigma_i)\mathfrak{G}_{\sigma_i} = \varphi(\sigma_i)$. Thus $\varphi_1(\sigma_i) \subseteq \varphi(\sigma_i)$ for all $\sigma_i \in Supp(\varphi)$ and therefore $\underline{\underline{f}}(\sigma_i) = Fit(\varphi_1(\sigma_i)) \subseteq \varphi(\sigma_i)$, i.e. $\underline{\underline{f}} \leqslant \varphi$ for every full H_σ-function φ of $\mathfrak{F}$. Therefore $LR_\sigma(\underline{\underline{f}}) \subseteq \mathfrak{F}$.

Conversely, let $G \in \mathfrak{F}$. Since $(G^{\mathfrak{G}_{\sigma_i'}})^{\mathfrak{G}_{\sigma_i}} = G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}}$ by Lemma 2.2, $G^{\mathfrak{G}_{\sigma_i'}} \in \varphi_1(\sigma_i) \subseteq \underline{\underline{f}}(\sigma_i)$. Now, by Lemma 2.1, $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \leq G^{\mathfrak{G}_{\sigma_i}}$. Hence $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in \underline{\underline{f}}(\sigma_i)$ and so $G \in \underline{\underline{f}}(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i}$ for all $\sigma_i \in \sigma(\mathfrak{F})$. By Lemma 3.1(b), $G \in LR_\sigma(\underline{\underline{f}})$. Hence $\mathfrak{F} \subseteq LR_\sigma(\underline{\underline{f}})$. This shows that $\underline{\underline{f}}$ is an H_σ-function of $\mathfrak{F}$. Moreover, clearly, that $\underline{\underline{f}}$ is a unique minimal full H_σ-function of $LR_\sigma(\underline{\underline{f}})$.

The proposition is proved. □

5. Proof of Theorem 1.1

Proof of Theorem 1.1. By Lemma 3.6, every σ-local Fitting class $\mathfrak{F}$ can be defined by a full integrated Lockett H_σ-function F. We show that F is a unique full integrated Lockett H_σ-function of $\mathfrak{F}$. Suppose that $\mathfrak{F} = LR_\sigma(F) = LR_\sigma(F_1)$, where F and F_1 are two full integrated Lockett H_σ-functions of G. Suppose that $F \nleqslant F_1$, so for some i we have $F(\sigma_i) \nsubseteq F_1(\sigma_i)$. Let $G \in F(\sigma_i) \setminus F_1(\sigma_i)$ and $W = G \wr A$ be the regular wreath product of G and a σ_i-group A. If $G^\natural$ is the base group of W, then $G^\natural \in F(\sigma_i)$ since $F(\sigma_i)$ is a Fitting class. Hence $G^\natural \leq W_{F(\sigma_i)}$. Since $W/G^\natural \in \mathfrak{G}_{\sigma_i}$, we have that $W/W_{F(\sigma_i)} \cong (W/G^\natural)/(W_{F(\sigma_i)}/G^\natural) \in \mathfrak{G}_{\sigma_i}$, and so $W \in F(\sigma_i)\mathfrak{G}_{\sigma_i} = F(\sigma_i)$. Since $F(\sigma_i)$ is a Lockett class and $G \notin F_1(\sigma_i)$, $W_{F_1(p)} = (G_{F_1(p)})^\natural$ by Lemma 2.6. Now by Lemma 2.7, we have

$$W/W_{F_1(\sigma_i)} \cong (G/G_{F_1(\sigma_i)}) \wr A \notin \mathfrak{G}_{\sigma_i'}.$$

But as $W \in F(\sigma_i) \subseteq \mathfrak{F} = LR_\sigma(F_1)$ and F_1 is a full integrated H_σ-function of $\mathfrak{F}$, we have that $W \in F_1(\sigma_i)\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'} = F_1(\sigma_i)\mathfrak{G}_{\sigma_i'}$, and so $W/W_{F_1(\sigma_i)} \in \mathfrak{G}_{\sigma_i'}$, a contradiction. Hence $F \leqslant F_1$. With a similar argument, $F_1 \leqslant F$. Hence $F_1 = F$.

The theorem is proved. □

The following corollary follows directly from Theorem 1.1 and Proposition 4.1.

Corollary 5.1. *Let $\mathfrak{F} = LR_\sigma(F) = LR_\sigma(\underline{f})$ and $\mathfrak{H} = LR_\sigma(H) = LR_\sigma(\underline{h})$, where F and H are the unique full integrated Lockett H_σ-functions of $\mathfrak{F}$ and $\mathfrak{H}$, and $\underline{f}$, $\underline{h}$ are the unique* *minimal H_σ-functions of $\mathfrak{F}$ and $\mathfrak{H}$, respectively. Then any two of the following statements are equivalent:*

(a) $\mathfrak{F} \subseteq \mathfrak{H}$;
(b) $\underline{f} \leqslant \underline{h}$;
(c) $F \leqslant H$.

Remarks 5.2. In Theorem 1.1, the uniqueness of the full integrated H_σ-function F of a σ-local Fitting class $\mathfrak{F}$ holds only under the condition that $F(\sigma_i)$ ($\sigma_i \in \sigma(\mathfrak{F})$) is a Lockett class. To illustrate this, we give the following example.

Recall that a Fitting class $\mathfrak{F}$ is said to be normal in $\mathfrak{S}$, where $\mathfrak{S}$ is the Fitting class of all soluble groups, if the $\mathfrak{F}$-radical of G is an $\mathfrak{F}$-maximal subgroup of G for every group $G \in \mathfrak{S}$. By Blessenohl-Gaschütz Theorem [26], the intersection of all nonidentity normal Fitting classes is a nonidentity normal Fitting class $\mathfrak{S}_*$. Let $\sigma = \sigma^1$ and $\mathfrak{F} = \mathfrak{S}$. Then $\mathfrak{F} = LR(f)$, where f is an H_σ-function of $\mathfrak{F}$ such that $f(p) = \mathfrak{S}$ for all primes p. Obviously, f is a full integrated H-function of $\mathfrak{F}$ and the value $f(p)$ is a Lockett class. Let φ be the H_σ-function such that $\varphi(p) = \mathfrak{S}_*\mathfrak{N}_p$ for all primes p. Then $LR(\varphi) = \bigcap_{p\in\mathbb{P}} \mathfrak{S}_*\mathfrak{N}_p\mathfrak{N}_p\mathfrak{G}_{p'} = \mathfrak{S}_*(\bigcap_{p\in\mathbb{P}} \mathfrak{N}_p\mathfrak{S}_{p'}) = \mathfrak{S}_*\mathfrak{N}$. By Cossey Theorem [27], $\mathfrak{S}_*\mathfrak{N} = \mathfrak{S}$ and so φ is also a full integrated H-function of $\mathfrak{F}$. Moreover, obviously $\varphi(p) \neq \mathfrak{S} = f(p)$ for all $p \in \mathbb{P}$ (see [28, Theorem 3.5(b)]). On the other hand, by Lemma 2.5(a)(c), $\varphi^*(p) = (\mathfrak{S}_*\mathfrak{N}_p)^* = (\mathfrak{S}_*)^*\mathfrak{N}_p = \mathfrak{S}^*\mathfrak{N}_p = \mathfrak{S}\mathfrak{N}_p = \mathfrak{S}$. Hence $\varphi(p) \neq \varphi^*(p)$, and so φ is not the Lockett function.

6. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $LR_\sigma(m)$ be a σ-local Fitting class with the local H_σ-function such that

$$m(\sigma_i) = \begin{cases} \mathfrak{F} \diamond h(\sigma_i) & \text{if } \sigma_i \in \sigma(\mathfrak{H}) = \Pi; \\ f(\sigma_i) & \text{if } \sigma_i \in \sigma \setminus \Pi. \end{cases}$$

We prove that $\mathfrak{M} = LR_\sigma(m)$. Let $G \in \mathfrak{M} = \mathfrak{F} \diamond \mathfrak{H}$. First assume that $\sigma_i \in \sigma(G/G_\mathfrak{F})$. By Lemma 2.3, $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}}/(G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}})_\mathfrak{F} = G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}}/(G_\mathfrak{F} \cap G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}}) \cong G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}}G_\mathfrak{F}/G_\mathfrak{F}$. Then since $\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}$ is a formation, $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}}/(G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}})_\mathfrak{F} \cong (G/G_\mathfrak{F})^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}}$ by Lemma 2.8. Since $G \in \mathfrak{F} \diamond \mathfrak{H}$, $G/G_\mathfrak{F} \in \mathfrak{H}$. Hence $(G/G_\mathfrak{F})^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}} \in h(\sigma_i)$ by Lemma 3.1(b), and so $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}}/(\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i})_\mathfrak{F} \in h(\sigma_i)$. Therefore $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}} \in \mathfrak{F} \diamond h(\sigma_i)$. Note that from $G/G_\mathfrak{F} \in \mathfrak{H}$ we have $\sigma_i \in \sigma(\mathfrak{H})$. Thus $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}} \in m(\sigma_i)$ for all $\sigma_i \in \sigma(\mathfrak{H}) \cap \sigma(G/G_\mathfrak{F})$.

Now assume that $\sigma_i \in \sigma(G) \setminus \sigma(G/G_\mathfrak{F})$. Then $G/G_\mathfrak{F}$ is a σ'_i-group and so $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}} = (G_\mathfrak{F})^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}}$ by Lemma 2.9. Moreover, since $G_\mathfrak{F} \in \mathfrak{F}$, $(G_\mathfrak{F})^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}} \in f(p)$ by Lemma 3.1(b). Hence $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}} \in f(p)$. If $\sigma_i \in \sigma(\mathfrak{H})$, then $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma'_i}} \in f(\sigma_i) \subseteq \mathfrak{F} \subseteq \mathfrak{F} \diamond h(\sigma_i) = m(\sigma_i)$.

Suppose that $\sigma_i \notin \sigma(\mathfrak{H})$. Then $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in f(\sigma_i) = m(\sigma_i)$. Thus $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in m(\sigma_i)$ for all $\sigma_i \in \sigma(G)$. Consequently, $\mathfrak{M} \subseteq LR_\sigma(m)$ by Lemma 3.1(b).

Conversely, we prove that $LR_\sigma(m) \subseteq \mathfrak{M}$. Suppose that $G \in LR_\sigma(m) \setminus \mathfrak{M}$ and G is a counterexample of minimal order. Then G has a unique maximal normal subgroup $M = G_{\mathfrak{M}} < G$. Since $\mathfrak{F} \subseteq \mathfrak{M}$ and $G \notin \mathfrak{M}$, we have that $G_{\mathfrak{F}} < G$ and $G_{\mathfrak{F}} \leq M$. Hence $M_{\mathfrak{F}} = G_{\mathfrak{F}} \cap M = G_{\mathfrak{F}}$ by Lemma 2.3. Moreover, from $M \in \mathfrak{M}$ we have $M/M_{\mathfrak{F}} = M/G_{\mathfrak{F}} \in \mathfrak{H}$.

We claim that G/M is a σ_i-group for some σ_i. In fact, assume that G/M is not σ-primary and $\sigma_i, \sigma_j \in \sigma(G/M)$. Since $\overline{G} = G/M$ is a simple group, $\overline{G}^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} = \overline{1}$ or $\overline{G}$. If $\overline{G}^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} = 1$, then $\overline{G} \in \mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ and so $\overline{G}/\overline{G}_{\mathfrak{G}_{\sigma_i}} \in \mathfrak{G}_{\sigma_i'}$. But $\overline{G}_{\mathfrak{G}_{\sigma_i}} = \overline{1}$ or $\overline{G}$. If $\overline{G}_{\mathfrak{G}_{\sigma_i}} = \overline{1}$, then G/M is a σ_i'-group, a contradiction. If $\overline{G}_{\mathfrak{G}_{\sigma_i}} = \overline{G}$, then G/M is a σ_i-group, a contradiction also. Hence G/M is a σ_i-group for some σ_i.

If $\sigma_i \in \sigma(\mathfrak{H})$, then $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in m(\sigma_i) = \mathfrak{F} \diamond h(\sigma_i)$ by Lemma 3.1(b). Hence $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}}/(G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}})_{\mathfrak{F}} \in h(\sigma_i)$. As shown above, $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}}/(G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}})_{\mathfrak{F}} \cong (G/G_{\mathfrak{F}})^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}}$. Therefore $(G/G_{\mathfrak{F}})^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in h(\sigma_i)$. Now we prove that $(G/G_{\mathfrak{F}})^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in h(\sigma_j)$ for all $\sigma_j \in \sigma(\mathfrak{H})$ different from σ_i. In fact, since $M \in \mathfrak{M} = \mathfrak{F} \diamond \mathfrak{H}$, $M/G_{\mathfrak{F}} \in \mathfrak{H}$. Hence $\sigma(M/G_{\mathfrak{F}}) \subseteq \sigma(\mathfrak{H})$. Then as $\sigma(G/M) = \{\sigma_i\} \subseteq \sigma(\mathfrak{H})$, we have that $\sigma(G/G_{\mathfrak{F}}) \subseteq \sigma(\mathfrak{H})$. Hence $(G/G_{\mathfrak{F}})^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in h(\sigma_j)$ for all $\sigma_j \in \sigma(\mathfrak{H})$ where $\sigma_j \neq \sigma_i$. Thus $(G/G_{\mathfrak{F}})^{\mathfrak{G}_{\sigma_k}\mathfrak{G}_{\sigma_k'}} \in h(\sigma_k)$ for all $\sigma_k \in \sigma(G/G_{\mathfrak{F}})$. It follows from Lemma 3.1(b) that $G/G_{\mathfrak{F}} \in \mathfrak{H}$. Consequently $G \in \mathfrak{F} \diamond \mathfrak{H} = \mathfrak{M}$, a contradiction.

Hence $\sigma_i \notin \sigma(\mathfrak{H})$. As G/M is a σ_i-group, it follows that $G = G^{\mathfrak{G}_{\sigma_i'}}$ (in fact, if $G^{\mathfrak{G}_{\sigma_i'}} < G$, then $(G/G^{\mathfrak{G}_{\sigma_i'}})/(M/G^{\mathfrak{G}_{\sigma_i'}}) \cong G/M \in \mathfrak{G}_{\sigma_i'}$, a contradiction). Then $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} = (G^{\mathfrak{G}_{\sigma_i'}})^{\mathfrak{G}_{\sigma_i}} = G^{\mathfrak{G}_{\sigma_i}} \in m(\sigma_i) = f(\sigma_i)$ by Lemma 2.2 and the definition of m. If $G^{\mathfrak{G}_{\sigma_i}} = G$, then $G \in f(\sigma_i) \subseteq \mathfrak{F} \subseteq \mathfrak{F} \diamond \mathfrak{H} = \mathfrak{M}$, a contradiction. Hence $G^{\mathfrak{G}_{\sigma_i}} < G$. Since $M/G^{\mathfrak{G}_{\sigma_i}} \trianglelefteq G/G^{\mathfrak{G}_{\sigma_i}} \in \mathfrak{G}_{\sigma_i}$, $M/G^{\mathfrak{G}_{\sigma_i}}$ is a σ_j'-group for every $\sigma_j \in \sigma(G)$ different from σ_i. Hence $M^{\mathfrak{G}_{\sigma_j}\mathfrak{G}_{\sigma_j'}} = (G^{\mathfrak{G}_{\sigma_i}})^{\mathfrak{G}_{\sigma_j}\mathfrak{G}_{\sigma_j'}}$ by Lemma 2.9. Since $G \in LR_\sigma(m)$, by Lemma 3.1(b) $G^{\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}} \in m(\sigma_i) = f(\sigma_i) \subseteq \mathfrak{F}$. Hence $(G^{\mathfrak{G}_{\sigma_i}})^{\mathfrak{G}_{\sigma_j}\mathfrak{G}_{\sigma_j'}} \in f(\sigma_j)$ for every $\sigma_j \in \sigma(G)$ different from σ_i. Analogously, G/M is a σ_j'-group and $G^{\mathfrak{G}_{\sigma_j}\mathfrak{G}_{\sigma_j'}} = M^{\mathfrak{G}_{\sigma_j}\mathfrak{G}_{\sigma_j'}} \in f(\sigma_j)$ for every $\sigma_j \in \sigma(G)$ different from σ_i. Thus $G^{\mathfrak{G}_{\sigma_k}\mathfrak{G}_{\sigma_k'}} \in f(\sigma_k)$ for all $\sigma_k \in \sigma(G)$. It follows from Lemma 3.1(b) that $G \in \mathfrak{F} \subseteq \mathfrak{F} \diamond \mathfrak{H} = \mathfrak{M}$. The final contradiction completes the proof of the theorem. □

7. On n-multiply σ-local Fitting classes

In this section, we generalize the theory of σ-local Fitting classes to n-multiply σ-local Fitting classes, which is the dual theory of the paper [12] about n-multiply local formations.

Definition 7.1. Following [18], every Fitting class can be considered as 0-multiply σ-local. Let $n > 0$. Then a Fitting class $\mathfrak{F}$ is called n-multiply σ-local if it has an H_σ-function f

such that every nonempty value $f(\sigma_i)$ of f is $(n-1)$-multiply local. A Fitting class $\mathfrak{F}$ is said to be totally σ-local if it is n-multiply local for all natural number n.

Proposition 7.2. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be n-multiply σ-local Fitting classes. Then $\mathfrak{M} = \mathfrak{F} \diamond \mathfrak{H}$ is a n-multiply σ-local Fitting class.*

Proof. Let $n = 0$. Then the statement holds for $n = 0$ since product of any two Fitting classes is also a Fitting class. Now assume that $n > 0$ and that the statement is true for $n-1$. Then $\mathfrak{F} = LR_\sigma(f)$ and $\mathfrak{H} = LR_\sigma(h)$ for some H_σ-functions f and h such that all values of f and h are $(n-1)$-multiply σ-local Fitting classes. Let $\mathfrak{M} = \mathfrak{F} \diamond \mathfrak{H}$. In view of Lemma 3.1, we assume without loss of generality that both functions f and h are integrated. Then, by Theorem 1.2, $\mathfrak{M} = LR_\sigma(m)$, where

$$m(\sigma_i) = \begin{cases} \mathfrak{F} \diamond h(\sigma_i) & \text{if } \sigma_i \in \Pi; \\ f(\sigma_i) & \text{if } \sigma_i \in \Pi'. \end{cases}$$

It is clear that $\mathfrak{F}$ is $(n-1)$-multiply σ-local, so $\mathfrak{F} \diamond h(\sigma_i)$ is $(n-1)$-multiply σ-local by inductive hypothesis. Hence $m(\sigma_i)$ is $(n-1)$-multiply σ-local for every σ_i. Therefore $\mathfrak{M}$ is n-multiply σ-local. □

Proposition 7.3. *The intersection of every nonempty set of σ-local Fitting classes is a σ-local Fitting class, and the intersection of every nonempty set of n-multiply σ-local Fitting classes is a n-multiply σ-local Fitting class.*

Proof. Let $\mathfrak{F} = \cap_{i \in I} \mathfrak{F}_j$, where $\mathfrak{F}_j = LR_\sigma(f_j)$ for some integrated H_σ-function f_j, $j \in I$. Let $f = \cap_{j \in I} f_j$. Evidently, f is an H_σ-function. We show that $\mathfrak{F} = LR_\sigma(f)$. Since $f \leq f_j$ for all $j \in I$, $LR_\sigma(f) \subseteq \mathfrak{F}$. Let $G \in \mathfrak{F}$. Then $G/G_{f_j(\sigma_i)} \in \mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ for all $j \in I$ and $\sigma_i \in \sigma(G)$. Since $\mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ is a formation and $\cap_{j \in I} G_{f_j(\sigma_i)} = G_{\cap_{j \in I} f(\sigma_i)} = G_{f(\sigma_i)}$, we have $G/\cap_{j \in I} G_{f_j(\sigma_i)} = G/G_{f(\sigma_i)} \in \mathfrak{G}_{\sigma_i}\mathfrak{G}_{\sigma_i'}$ for all $\sigma_i \in \sigma(G)$. Hence $G \in LR_\sigma(f)$ by Lemma 3.1(b). This shows that $\mathfrak{F} = LR_\sigma(f)$. Hence $\mathfrak{F}$ is a σ-local Fitting class. The second statement can be similarly proved by induction.

The proposition is proved. □

Theorem 7.4. *Let $\mathfrak{F} = LR_\sigma(\varphi)$, where φ is the full integrated H_σ-function of $\mathfrak{F}$, and let $\Pi = \sigma(\mathfrak{F})$. Then*

(a) *The Fitting class $\mathfrak{F}$ is n-multiply σ-local if and only if the Fitting class $\varphi(\sigma_i)$ is $(n-1)$-multiply σ-local for all $\sigma_i \in \Pi$.*
(b) *The Fitting class $\mathfrak{F}$ is totally σ-local if and only if the Fitting class $\varphi(\sigma_i)$ is totally σ-local for all $\sigma_i \in \Pi$.*

Proof. (a) It is enough to show that if $\mathfrak{F}$ is n-multiply σ-local, then every value of φ is $(n-1)$-multiply σ-local. Since $\mathfrak{F}$ is n-multiply σ-local, $\mathfrak{F} = LR_\sigma(f)$ for some H_σ-function f

such that all values $f(\sigma_i)$ are $(n-1)$-multiply σ-local. Hence $\mathfrak{F} = LR_\sigma(\varphi)$ by Lemma 3.2, where $\varphi(\sigma_i) = (f(\sigma_i) \cap \mathfrak{F})\mathfrak{G}_{\sigma_i}$ for every $\sigma_i \in \Pi$. It is clear that a n-multiply σ-local Fitting class is $(n-1)$-multiply σ-local, so $\mathfrak{F}$ is $(n-1)$-multiply σ-local. Hence the Fitting class $f(\sigma_i) \cap \mathfrak{F}$ is also $(n-1)$-multiply σ-local by Proposition 7.3. Note that $\mathfrak{G}_{\sigma_i}$ is a totally σ-local Fitting class with the H_σ-function h such that $h(\sigma_i) = \mathfrak{G}_{\sigma_i}$ and $h(\sigma_j) = \emptyset$ for $j \neq i$. It follows from Proposition 7.2 that $(f(\sigma_i) \cap \mathfrak{F})\mathfrak{G}_{\sigma_i}$ is $(n-1)$-multiply σ-local. Therefore, every value $\varphi(\sigma_i)$ of φ is $(n-1)$-multiply σ-local.

(b) This assertion is a corollary of the statement (a) of the theorem.

The theorem is proved. □

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