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On the dual theory of a result of Bryce and Cossey ☆

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ABSTRACT

In the theory of formations of finite soluble groups, a well known result of Bryce and Cossey is: a local formation $\mathfrak{F}$ is a Fitting class if and only if every value of the canonical formation function F of $\mathfrak{F}$ is a Fitting class. In this paper, we give the dual theory of the result of Bryce and Cossey. We proved that an ω-local (in particular, local) Fitting class $\mathfrak{F}$ is a formation if and only if every value of the canonical ω-local (in particular, local) Hartley function F of $\mathfrak{F}$ is a formation.

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1. Introduction

Throughout this paper, all groups are finite. All unexplained notions and terminologies are standard, and the reader is referred to [1,2] if necessary. Recall that a class $\mathfrak{F}$ of groups is said to be: (i) a *formation* if it is closed under taking homomorphic images and subdirect products; (ii) a *Fitting class* if it is closed under taking normal subgroups and products of normal $\mathfrak{F}$-subgroups. If a class of groups is both a formation and a Fitting class, then it is said to be a *Fitting formation*.

Clearly, for a nonempty formation $\mathfrak{F}$, every group G has a least normal subgroup $G^{\mathfrak{F}}$ such that $G/G^{\mathfrak{F}} \in \mathfrak{F}$; for a nonempty Fitting class $\mathfrak{F}$, every group G has a largest normal $\mathfrak{F}$-subgroup $G_{\mathfrak{F}}$. The subgroups $G^{\mathfrak{F}}$ and $G_{\mathfrak{F}}$ are called $\mathfrak{F}$-residual and $\mathfrak{F}$-radical of G respectively.

Recall that the product $\mathfrak{F}\mathfrak{H}$ of two classes of groups $\mathfrak{F}$ and $\mathfrak{H}$ is the class (G: G has a normal subgroup $N \in \mathfrak{F}$ with $G/N \in \mathfrak{H}$); the product $\mathfrak{F} \circ \mathfrak{H}$ of two formation $\mathfrak{F}$ and $\mathfrak{H}$ is a class $(G : G^{\mathfrak{H}} \in \mathfrak{F})$ and the product $\mathfrak{F} \diamond \mathfrak{H}$ of two Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ is a class $(G : G/G_{\mathfrak{F}} \in \mathfrak{H})$. It is well known that if $\mathfrak{F}$ is closed under taking normal subgroups and $\mathfrak{H}$ is closed under taking homomorphic images, then $\mathfrak{F} \circ \mathfrak{H} = \mathfrak{F}\mathfrak{H} = \mathfrak{F} \diamond \mathfrak{H}$ (see p. 338 and p. 566 in [1]). Also, it is known that the product of two formations (resp. the product of two Fitting classes) is also a formation (resp. a Fitting class) and the multiplication of formations (resp. Fitting classes) satisfies the associative law (see [1, IV, Theorem 1.8(a)(c) and IX, Theorem (1.12)(a)(c)]).

Let π be a set of primes. We denote by π' the set $\mathbb{P}\backslash\pi$, where $\mathbb{P}$ is the set of all primes. The symbols $\mathfrak{S}$, $\mathfrak{E}_{\pi}$, $\mathfrak{E}_{p'}$, $\mathfrak{N}_p$ denote the classes of all soluble groups, all π-groups, all p'-groups, all p-groups, respectively, where p' denotes $\{p\}'$. Let G be a group and $\mathfrak{X}$ a class of groups. We define $\sigma(G) = \{p : p \in \mathbb{P}, p||G|\}$ and $\sigma(\mathfrak{X}) = \bigcup\{\sigma(X) : X \in \mathfrak{X}\}$. A function $f : \mathbb{P} \to \{formations\}$ is called a *formation function*. Let f be an arbitrary formation function and $LF(f) = \mathfrak{E}_{\pi} \cap (\bigcap_{p\in\pi} \mathfrak{E}_{p'}\mathfrak{N}_p f(p))$, where π denotes the set $\{p \in \mathbb{P} : f(p) \neq \varnothing\}$ (which is called the support of f).

A formation $\mathfrak{F}$ is called local if $\mathfrak{F} = LF(f)$ for some formation function f. In this case, we say that f is a formation function of $\mathfrak{F}$. By [1, IX Theorem 3.7], a local formation $\mathfrak{F}$ can be defined by a unique formation function F such that $F(p) = \mathfrak{N}_p F(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$. This formation function F is called a *canonical formation function* of $\mathfrak{F}$. Note that if all groups in a formation $\mathfrak{F}$ are soluble, then $\mathfrak{F}$ is said to be a soluble formation.

In the theory of classes of groups, a basic result is the following well known Bryce–Cossey Theorem:

Theorem 1.1 *([3, Theorem 3], see also [4, Theorem 4.7 and 4.10]). A soluble local formation $\mathfrak{F}$ is a Fitting class if and only if every value of the canonical formation function F of $\mathfrak{F}$ is a Fitting class.*

Recall that if ω is a nonempty set of primes, by Shemetkov and Skiba in [5], a function $f : \omega \cup \{\omega'\} \to \{Fitting\ classes\}$ is called an ω-local Hartley function or simply an ω-local

H-function. For a ω-local H-function f, we put $Supp(f) = \{a \in \omega \cup \{\omega'\} : f(a) \neq \phi\}$. For an arbitrary ω-local H-function f, let $\pi_1 = Supp(f) \cap \omega$ and $\pi_2 = \omega \backslash \pi_1$. By using [6] (to compare with [5]), we define the class

$$LR_\omega(f) = (\bigcap_{p\in\pi_2} \mathfrak{E}_{p'}) \cap (\bigcap_{p\in\pi_1} f(p)\mathfrak{N}_p\mathfrak{E}_{p'}) \cap f(\omega')\mathfrak{E}_\omega. \tag{1}$$

A Fitting class $\mathfrak{F}$ is called ω-*local* if $\mathfrak{F} = LR_\omega(f)$ for some ω-*local* H-function f. If an ω-*local* H-function f of $\mathfrak{F}$ satisfies that $f(a) \subseteq \mathfrak{F}$ for all $a \in \omega \cup \{\omega'\}$, then f is called *integrated* H-function of $\mathfrak{F}$. Note that if $\omega = \mathbb{P}$, then an ω-local Fitting class is called a *local Fitting class* (see [4] p. 87). It is easy to see that $\mathfrak{S}$, $\mathfrak{S}_\pi$, $\mathfrak{S}_{p'}$, $\mathfrak{N}_p$ and $\mathfrak{F} = (1)$ are all local Fitting classes, where the class (1) is the class of all identity groups. By Lemma 22 and Lemma 25 in [5], every ω-local Fitting class $\mathfrak{F}$ can be defined by a H-function F such that $F(\omega') = \mathfrak{F}$ and $F(p)\mathfrak{N}_p = F(p) \subseteq \mathfrak{F}$ for all $p \in \omega$.

By Theorem in [10] (see Lemma 2.3 below and also Lemma 9.10 in [2]), every ω-local Fitting class $\mathfrak{F}$ can be defined by a unique maximal integrated ω-local H-function F such that $F(\omega') = \mathfrak{F}$ and $F(p)\mathfrak{N}_p = F(p) \subseteq \mathfrak{F}$ for all $p \in \omega$ and the value $F(p)$ (for every $p \in \omega$) is a Lockett class. This H-function is called the canonical ω-local Hartley function of $\mathfrak{F}$ (or simply the canonical).

In connection with above, the following dual question of Bryce–Cossey Theorem naturally arises:

Question I. Is it true that an ω-local (in particular local) Fitting class $\mathfrak{F}$ is a formation if and only if every value of the canonical ω-local (local) Hartley function F of $\mathfrak{F}$ is a formation?

Note that our discussion is not limited to the soluble groups.

It is well known that the operation "$*$" defined by Lockett in [7] plays an important role in investigations of the structure of classes of groups and canonical subgroups (see [1, Chapter IX-X] and [2, Chapter 5, Section 5.9]). In fact, every nonempty Fitting class $\mathfrak{F}$ can be connected with a Fitting class $\mathfrak{F}^*$, where $\mathfrak{F}^*$ is the smallest Fitting class containing $\mathfrak{F}$ such that $(G \times H)_{\mathfrak{F}^*} = G_{\mathfrak{F}^*} \times H_{\mathfrak{F}^*}$ for all groups G and H. A Fitting class $\mathfrak{F}$ is called a Lockett class if $\mathfrak{F} = \mathfrak{F}^*$.

As the duality of above concept in [7], Doerk and Hawkes [8] defined the operation "^{0}" on the set of formations: For every nonempty formation $\mathfrak{F}$, let $\mathfrak{F}^0$ be the least formation containing $\mathfrak{F}$ such that $(G \times H)^{\mathfrak{F}^0} = G^{\mathfrak{F}^0} \times H^{\mathfrak{F}^0}$ for all groups G and H. Note that if $\mathfrak{F}$ is a soluble formation, then $\mathfrak{F} = \mathfrak{F}^0$ (see Theorem 1.2 in [8]); if $\mathfrak{F}$ contains non-soluble groups, then $\mathfrak{F} \neq \mathfrak{F}^0$ in general (see [8, Proposition 2.3])

Definition 1.2. A nonempty formation $\mathfrak{F}$ is called a Lockett formation if $\mathfrak{F} = \mathfrak{F}^0$.

Example 1.3. Let $\mathfrak{F}$ be the product of Fitting classes $\mathfrak{S}_*$ and $\mathfrak{N}_p$, where $\mathfrak{S}_*$ is the least normal Fitting class (note that a Fitting class $\mathfrak{X}$ is said to be *normal* in class $\mathfrak{S}$ if

$(1) \neq \mathfrak{X} \subseteq \mathfrak{S}$ and $G_{\mathfrak{X}}$ is $\mathfrak{X}$-maximal in G for all groups G in $\mathfrak{S}$). Let $\mathfrak{F} = \mathfrak{S}_*\mathfrak{N}_p$. Obviously, $\mathfrak{F} \neq \mathfrak{S}_*$ (see [9, Lemma 3.3]). Let

$$\mathfrak{F}(F^p) = \begin{cases} Fit(F^p(G) : G \in \mathfrak{F}), \text{ if } p \in \sigma(\mathfrak{F}) \\ \phi, \text{ if } p \notin \sigma(\mathfrak{F}), \end{cases}$$

where $F^p(G) = G^{\mathfrak{N}_p\mathfrak{S}_{p'}}$, and $Fit(F^p(G) : G \in \mathfrak{F})$ is the least Fitting class containing the class $(F^p(G) : G \in \mathfrak{F})$. It is clear that $\mathfrak{F}(F^p) \subseteq \mathfrak{S}_*$, therefore $\mathfrak{F}(F^p)\mathfrak{N}_p \subseteq \mathfrak{S}_*\mathfrak{N}_p = \mathfrak{F}$. Consequently, by Theorem 9 in [5], we have that the Fitting class $\mathfrak{F}$ is ω-local for $\omega = \{p\}$. Since $\mathfrak{S}_*$ is a normal Fitting class and by Proposition X (3.11) in [1], $\mathfrak{F}$ is a normal Fitting class. If $\mathfrak{F}$ is a formation, then by Theorem X (3.7) and Proposition X (1.25) in [1], $\mathfrak{F} = \mathfrak{F}^* = \mathfrak{S}$, a contradiction. This shows that $\mathfrak{F}$ is not a formation, in particularly, $\mathfrak{F}$ is not a Lockett formation.

Base on the above example, the following question also naturally arises:

Question II. Is it true that an nonempty ω-local Fitting class $\mathfrak{F}$ is Lockett formation if and only if every nonempty value of the canonical Hartley function of $\mathfrak{F}$ is a Lockett formation?

The following Theorems A and B completely resolve Questions I and II.

Theorem A. *An ω-local Fitting class $\mathfrak{F}$ is a formation if and only if every value of the canonical ω-local Hartley function of $\mathfrak{F}$ is a formation.*

Theorem B. ***A nonempty** ω-local Fitting class $\mathfrak{F}$ is a Lockett formation if and only if every **nonempty** value of the canonical ω-local Hartley function of $\mathfrak{F}$ is a Lockett formation.*

2. Preliminaries

Lemma 2.1 *(see [1, IX, Lemma (1.1)(a)]). Let $\mathfrak{F}$ be a Fitting class and G a group. **If** N **is a subnormal subgroup of** G**,** then $N_{\mathfrak{F}} = N \cap G_{\mathfrak{F}}$.*

Recall that a Fitting class $\mathfrak{F}$ is called a *Lockett class* if $(G \times H)_{\mathfrak{F}} = G_{\mathfrak{F}} \times H_{\mathfrak{F}}$ for all groups G and H. A Fitting class $\mathfrak{F}$ is called a *Fischer class* if $K \trianglelefteq G \in \mathfrak{F}$ and H/K is a nilpotent subgroup of G/K, then $H \in \mathfrak{F}$.

Lemma 2.2 *(see [1, Proposition X(1.25) and Proposition IX(3.5)]). Every Q-closed Fitting class (in particularly, every Fitting formation) and every Fischer class are Lockett class.*

Lemma 2.3 *(see [10, Theorem]). Let $\mathfrak{F}$ be an ω-local Fitting class. Then $\mathfrak{F}$ can be defined by a unique maximal integrated H-function F such that $F(p)\mathfrak{N}_p = F(p) \subseteq \mathfrak{F}$ and $F(p)$ is a Lockett class for all $p \in \omega$ and $f(\omega') = \mathfrak{F}$.*

Let G and H be groups. $G \wr H$ denotes the regular wreath product of G with H. If $K \leq G$, we denote by K^* the subgroup of the base group of $K \wr H$ which is isomorphic to the direct product of $|H|$ copies of K. In particular, G^* denotes the base group of $G \wr H$.

Lemma 2.4 *(see [1, A, Lemma (18.2)(d)]). Let $W = G \wr H$, if $K \trianglelefteq G$ and K^* is the base group of $K \wr H$, then $K^* \trianglelefteq W$ and $W/K^* \cong (G/K) \wr H$.*

Lemma 2.5 *(see [1, Proposition X (2.1)(a)]). Let $\mathfrak{F}$ be a Lockett class. If $G \notin \mathfrak{F}$, then $(G \wr H)_{\mathfrak{F}} = (G_{\mathfrak{F}})^*$ for any group H.*

The following facts about Lockett formations will be needed.

Lemma 2.6. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be nonempty formations, then*

(a) $\mathfrak{F} \subseteq \mathfrak{F}^0 = (\mathfrak{F}^0)^0$ *([8, Theorem 3.9 (a)]);*
(b) $\mathfrak{F}^0 \subseteq E_\phi\mathfrak{F}$, *where* $E_\phi\mathfrak{F} = (G :$ *there exists* $N \trianglelefteq G$ *with* $N \leq \Phi(G)$ *and* $G/N \in \mathfrak{F})$ *([8, Proposition (3.16)]);*
(c) *If* $\mathfrak{F} \subseteq \mathfrak{H}$, *then* $\mathfrak{F}^0 \subseteq \mathfrak{H}^0$ *([8, Theorem 3.9(6)]);*
(d) $[G^{\mathfrak{F}}, AutG] \leq G^{\mathfrak{F}^0}$, *and if* $\mathfrak{F} \leq \mathfrak{H}$, *then* $\mathfrak{H} \subseteq \mathfrak{F}^0$ *if and only if* $[G^{\mathfrak{F}}, AutG] \leq G^{\mathfrak{H}}$ *for every group G ([11, Lemma 1.2 and Proposition 2.1]);*
(e) *If* $\{\mathfrak{F}_\lambda\}_{\lambda\in\Lambda}$ *is a family of nonempty formations, then* $(\cap_{\lambda\in\Lambda}\mathfrak{F}_\lambda)^0 = \cap_{\lambda\in\Lambda}(\mathfrak{F}_\lambda)^0$ *([8, Lemma 3.12]);*
(f) *A nonempty formation* $\mathfrak{F}$ *is a Lockett formation if and only if* $(G \times H)^{\mathfrak{F}} = G^{\mathfrak{F}} \times H^{\mathfrak{F}}$ *for all groups G and H ([8, Theorem 3.10]).*

3. The proof of Theorem A

Recall that a class $\mathfrak{F}$ of groups is called homomorph (or Q-closed) if every homomorphic image of an $\mathfrak{F}$-group is an $\mathfrak{F}$-group.

Lemma 3.1. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be Fitting classes and $\mathfrak{F} \diamond \mathfrak{H}$ a Fitting product of $\mathfrak{F}$ and $\mathfrak{H}$. Then*

(a) *If* $\mathfrak{F}$ *and* $\mathfrak{H}$ *are both homomorphs, then* $\mathfrak{F}\mathfrak{H} = \mathfrak{F} \diamond \mathfrak{H}$ *is a homomorph;*
(b) *If* $\mathfrak{F}$ *and* $\mathfrak{H}$ *are classes that are closed under taking subdirect products and* $\mathfrak{H}$ *is a homomorph, then the class* $\mathfrak{F} \diamond \mathfrak{H} = \mathfrak{F}\mathfrak{H}$ *is closed under taking subdirect products.*

Proof. (a) By [1, IX,(1.11)], $\mathfrak{F}\mathfrak{H} = \mathfrak{F} \diamond \mathfrak{H}$, so we only need to prove that the product is a homomorph. Let $N \trianglelefteq G \in \mathfrak{F}\mathfrak{H}$. Since $G_{\mathfrak{F}}$ is the $\mathfrak{F}$-subgroup of G and $\mathfrak{F}$ is a homomorph, $G_{\mathfrak{F}}/N_{\mathfrak{F}} \in \mathfrak{F}$. By Lemma 2.1, we have that $G_{\mathfrak{F}}N/N \cong G_{\mathfrak{F}}/N_{\mathfrak{F}}$. Hence $G_{\mathfrak{F}}N/N \in \mathfrak{F}$ and so $G_{\mathfrak{F}}N/N \leq (G/N)_{\mathfrak{F}}$. Since $G/G_{\mathfrak{F}} \in \mathfrak{H}$ and $\mathfrak{H}$ is a homomorph, $G/G_{\mathfrak{F}}N \cong (G/G_{\mathfrak{F}})/(G_{\mathfrak{F}}N/G_{\mathfrak{F}}) \in \mathfrak{H}$. Then $((G/N)/(G_{\mathfrak{F}}N/N))/((G/N)_{\mathfrak{F}}/(G_{\mathfrak{F}}N/N)) \cong (G/N)/(G/N)_{\mathfrak{F}} \in \mathfrak{H}$. Hence $G/N \in \mathfrak{F}\mathfrak{H}$.

(b) Let G be a group and N_i a normal subgroup of G such that $G/N_i \in \mathfrak{F}\mathfrak{H}$ $(i = 1, 2)$. Without loss of generality, we may assume that $N_1 \cap N_2 = 1$ ([1, Proposition II(2.6)]). We need show that $G \in \mathfrak{F}\mathfrak{H}$. Let $K_i/N_i = (G/N_i)_{\mathfrak{F}}$ for $i = 1, 2$. Since $K_1 \cap K_2/K_1 \cap N_2 \cong (K_1 \cap K_2)N_2/N_2$ and $K_2/N_2 \in \mathfrak{F}$, we have that $K_1 \cap K_2/K_1 \cap N_2 \in \mathfrak{F}$. Analogously, $K_1 \cap K_2/K_2 \cap N_1 \in \mathfrak{F}$. Hence $K_1 \cap K_2/(K_1 \cap K_2) \cap (N_1 \cap N_2) = K_1 \cap K_2 \in \mathfrak{F}$, and so $K_1 \cap K_2 = (K_1 \cap K_2)_{\mathfrak{F}} \leq G_{\mathfrak{F}}$. Since $G/N_i \in \mathfrak{F}\mathfrak{H}$ and $G/K_i \cong (G/N_i)/(K_i/N_i)$, we have that $G/K_i \in \mathfrak{H}$. Now as the class $\mathfrak{H}$ is closed under taking subdirect products, $G/K_1 \cap K_2 \in \mathfrak{H}$. But since $\mathfrak{H}$ is a homomorph, $G/G_{\mathfrak{F}} \cong (G/K_1 \cap K_2)/(G_{\mathfrak{F}}/K_1 \cap K_2) \in \mathfrak{H}$. Therefore $G \in \mathfrak{F}\mathfrak{H}$. The lemma is proved. □

Corollary 3.2. *The product of any two Fitting formations is a Fitting formation.*

Lemma 3.3. *Let $\mathfrak{F}$ be an ω-local Fitting class defined by the canonical H-function F of $\mathfrak{F}$, and let $W = G \wr Z_p$ be a regular wreath product of G with the cyclic group Z_p. If $p \in \omega$ and $G \in F(p)$, then $W \in \mathfrak{F}$.*

Proof. Let G^* be a base group of W. As $F(p)$ is a Fitting class, $G^* \in F(p)$ and so $G^* \leq W_{F(p)}$. But since $W/G^* \in \mathfrak{N}_p$, we have that $W \in F(p)\mathfrak{N}_p \subseteq \mathfrak{F}$. Hence $W \in \mathfrak{F}$. □

Proof of Theorem A. *Necessity.* Assume that the ω-local Fitting class $\mathfrak{F}$ is a formation. Then $\mathfrak{F}$ is a Lockett class by Lemma 2.2. Hence, by Lemma 2.3, all nonempty values of the canonical H-function F of $\mathfrak{F}$ are Lockett classes.

We need to prove that every value of F is a formation, i.e. $F(a)$ is a formation for all $a \in \omega \cup \{\omega'\}$.

Firstly, we show that all values of F are homomorphs. Let $Supp(f) = \{a \in \omega \cup \{\omega'\} : f(a) \neq \phi\}$, $\pi_1 = Supp(f) \cap \omega$ and $\pi_2 = \omega \backslash \pi_1$. If $a = \omega'$, then $F(\omega') = \mathfrak{F}$, so $f(\omega')$ is a homomorph. Suppose that $a \in \omega \backslash \pi_1$. Then $F(p) = \phi$ and so $F(a)$ is a homomorph for all $a \in \pi_2$.

Now we show that $F(p)$ is a homomorph for every $a \in \pi_1$. Assume $G \in F(p)$ and $G/N \notin F(p)$ for some $N \trianglelefteq G$. Let $W = G \wr Z_p$. Then $W = K \rtimes Z_p$, where K is the base group of W. Since $F(p)$ is a Fitting class and $F(p)\mathfrak{N} = F(p)$, clearly $W \in F(p) \subseteq \mathfrak{F}$, so $W \in \mathfrak{F}$. Let $W_1 = (G/N) \wr Z_p$. Since $\mathfrak{F}$ is a formation, by Lemma 2.4 $W_1 \cong W/N^* \in \mathfrak{F}$, where N^* is the base group of $N \wr Z_p$. Consequently, $W_1 \in \cap_{p \in \pi_1} F(p)\mathfrak{E}_{p'}$ (see the equation (1)). Then $W_1/(W_1)_{F(p)} \in \mathfrak{E}_{p'}$ and so $p \nmid |W_1/(W_1)_{F(p)}|$ for $p \in \pi_1$. On the other hand, since $G/N \notin F(p)$ and $F(p)$ is a Lockett class, $(W_1)_{F(p)} = ((G/N)_{F(p)})^*$ by Lemma 2.5. Now, using Lemma 2.4, we obtain $W_1/(W_1)_{F(p)} \cong (G/N)/(G/N)_{F(p)} \wr Z_p$. Hence $p||W_1/(W_1)_{F(p)}|$. This contradiction shows that $F(p)$ is a homomorph for all $p \in \pi_1$.

Now we prove that $F(a)$ is closed under taking subdirect product for all $a \in \omega \cup \{\omega'\}$. If $a = \omega'$ or $a \in \omega \setminus \pi_1$, then as above, we can see that $F(a)$ is closed under taking subdirect product.

Suppose that there exist $p \in \pi_1$ such that $G/N_i \in F(p)$, but $G = G/N_1 \cap N_2 \notin F(p)$ for some $N_i \trianglelefteq G$, where $i \in \{1, 2\}$. Without loss of generality, we may assume that

$N_1 \cap N_2 = 1$. Let $W = G \wr Z_p, W_i = (G/N_i) \wr Z_p$ and $(G/N_i)^*$ be the basic group of W_i. Then $W_i \cong (G/N_i)^* \rtimes Z_p$ and $W_i/(G/N_i)^* \in \mathfrak{N}_p$. Since $G/N_i \in F(p)$ and $F(p)$ is a Fitting class, $W_i \in F(p)\mathfrak{N}_p = F(p) \subseteq \mathfrak{F}$. Consequently, $W_i \in \mathfrak{F}$.

By Lemma 2.4, $W_i = (G/N_i) \wr Z_p \cong W/N_i^* = (G \wr Z_p)/N_i^*$, where N_i^* is the basic group of $N \wr Z_p$. Hence $W/N_i^* \in \mathfrak{F}$. Since $\mathfrak{F}$ is a formation, $W/N_1^* \cap N_2^* = W \in \mathfrak{F}$. It follows that $W \in F(p)\mathfrak{E}_{p'}$ by the equation (1), and so $p \nmid |W/W_{F(p)}|$.

On the other hand, $F(p)$ is a Lockett class by Lemma 2.3 and $G \notin F(p)$. By Lemma 2.5, $W_{F(p)} = (G_{F(p)})^*$. Hence $W/(G_{F(p)})^* = W/W_{F(p)} \cong (G/G_{F(p)}) \wr Z_p$ by Lemma 2.4, and so $p||W/W_{F(p)}|$. This contradiction shows that the class $F(p)$ is closed under taking subdirect product for all $p \in \pi_1$. Therefore $F(p)$ is a formation for all $p \in \omega \cup \{\omega'\}$.

Sufficiency. Suppose that all values of the canonical function F of $\mathfrak{F}$ are formations. Since $\mathfrak{F}$ is an ω-local Fitting class, $\mathfrak{F} = (\cap_{p\in\pi_2}\mathfrak{E}_{p'}) \cap (\cap_{\pi_1} F(p)\mathfrak{E}_{p'}) \cap F(\omega')\mathfrak{E}_\omega$.

Obviously, the classes $\mathfrak{G}_{p'}, \mathfrak{G}_\omega$ are Fitting formations, and by the hypothesis $F(p)$ and $F(\omega')$ are also Fitting formations. It is also clear that the intersection of Fitting formations is a Fitting formation. Besides by Lemma 3.1, the product of Fitting formations is a Fitting formation. Hence $\mathfrak{F}$ is a Fitting formation. The theorem is proved. □

If $\omega = \mathbb{P}$, we obtain the following Corollaries from Theorem A.

Corollary 3.4. *A local Fitting class is a formation if and only if every value of the canonical local Hartley function of $\mathfrak{F}$ is a formation.*

4. The proof of Theorem B

Lemma 4.1. *Every nonempty local formation is a Lockett formation.*

Proof. Let $\mathfrak{F}$ be a local formation. Then by Lemma 2.6(a)(b), $\mathfrak{F} \subseteq \mathfrak{F}^0$ and $\mathfrak{F}^0 \subseteq E_\Phi\mathfrak{F}$, where $E_\phi\mathfrak{F} = (G$: there exists $N \trianglelefteq G$ with $N \leq \Phi(G)$ and $G/N \in \mathfrak{F})$. Since $\mathfrak{F}$ is local, by Gaschutz–Lubeseder–Schmid Theorem (see Theorem IV(4.6) in [1]), $E_\Phi\mathfrak{F} = \mathfrak{F}$. Hence $\mathfrak{F} = \mathfrak{F}^0$. The lemma is proved. □

Lemma 4.2 *(see [12], Lemma 3.2 (2)). Suppose that $\mathfrak{F}$, $\mathfrak{H}$ be any nonempty formations. If $\mathfrak{H}$ be a Lockett formation, then $\mathfrak{F}^0\mathfrak{H} = (\mathfrak{F}\mathfrak{H})^0$.*

Definition 4.3. Let $\mathfrak{F}$ be an ω-local Fitting class and f an ω-local H-function of $\mathfrak{F}$. Then we define the H-function f^0: $f^0(a) = (f(a))^0$ for all $a \in \omega \cup \{\omega'\}$.

Proof of Theorem B. Let $\mathfrak{F}$ be an nonempty ω-local Fitting class. Then $\mathfrak{F} = (\cap_{p\in\pi_2}\mathfrak{E}_{p'}) \cap (\cap_{p\in\pi_1} F(p)\mathfrak{E}_{p'}) \cap \boldsymbol{F}(\omega')\mathfrak{E}_\omega$ for the canonical H-function F of $\mathfrak{F}$, where $\pi_1 = Supp(f) \cap \omega$ and $\pi_2 = \omega \backslash \pi_1$.

Assume that $\mathfrak{F}$ is a Lockett formation, that is, $\mathfrak{F} = \mathfrak{F}^0$. By Theorem A, every value of F is also a formation.

Now we prove that all nonempty values of F are Lockett formations.

Suppose that $a = \omega'$, then $F(a) = \mathfrak{F} = \mathfrak{F}^0$ and so $F(a)$ is a Lockett formation.

Let $a = p \in \omega$. Since $F(p) \subseteq \mathfrak{F}$ and $\mathfrak{F}$ is a Lockett formation, by Lemma 2.6(c) $F^0(p) = (F(p))^0 \subseteq \mathfrak{F}^0 = \mathfrak{F}$. Since F is the canonical H-function F of $\mathfrak{F}$, $F(p)\mathfrak{N}_p = F(p)$ for all $p \in \omega$ and $F(\omega') = \mathfrak{F}$. It is clear that the classes $\mathfrak{G}_{p'}$, $(\cap_{p\in\pi_2}\mathfrak{E}_{p'})$, $\mathfrak{N}_p$ and $\mathfrak{G}_\omega$ are all local formations. Hence they are Lockett formations by Lemma 4.1. Then $(\cap_{p\in\pi_2}\mathfrak{E}_{p'})^0 = (\cap_{p\in\pi_2}\mathfrak{E}_{p'})$, $(F(p))^0 = (F(p)\mathfrak{N}_p)^0 = F^0(p)\mathfrak{N}_p$ and $(F(\omega')\mathfrak{G}_\omega)^0 = F(\omega')^0\mathfrak{G}_\omega$ by Lemma 4.2. It follows from Lemma 2.6(e) and the equation (1) that $\mathfrak{F} = \mathfrak{F}^0 = (\bigcap_{p\in\pi_2}\mathfrak{E}_{p'}) \cap (\bigcap_{p\in\pi_1} F^0(p)\mathfrak{E}_{p'}) \cap F^0(\omega')\mathfrak{E}_\omega$ and F^0 is an integrated ω-local H-function of $\mathfrak{F}$. But since the canonical H-function F is such a unique of maximal integrated ω-local H-function of $\mathfrak{F}$, we have that $F^0 \leq F$. On the other hand, by the definition of $\mathfrak{F}^0$ and Definition 4.3, we know that $F \leq F^0$. Therefore we obtain that $F = F^0$. Consequently, all values of F are Lockett formations.

Conversely, suppose that all nonempty values of the canonical ω-local H-function F of $\mathfrak{F}$ are Lockett formations. Then $F = F^0$ and $\mathfrak{F}$ is a formation by Theorem A. Since $\mathfrak{F}^0 = ((\cap_{p\in\pi_2}\mathfrak{E}_{p'}) \cap (\cap_{p\in\pi_1} F(p)\mathfrak{E}_{p'}) \cap (F(\omega')\mathfrak{E}_\omega))^0$, by Lemma 2.6(e) $\mathfrak{F}^0 = (\cap_{p\in\pi_2}\mathfrak{E}_{p'})^0 \cap (\cap_{p\in\pi_1} F(p)\mathfrak{E}_{p'})^0 \cap (F(\omega')\mathfrak{E}_\omega)^0$. With a similar argument as above, we have that $\mathfrak{F}^0 = (\bigcap_{p\in\pi_2}\mathfrak{E}_{p'}) \cap (\bigcap_{p\in\pi_1} F^0(p)\mathfrak{E}_{p'}) \cap F^0(\omega')\mathfrak{E}_\omega = (\bigcap_{p\in\pi_2}\mathfrak{E}_{p'}) \cap (\bigcap_{p\in\pi_1} F(p)\mathfrak{E}_{p'}) \cap F(\omega')\mathfrak{E}_\omega$ since $F = F^0$. Thus $\mathfrak{F}^0 = \mathfrak{F}$. This shows that $\mathfrak{F}$ is a Lockett formation. The theorem is proved. □

If we put $\omega = \mathbb{P}$, then form Theorem B we have

Corollary 4.4. *A nonempty local Fitting class $\mathfrak{F}$ is a Lockett formation if and only if every nonempty value of the canonical H-function F of $\mathfrak{F}$ is a Lockett formation.*

The following result direct follows from Lemma 2.3 and Theorem B.

Corollary 4.5. *If a local Fitting class $\mathfrak{F}$ is a Lockett formation, then $\mathfrak{F}$ can be defined by the canonical local H-function F such that every value of F is both a Lockett formation and a Lockett class.*

5. A characterization of hereditary ω-local Fitting classes

As a continuation of the study of Theorem A, in this section, we discuss a characterization of hereditary ω-local Fitting classes

Let $\mathfrak{X}$ be a class of groups, we say that $\mathfrak{X}$ is *hereditary* if it is closed under taking subgroups.

Lemma 5.1. *Let $\mathfrak{F}$ and $\mathfrak{H}$ are Fitting classes and $\mathfrak{F} \diamond \mathfrak{H}$ is a Fitting product of $\mathfrak{F}$ with $\mathfrak{H}$. If $\mathfrak{F}$ and $\mathfrak{H}$ are hereditary and $\mathfrak{H}$ is a homomorph, then $\mathfrak{F} \diamond \mathfrak{H} = \mathfrak{F}\mathfrak{H}$ is hereditary.*

Proof. Let $G \in \mathfrak{F}\mathfrak{H}$ and $H \leq G$. Since $G_{\mathfrak{F}} \cap H \leq H, G_{\mathfrak{F}} \cap H \in \mathfrak{F}$. Note that $HG_{\mathfrak{F}}/G_{\mathfrak{F}}$ is a subgroup of the $\mathfrak{H}$-group $G/G_{\mathfrak{F}}$. Hence $H/H \cap G_{\mathfrak{F}} \cong HG_{\mathfrak{F}}/G_{\mathfrak{F}} \in \mathfrak{H}$. As $\mathfrak{H}$ is a homomorph, from the isomorphism $(H/H \cap G_{\mathfrak{F}})/(H_{\mathfrak{F}}/H \cap G_{\mathfrak{F}}) \cong H/H_{\mathfrak{F}}$, we obtain that $H/H_{\mathfrak{F}} \in \mathfrak{H}$, and so $H \in \mathfrak{F}\mathfrak{H}$. The Lemma is proved. □

Theorem 5.2. *An ω-local Fitting class $\mathfrak{F}$ is hereditary if and only if every value of the canonical ω-local Hartley function of $\mathfrak{F}$ is hereditary.*

Proof. Let $\mathfrak{F}$ be a hereditary ω-local Fitting class and F is the canonical ω-local H-function of $\mathfrak{H}$. We show that all values of F is hereditary. Since $\mathfrak{F}$ is hereditary, $\mathfrak{F}$ is a Lockett class by Lemma 2.2. Therefore, by Lemma 2.3, all nonempty values of F are Lockett classes.

Let $a \in \omega \cup \{\omega'\}$. If $a = \omega'$, then $F(\omega') = \mathfrak{F}$ and so $f(a)$ is a hereditary. Suppose that $a \in \omega \backslash \pi_1$, where $\pi_1 = Supp(f) \cap \omega$. Then $F(a) = \phi$ and the class $F(a)$ is hereditary.

Now we show that $F(p)$ is a hereditary class for all $p \in \pi_1$. Assume that $F(p)$ is not hereditary for some $p \in \pi_1$. The there exists a group G and a subgroup H of G such that G is a $F(p)$-group but $H \notin F(p)$. Let $W = G \wr Z_p$. Since $G \in F(p)$, $W \in \mathfrak{F}$ by Lemma 3.3. Let $W_1 = H \wr Z_p$. Since $W_1 \leq W$ and $\mathfrak{F}$ is hereditary, $W_1 \in \mathfrak{F}$ and so $W_1 \in F(p)\mathfrak{E}_{p'}$. Then $W_1/(W_1)_{F(p)}$ is a p'-group and so $p \nmid |W_1/(W_1)_{F(p)}|$. On the other hand, since $F(p)$ is a Lockett class and $H \notin F(p)$, by Lemma 2.5 $(W_1)_{F(p)} = (H_{F(p)})^*$, where $(H_{F(p)})^*$ is the base group of $(H_{F(p)}) \wr Z_p$. Hence by Lemma 2.4, $W_1/(W_1)_{F(p)} \cong (H/H_{F(p)}) \wr Z_p$. It follows that $p||W_1/(W_1)_{F(p)}|$. This contradiction shows that the class $F(p)$ is hereditary for all $p \in \pi_1$.

Conversely, suppose that every value $F(a)$ of the canonical function F of $\mathfrak{F}$ is hereditary for $a \in \omega \cup \{\omega'\}$. Note that $\mathfrak{F} = (\cap_{p \in \pi_2} \mathfrak{E}_{p'}) \cap (\cap_{p \in \pi_1} F(p)\mathfrak{E}_{p'}) \cap F(\omega')\mathfrak{E}_{\omega}$, $\mathfrak{E}_{p'}, \mathfrak{E}_{\omega}$ are hereditary and the intersection of hereditary classes is hereditary. Hence the class $\mathfrak{F}$ is hereditary by Lemma 5.1. This completes the proof. □

In the case of $\omega = \mathbb{P}$, we obtain:

Corollary 5.3. *A local Fitting class $\mathfrak{F}$ is hereditary if and only if all values of the canonical Hartley function F of $\mathfrak{F}$ are hereditary.*

Following [13], we define in [14], the concept of multiply locality of the Fitting class as follows: Every Fitting class can be considered as 0-multiply local. Let $n > 0$, then a Fitting class $\mathfrak{F}$ is called n-multiply local if it has a local H-function f such that every nonempty value of $f(p)$ is $(n-1)$-multiply local. A Fitting class is said to be *totally local* if it is n-multiply local for all natural number.

A Fitting class $\mathfrak{F}$ is said to be a soluble Fitting class if all groups in $\mathfrak{F}$ are soluble. By Theorem 1 in [15] and Theorem 1 in [3], a soluble Fitting class $\mathfrak{F}$ is hereditary if and only if $\mathfrak{F}$ is a primitive local formation (see also Theorem XI.(1.7) and Theorem XI.(1.2) in [1]). Moreover, in [15], the author proved that a soluble Fitting class $\mathfrak{F}$ is totally local (as well $\mathfrak{F}$ is a primitive local formation [3]) if and only if $\mathfrak{F}$ is hereditary. Hence by Theorem A and Theorem 5.2, we obtain the basic result of Bryce and Cossey as the following Corollary:

Corollary 5.4 *(see [3, Theorem 3]). Let $\mathfrak{F}$ be a soluble local formation, then $\mathfrak{F}$ is a subgroup closed Fitting formation if and only if it can be locally defined by subgroup closed Fitting formations (i.e. primitive local formations).*

Corollary 5.5 *([15], also [14, Theorem]). A soluble local Fitting class is hereditary if and only if $\mathfrak{F}$ is totally local.*

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