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ON THE PROBLEM OF FRATTINI DUALITY IN THE THEORY OF FITTING CLASSES

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We study the application of the Frattini duality to the description of multiple local Fitting classes. In particular, we establish a necessary and sufficient condition for the local Fitting class to be a formation.

1. Introduction

In the present paper, unless otherwise specified, we consider only finite groups. Recall that a *class of groups* is a set of groups that contains, parallel with each group, all its isomorphic groups. A mapping τ of a set of classes of groups into a set of classes of groups is called the *operation of closure* (see [1], II, Definition 1.4) if, for any classes of groups $\mathfrak{X}$ and $\mathfrak{Y}$, the following conditions are satisfied:

(1) $\mathfrak{X} \subseteq \tau\mathfrak{X}$;

(2) $\tau\mathfrak{X} = \tau(\tau\mathfrak{X})$;

(3) if $\mathfrak{X} \subseteq \mathfrak{Y}$, then $\tau\mathfrak{X} \subseteq \tau\mathfrak{Y}$.

In what follows, we use the conventional notation of the operations of closure: S_n, Q, R_0, N_0, and E_ϕ. For the class of groups $\mathfrak{X}$, these operations are defined as follows:

$$\mathrm{S}_n\mathfrak{X} = (G\colon\ G \trianglelefteq\trianglelefteq H \ \text{ for a certain group } \ H \in \mathfrak{X}),$$

$$\mathrm{Q}\mathfrak{X} = (G\colon\ \exists H \in \mathfrak{X} \ \text{ is an epimorphism of } \ H \ \text{ onto } \ G),$$

$$\mathrm{R}_0\mathfrak{X} = \Big(G\colon\ \exists N_i \trianglelefteq G \ \ (i = 1, \ldots, r), \ \ G/N_i \in \mathfrak{X} \ \text{ and } \ \bigcap_{i=1}^{r} N_i = 1\Big),$$

$$\mathrm{N}_0\mathfrak{X} = \big(G\colon\ \exists K_i \trianglelefteq\trianglelefteq G \ \ (i = 1, \ldots, r), \ \ K_i \in \mathfrak{X} \ \text{ and } \ G = \langle K_1, \ldots, K_r\rangle\big),$$

$$\mathrm{E}_\phi\mathfrak{X} = \big(G\colon\ \exists N \trianglelefteq G, N \leq \Phi(G) \ \text{ and } \ G/N \in \mathfrak{X}\big), \quad \text{where} \quad \Phi(G) \ \text{ is a Frattini subgroup of } \ G.$$

The class of groups $\mathfrak{X}$ is called τ-*closed* if $\tau\mathfrak{X} = \mathfrak{X}$. If $\mathfrak{X}$ is simultaneously Q-closed and R_0-closed, then the class $\mathfrak{X}$ is called a *formation*. In the case where $\mathfrak{X}$ is simultaneously S_n-closed and N_0-closed, $\mathfrak{X}$ is called a *Fitting class*. The formation $\mathfrak{X}$ is called *saturated* if it is E_ϕ-closed, i.e., the condition $G/\Phi(G) \in \mathfrak{X}$ implies that $G \in \mathfrak{X}$.

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Let $\mathbb{P}$ be the set of all prime numbers. The mappings

$$f\colon \mathbb{P} \to \{\text{formations of groups}\} \qquad \text{and} \qquad h\colon \mathbb{P} \to \{\text{Fitting classes}\}$$

are called a *formation function* [1] (IV, 3.1(a)) and a *Hartley function* (or briefly an *H-function* [2]), respectively.

The symbols $\mathrm{Supp}(f)$ and $\mathrm{Supp}(h)$ denote the sets $\{p \in \mathbb{P}\colon f(p) \neq \varnothing\}$ and $\{p \in \mathbb{P}\colon h(p) \neq \varnothing\}$, respectively, and the symbols $\mathfrak{E}_\pi$, $\mathfrak{N}_p$, and $\mathfrak{E}_{p'}$ stand for the class of all π-groups ($\pi \subseteq \mathbb{P}$), the class of all p-groups, and the class of all p'-groups ($p' = \mathbb{P} \setminus \{p\}$), respectively.

If $\mathfrak{X}$ and $\mathfrak{H}$ are classes of groups, then their product is the class $\mathfrak{X}\mathfrak{H} = (G\colon \exists K \in \mathfrak{X}$ and $G/K \in \mathfrak{X})$.

Let $LF(f)$ and $LR(h)$ be classes of groups

$$\mathfrak{E}_\pi \cap \left(\bigcap_{p\in\pi} \mathfrak{E}_{p'}\mathfrak{N}_p f(p) \right) \qquad \text{and} \qquad \mathfrak{E}_\sigma \cap \left(\bigcap_{p\in\sigma} h(p)\mathfrak{N}_p\mathfrak{E}_{p'} \right),$$

where $\pi = \mathrm{Supp}(f)$ and $\sigma = \mathrm{Supp}(h)$.

Definition 1.1. *The class of groups $\mathfrak{F}$ is called:*

(1) a local formation [1] (IV, 3.1(c)) if $\mathfrak{F} = LF(f)$ for a certain formation function f;

(2) a local Fitting class [2] if $\mathfrak{F} = LR(h)$ for a certain H-function h.

The Gaschütz–Lubeseder–Schmid theorem [3, 4] (see also [1], IV, Theorem 4.6) is a fundamental result from the theory of formations of groups: a nonempty formation $\mathfrak{F}$ is local if and only if $\mathfrak{F}$ is saturated. This result was later developed by Skiba and Shemetkov [5, 6] who characterized partially local formations by means of partial saturation. However, in [7] (Theorem 2.1), it was established that it is impossible to get a dual analog of the Gaschütz–Lubeseder–Schmid theorem in the theory of Fitting classes by applying a subgroup $\Psi(G)$ of the group G dual to the Frattini subgroup of G. We recall that the subgroup $\Psi(G)$ was defined by Ito in [8] and studied by Gaschütz in [9] as a subgroup of the group G generated by all minimal subgroups of G. In this connection, for the characterization of Fitting classes, Doerk and Hauck [7, 10] proposed to use the Frattini duality in the sense of the following definition:

Definition 1.2 ([10], Definition 2.2; see also [1], XI, 6). *Let τ be an operation of closure, let G be a group, and let $\Psi_\tau(G)$ be the least normal subgroup of G such that $\tau(\Psi_\tau(G)\cap M) \supseteq \tau(M)$ for all $M \trianglelefteq\trianglelefteq G$. The Frattini class $\mathfrak{F}$ is called τ-saturated or E^{Ψ_τ}-closed if the condition $\Psi_\tau(G) \in \mathfrak{F}$ always gives $G \in \mathfrak{F}$.*

Note that if τ_1 and τ_2 are operations of closure, then $\tau_1 \leq \tau_2$ if and only if $\tau_1\mathfrak{X} \subseteq \tau_2\mathfrak{X}$ for all groups from the class $\mathfrak{X}$. The class of groups $\mathfrak{X}$ is called *soluble* if $\mathfrak{X} \subseteq \mathfrak{S}$, where $\mathfrak{S}$ is the Fitting class of all soluble groups.

In [1], Doerk and Hawkes formulated the following general problem for the characterization of τ-saturated Fitting classes:

Problem 1.1 [1, p. 829]. *What Fitting classes in the class of groups $\mathfrak{S}$ are τ-saturated for a given operation of closure τ ($\mathrm{S}_n \leq \tau$)?*

The main aim of the present paper is to determine a countable set of families of soluble Fitting classes for which $\tau \leq \mathrm{S}_n$, and each Fitting class of these families is τ-saturated. For the solution of this problem, we use the idea of the Hartley localization [11] and the multiple Skiba localization [12].

Assume that any Fitting class $\mathfrak{F}$ is 0-*tuply local* and that, for natural $m > 0$, the class $\mathfrak{F}$ is called m-*tuply local* [2] if it is defined by an H-function f all values of which are $(m-1)$-tuply local Fitting classes. The Fitting class $\mathfrak{F}$ is called *totally local* if it is n-tuply local for all $n \in \mathbb{N}$.

Definition 1.3. *Let $m \in \mathbb{N}$ and let τ_m be an operation that associates each class of groups $\mathfrak{X}$ with the intersection of all m-tuply local Fitting classes containing $\mathfrak{X}$ that are formations.*

It is easy to see that τ_m is the operation of closure and $\mathrm{S}_n \leq \tau_m$.

The next theorem gives an answer to Problem 1.1 for a countable set of families from the Fitting classes and, in particular, gives a classification of all local Fitting classes that are formations:

Theorem 1.1. *Suppose that $\mathfrak{F}$ is a soluble m-tuply local Fitting class $(m \geq 1)$ and that τ_m is the operation of closure from Definition 1.3. The class $\mathfrak{F}$ is a formation if and only if $\mathfrak{F}$ is τ_m-saturated.*

Corollary 1.1. *A soluble local Fitting class is a formation if and only if it is τ_1-saturated.*

Let τ_∞ be the operation of closure that associates each class of groups $\mathfrak{X}$ with a totally local Fitting class generated by $\mathfrak{X}$. In [2], it was shown that the soluble totally local Fitting class $\mathfrak{F}$ is exactly the Fitting class closed with respect to taking subgroups. Moreover, by the Bryce–Cossey theorem [13], this is equivalent to the statement that $\mathfrak{F}$ is a primitive saturated formation. Thus, Theorem 1.1 yields the following statements:

Corollary 1.2. *The soluble Fitting class $\mathfrak{F}$ is τ_∞-saturated if and only if $\mathfrak{F}$ is totally local.*

Corollary 1.3 (Doerk–Hauck theorem [7], Theorem 2.5)**.** *The soluble Fitting class $\mathfrak{F}$ is τ_∞-saturated if and only if $\mathfrak{F}$ is a formation closed with respect to taking subgroups.*

2. Necessary Information

Let $\mathfrak{X}$ be a class of groups and let τ be an operation of closure. By E^{Ψ_τ} we denote a class

$$(G\colon \exists N \trianglelefteq\trianglelefteq G \text{ such that } \Psi_\tau(G) \leq N \in \mathfrak{X})$$

(see [1], Definition XI, 6.10). If $\mathfrak{X} = \{G\}$, then we denote $\tau\{G\}$ simply by τG.

By the definition of formation (Fitting class), for any group G, one can find the least (greatest) normal subgroup $G^{\mathfrak{F}}$ $(G_{\mathfrak{F}})$ such that $G/G^{\mathfrak{F}} \in \mathfrak{F}$ $(G_{\mathfrak{F}} \in \mathfrak{F})$ in G. This subgroup is called a $\mathfrak{F}$-coradical ($\mathfrak{F}$-radical) of G, respectively. If $\mathfrak{F}$ and $\mathfrak{H}$ are Fitting classes, then the class $\mathfrak{F}\mathfrak{H} = (G\colon G/G_{\mathfrak{F}} \in \mathfrak{H})$ is the product of $\mathfrak{F}$ and $\mathfrak{H}$. It is known that the product $\mathfrak{F}\mathfrak{H}$ is a Fitting class and that the operation of multiplication of Fitting classes is associative (see [1], Theorem IX, 1.12(a), (c)).

The group G is called *comonolithic* if it has a unique maximal normal subgroup called a *comonolith* of G.

We repeatedly use the following properties of comonolithic groups established in the universe $\mathfrak{S}$ by Doerk [14]:

Lemma 2.1 ([14], Lemmas 1–3)**.** *The following assertions are true:*

(1) if $N \trianglelefteq G$, G/N is a comonolithic group and S is a minimal subnormal complement to N in G, then S is a comonolithic group;

(2) if N_1 and N_2 are normal subgroups of the group G such that $N_1N_2 \neq G$, $N_1 \cap N_2 = 1$, G/N_i, $i = 1, 2$, is a comonolithic group, and S is a minimal subnormal complement to N_1N_2 in G, then S is a comonolithic group such that $S/S \cap N_i \cong G/N_i$ for $i = 1, 2$. In addition, if G/N_1N_2 is a p-group, then $S/(S \cap N_i)(S \cap N_i)$ is a nontrivial cyclic p-group;

(3) if $\mathfrak{F}$ is a Fitting class, p is a prime number, G_1 is a group such that $(G_1)_\mathfrak{F}$ is a comonolith with index p in the group G_1, and G_2 is a comonolithic non-p-perfect group belonging to $\mathfrak{F}$, then there exists a comonolithic group S with the following properties:

(a) S has normal subgroups S_1 and S_2 such that $S_1 \cap S_2 = 1$, S_1S_2/S_2 is a cyclic nontrivial p-group, and $S/S_i \cong G_i$ for $i = 1, 2$;

(b) $S_\mathfrak{F}$ is a maximal normal subgroup of S with index p.

We also use the Lockett operation $*$ defined in [15].

Recall that if $\mathfrak{F}$ is a nonempty Fitting class, then the operation $*$ associates $\mathfrak{F}$ with the least Fitting class $\mathfrak{F}^*$ containing $\mathfrak{F}$ and such that

$$(G \times H)_{\mathfrak{F}^*} = G_{\mathfrak{F}^*} \times H_{\mathfrak{F}^*}$$

for all groups G and H. The Fitting class $\mathfrak{F}$ is called a Lockett class if $\mathfrak{F} = \mathfrak{F}^*$.

Lemma 2.2 ([16], Lemma 5)**.** *Every local Fitting class is a Lockett class.*

Lemma 2.3 (improved version of the quasi-R_0-lemma; [1], Theorem X, 1.24)**.** *The following assertions are equivalent:*

(1) $\mathfrak{F}$ is a Lockett class;

(2) for each group G with normal subgroups N_1 and N_2 such that G/N_1N_2 is a nilpotent group, the following condition is satisfied:

$$G \in \mathfrak{F} \Leftrightarrow G/N_1 \quad \text{and} \quad G/N_2 \in \mathfrak{F}.$$

By $G \,\mathrm{wr}\, H$ we denote the regular wreath product of the groups G and H. If $K \leq G$, then by K^* we denote a subgroup of the basis group $G \,\mathrm{wr}\, H$ isomorphic to the direct product $|H|$ of multipliers of the group K. By Z_n we denote a cyclic group of order n.

Lemma 2.4 {[1], A, 18.11(a)}**.** *If $p \in \mathbb{P}$, $n \in \mathbb{N}$, and $W = Z_{p^{n-1}} \,\mathrm{wr}\, Z_p$, then W contains a subnormal subgroup isomorphic to Z_{p^n}.*

Lemma 2.5 {[1], X, 2.1(a)}**.** *Suppose that $\mathfrak{F}$ is a Lockett class and G is a group. If $G \notin \mathfrak{F}$, then*

$$(G \,\mathrm{wr}\, H)_\mathfrak{F} = (G_\mathfrak{F})^*$$

for all groups H.

Lemma 2.6 {[1], Lemma A, 18.2}**.** *Suppose that $W = G \,\mathrm{wr}\, H$. If $K \trianglelefteq G$, then $K^* \trianglelefteq W$ and*

$$W/K^* \cong (G/K) \,\mathrm{wr}\, H.$$

If $\mathfrak{X}$ is a class of groups, then by $\sigma(\mathfrak{X})$ and $\mathrm{Char}(\mathfrak{X})$ we denote, respectively, the set of all prime divisors of all groups from $\mathfrak{X}$ and the characteristic of the class $\mathfrak{X}$, i.e., the set $\{p \in \mathbb{P} \colon Z_p \in \mathfrak{X}\}$.

Lemma 2.7. *If $\mathfrak{F} = LR(f)$ for some H-function f with support of the function π, then the following assertions are true:*

(1) $\pi = \sigma(\mathfrak{F}) = \mathrm{Char}(\mathfrak{F})$ *[18] (Lemma 2.3);*

(2) $LR(f) = LR(f^*)$, *where f^* is an H-function such that $f^*(p) = (f(p))^*$ for all $p \in \pi$ [17] (Theorem 1).*

3. Proof of Theorem 1.1

Lemma 3.1. *For any local Fitting class $\mathfrak{F}$ and any comonolithic group $G \in \mathfrak{F}$ with comonolith M of index p in G, the regular wreath product $G \,\mathrm{wr}\, Z_p \in \mathfrak{F}$.*

Proof. Let G be a comonolithic $\mathfrak{F}$-group with comonolith M of index p in G and let $W = G \,\mathrm{wr}\, Z_p$. Since $\mathfrak{F}$ is a local Fitting class, we have

$$G \in \mathfrak{E}_\pi \cap \left(\bigcap_{p \in \pi} f(p)\mathfrak{N}_p\mathfrak{E}_{p'} \right),$$

where $\pi = \mathrm{Supp}(f)$.

It is clear that $O^{p'}(G) \neq 1$. If $O^{p'}(G) \lneqq G$ and M is the comonolith of the group G, then $O^{p'}(G) \leq M$ and the index $|G : M|$ is a p'-number, which contradicts the choice of the group G. Hence, $G = O^{p'}(G)$ for all $p \in \pi$ and, by virtue of the assertion (1) of Lemma 2.7, we get $W \in \mathfrak{E}_\pi$.

It remains to show that $W \in f(p)\mathfrak{N}_p\mathfrak{E}_{p'}$ for all $p \in \pi$.

Since $G \in \mathfrak{F}$, we have $G \in f(p)\mathfrak{N}_p\mathfrak{E}_{p'}$ for any $p \in \pi$. According to the assertion (2) of Lemma 2.7 and Theorem X.1.8(a) in [7], we get

$$f(p)\mathfrak{N}_p\mathfrak{E}_{p'} = f^*(p)\mathfrak{N}_p\mathfrak{E}_{p'}.$$

Thus, without loss of generality, we can assume that f is an H-function $\mathfrak{F}$ such that $f(p)$ is a Lockett class for any $p \in \pi$.

If $G \in f(p)$, then it is clear that $W \in f(p)\mathfrak{N}_p\mathfrak{E}_{p'}$. Let $G \in f(p)\mathfrak{N}_p \setminus f(p)$. Then, by virtue of Lemma 2.5, we get $W_{f(p)} = (G_{f(p)})^*$. Hence, by Lemma 2.6,

$$W/W_{f(p)} \cong (G/G_{f(p)}) \,\mathrm{wr}\, Z_p \qquad \text{and} \qquad W \in f(p)\mathfrak{N}_p \subseteq f(p)\mathfrak{N}_p\mathfrak{E}_{p'}.$$

Let

$$G \in f(p)\mathfrak{N}_p\mathfrak{E}_{p'} \setminus f(p)\mathfrak{N}_p.$$

Similarly, by using Lemmas 2.5 and 2.6, we obtain

$$W/W_{f(p)\mathfrak{N}_p} \cong (G/G_{f(p)\mathfrak{N}_p}) \,\mathrm{wr}\, Z_p.$$

By using the condition $G = O^{p'}(G) \neq 1$, we get

$$W \in f(p)\mathfrak{N}_p\mathfrak{E}_{p'}.$$

Hence, $W \in f(p)\mathfrak{N}_p\mathfrak{E}_{p'}$ for all $p \in \pi$ and $W \in \mathfrak{F}$.

The lemma is proved.

Proof of Theorem 1.1. Let $\mathfrak{F}$ be an m-tuply local Fitting class, which is a formation. We prove that $\mathfrak{F}$ is τ_m-saturated. Let G be a group of the least order such that $\Psi_{\tau_m}(G) \in \mathfrak{F}$ and $G \notin \mathfrak{F}$ and let M be any maximal normal subgroup of G.

We now show that $\Psi_{\tau_m}(M) \leq \Psi_{\tau_m}(G)$. Let $K \trianglelefteq \trianglelefteq M$. Then it is clear that $K \trianglelefteq \trianglelefteq G$. Therefore,

$$\tau_m K \leq \tau_m(K \cap \Psi_{\tau_m}(G)) = \tau_m(K \cap (M \cap \Psi_{\tau_m}(G))).$$

This implies that

$$\Psi_{\tau_m}(M) \leq M \cap \Psi_{\tau_m}(G) \leq \Psi_{\tau_m}(G)$$

and, hence, $\Psi_{\tau_m}(M) \in \mathfrak{F}$. By induction, $M \in \mathfrak{F}$ and $M = G_{\mathfrak{F}}$. Thus, the group G is comonolithic.

Since $G \in \tau_m G$ and $G_{\mathfrak{F}} \trianglelefteq G$, we have $G_{\mathfrak{F}} \in \tau_m G$ and $\tau_m G_{\mathfrak{F}} \subseteq \tau_m G$.

If $\tau_m G_{\mathfrak{F}} = \tau_m G$, then $\tau_m G \subseteq \tau_m \mathfrak{F} = \mathfrak{F}$. Therefore, $G \in \mathfrak{F}$, which contradicts the choice of the group G.

Let $\tau_m G_{\mathfrak{F}} \subsetneqq \tau_m G$. Then $G = \Psi_{\tau_m}(G) \in \mathfrak{F}$. The obtained contradiction completes the proof of the fact that the Fitting class $\mathfrak{F}$ is τ_m-saturated.

We prove the converse assertion. Let $\mathfrak{F}$ be a τ_m-saturated m-tuply local Fitting class. It is necessary to show that $\mathfrak{F}$ is a formation, i.e., the class $\mathfrak{F}$ is simultaneously Q-closed and R_0-closed.

We first prove that $\mathfrak{F}$ is a Q-*closed Fitting class.*

Assume that $\mathfrak{F}$ is not a Q-closed class. Let G be a group of the least order such that $G \in \mathfrak{F}$ and $G/K \notin \mathfrak{F}$ for a certain normal subgroup K of the group G. Then there exists a subnormal subgroup H/K in G/K all proper normal subgroups of which are $\mathfrak{F}$-groups. Let L/K be a $\mathfrak{F}$-radical of the group H/K. According to the choice of G, we can assume that $L = G$. Hence, the group G/K is comonolithic and has the comonolith $(G/K)_{\mathfrak{F}}$. In view of the solubility of G/K, the index $|G/K : (G/K)_{\mathfrak{F}}| = p$, where $p \in \mathbb{P}$. Let S be the minimal subnormal complement to K in G. Then it follows from $G \in \mathfrak{F}$ that $S \in \mathfrak{F}$. Hence, by the assertion (1) of Lemma 2.1, the group S is comonolithic. Moreover, $G/K \cong S/S \cap K \notin \mathfrak{F}$. Due to the choice of the group G, we can assume that $S = G$. Therefore, G is a comonolithic $\mathfrak{F}$-group. Since $G \in \tau_m G$ and the Fitting class $\tau_m G$ is Q-closed, $G/K \in \tau_m G$ and the inclusion

$$\tau_m(G/K) \subseteq \tau_m G \tag{3.1}$$

is true.

Let M be a comonolith of the group G. Then $\tau_n M \subseteq \tau_n G$. Assume that the following equality holds:

$$\tau_n M = \tau_n G. \tag{3.2}$$

Let $\overline{G} = G/K$. It is clear that the group $\overline{G}$ is comonolithic, $\overline{G}_{\mathfrak{F}}$ is its comonolith and, moreover, the index $|\overline{G} : \overline{G}_{\mathfrak{F}}| = p$ for a certain prime p. In addition, G is a comonolithic non-p-perfect $\mathfrak{F}$-group. Hence, according to the assertion (3) of Lemma 2.1, there exists a comonolithic group R with the following properties:

(a) R has normal subgroups R_1 and R_2 such that $R_1 \cap R_2 = 1$, R/R_1R_2 is a cyclic nontrivial p-group, $R/R_1 \cong G$, $R/R_2 \cong \overline{G}$, and $R_{\mathfrak{F}}/R_1 \cong M$, $R_{\mathfrak{F}}/R_1 \cong \overline{G}_{\mathfrak{F}}$;

(b) $R_{\mathfrak{F}}$ is a maximal normal subgroup with index p of the group R.

Then $R/R_1 \in \tau_m G$ and, according to (3.1), $R/R_2 \in \tau_m G$.

On the other hand, in view of the property (a), we conclude that G is a homomorphic image of the group R. Hence,

$$G \in \mathrm{Q}(\tau_m R) = \tau_m R \qquad \text{and} \qquad \tau_m G \subseteq \tau_m R.$$

Thus, the equality $\tau_m R = \tau_m G$ is true. Reasoning similarly, from $M \in \mathrm{Q}R_{\mathfrak{F}} \subseteq \mathrm{Q}(\tau_m R_{\mathfrak{F}}) = \tau_m R_{\mathfrak{F}}$, we obtain

$$\tau_m M \subseteq \tau_m R_{\mathfrak{F}}.$$

Further, according to proposition (3.2), we get

$$\tau_m G = \tau_m M \subseteq \tau_m R_{\mathfrak{F}} \subseteq \tau_m R = \tau_m G.$$

Therefore, $\tau_m R_{\mathfrak{F}} = \tau_m R$ and $\Psi_{\tau_m}(R) \leq R_{\mathfrak{F}}$. Hence, $\Psi_{\tau_m}(R) \in \mathfrak{F}$. Since the Fitting class $\mathfrak{F}$ is τ_m-saturated, we conclude that $R \in \mathfrak{F}$, which contradicts the property (b). This proves that equality (3.2) is impossible. Hence, the inclusion

$$\tau_n M \subsetneqq \tau_n(G) \tag{3.3}$$

is true.

Now let $W = G \operatorname{wr} Z_p$. Then $M^* = M \times \ldots \times M$ is a subgroup of the basis group G^* of the group W. In this case, $M = G_{\tau_m M}$ is the maximal normal subgroup with index p in G. Since $M \trianglelefteq G$, by Lemma 2.6 we have $W/M^* \cong (G/M) \operatorname{wr} Z_p$. Hence,

$$W/M^* \cong Z_p \operatorname{wr} Z_p.$$

Further, applying Lemma 2.4, we conclude that the wreath product $Z_p \operatorname{wr} Z_p$ has a cyclic subgroup Z_{p^2} such that the intersection of the basis group $Z_p \operatorname{wr} Z_p$ with Z_{p^2} is a cyclic group of order p. Let $\overline{Z}_{p^2}$ be the complete preimage of Z_{p^2} in W. Since $W/M^* \in \mathfrak{N}_p \subseteq \mathfrak{N}$, we have $\overline{Z}_{p^2} \trianglelefteq\trianglelefteq W$. In addition, by virtue of the isomorphism $W/M^* \cong Z_p \operatorname{wr} Z_p$, we conclude that $\overline{Z}_{p^2}/M^* \cong Z_{p^2}$ and $(\overline{Z}_{p^2} \cap G^*)/M^*$ is a subgroup of order p of the group $\overline{Z}_{p^2}/M^*$. Since $G \in \tau_m G$ and $\tau_m G$ is a local Fitting class, by Lemma 2.5, we get $W \in \tau_m G$. By using $\overline{Z}_{p^2} \trianglelefteq\trianglelefteq W$, we obtain $\overline{Z}_{p^2} \in \tau_m G$. Therefore, $\tau_m \overline{Z}_{p^2} \subseteq \tau_n G$.

On the other hand, in view of Lemma 2.2 and the locality of the class $\tau_m \overline{Z}_{p^2}$, this is a Lockett class. Since $\overline{Z}_{p^2} \nsubseteq G^*$, by applying Lemma 2.5, we obtain $W \in \tau_m \overline{Z}_{p^2}$. Hence, the group $G \in \tau_m \overline{Z}_{p^2}$ and the inclusion $\tau_m G \subseteq \tau_m \overline{Z}_{p^2}$ is true. This means that the equality

$$\tau_m \overline{Z}_{p^2} = \tau_m G \tag{3.4}$$

is proved.

As already shown, $W \in \tau_m G$. Hence, $\tau_m W \subseteq \tau_m G$. Since $\overline{Z}_{p^2} \trianglelefteq\trianglelefteq W$, we get $\tau_m \overline{Z}_{p^2} \subseteq \tau_m W$. Further, by using (3.4), we arrive at the equality

$$\tau_m \overline{Z}_{p^2} = \tau_m W. \tag{3.5}$$

Let F be the minimal subnormal complement to M^* in the group $\overline{Z}_{p^2}$. Then $\tau_m F \subseteq \tau_m \overline{Z}_{p^2}$. Assume that $F \subseteq G^*$. In this case, $\overline{Z}_{p^2} \subseteq G^*$, which is impossible because $(\overline{Z}_{p^2} \cap G^*)/M^*$ is a subgroup of order p in $\overline{Z}_{p^2}/M$. Hence, $F \nleq G^*$. Since the class $\tau_m F$ is local, by Lemma 2.2, $\tau_m F$ is a Lockett class. Therefore, according to Lemma 2.5, $W \in \tau_m F$. Thus, by using (3.5), we arrive at the inclusion $\tau_m \overline{Z}_{p^2} \subseteq \tau_m F$. Hence, the equalities

$$\tau_m F = \tau_m \overline{Z}_{p^2} = \tau_m G \tag{3.6}$$

are true.

Since $\overline{Z}_{p^2}/M^*$ is a comonolithic group, by the assertion (1) of Lemma 2.1, the complement F in M^* is also a comonolithic group. In view of the isomorphism $F/F \cap M^* \cong \overline{Z}_{p^2}/M^*$, the group $F/F \cap M^*$ is cyclic of order p^2 and $F \cap G^*$ is the maximal normal subgroup of F.

Further, we prove the validity of the equality

$$\tau_m F = \tau_m(F \cap G^*). \tag{3.7}$$

Let $(\overline{Z}_{p^2})_{\tau_m(F\cap G^*)} \nleq G^*$. Note that, by Lemma 2.2, $\tau_m(F \cap G^*)$ is a Lockett class. Hence, by Lemma 2.5, we get $W \in \tau_m(F\cap G^*)$. Since $\overline{Z}_{p^2} \trianglelefteq\trianglelefteq W$, we have $\overline{Z}_{p^2} \in \tau_m(F\cap G^*)$. Thus, by using $F \trianglelefteq\trianglelefteq \overline{Z}_{p^2}$, we conclude that $F \in \tau_m(F \cap G^*)$ and, therefore, $\tau_m F \subseteq \tau_m(F \cap G^*)$. Since the reverse inclusion is obvious, equality (3.7) is true in this case.

Assume that

$$(\overline{Z}_{p^2})_{\tau_m(F\cap G^*)} \leq G^*.$$

If $(\overline{Z}_{p^2})_{\tau_m(F\cap G^*)} = G^*$, then we arrive at a contradiction with $G^* \nleq \overline{Z}_{p^2}$. Therefore,

$$(\overline{Z}_{p^2})_{\tau_m(F\cap G^*)} \leq (G^*)_{\tau_m M} = M^* < G.$$

Note that the indicated relation is obtained by using the reasoning according to which, by Lemma 2.2, the radicals of direct products of the groups of local Fitting classes coincide with the direct products of radicals of these groups for these classes and, hence, there exists a one-to-one correspondence between the radicals of the groups G and G^*.

However, $F \cap G^* \nleq M^*$ and, therefore, the case $(\overline{Z}_{p^2})_{\tau_m(F\cap G^*)} \leq M^*$ is impossible.

Thus, $(\overline{Z}_{p^2})_{\tau_m(F\cap G^*)} \nleq G^*$ and equality (3.7) is proved. Applying equality (3.5), we get

$$\Psi_{\tau_m}(F) \leq F \cap G^* \in \mathfrak{F}.$$

With regard for the τ_m-saturation of $\mathfrak{F}$, we conclude that $F \in \mathfrak{F}$.

According to (3.6), we replace $\tau_m G$ in equality (3.1) with $\tau_m F$ and equality (3.2) with equality (3.7). Further, for the groups $\overline{G}$ and F, we use the reasoning similar to the reasoning used for the groups $\overline{G}$ and G. As a result, in view of the assertion (3) of Lemma 2.1, we construct the comonolithic group $\widetilde{R}$, which is not a $\mathfrak{F}$-group. However, $\Psi_{\tau_m}(\widetilde{R}) \in \mathfrak{F}$ and, hence, in view of the τ_m-saturation of $\mathfrak{F}$, we get $\widetilde{R} \in \mathfrak{F}$. The obtained contradiction completes the proof of Q-closeness of the Fitting class $\mathfrak{F}$.

We now prove that $\mathfrak{F}$ *is an* R_0*-closed Fitting class.*

Let G be a counterexample of the minimal order. Then there exist normal subgroups K_1 and K_2 in G such that $K_1 \cap K_2 = 1$, $G/K_i \in \mathfrak{F}$, and $G \notin \mathfrak{F}$, $i = 1, 2$.

Let $K_1K_2 < G$. In this case, we show that the group G is comonolithic with comonolith $G_{\mathfrak{F}}$ of index p. Assume that L/K_1 is a maximal normal subgroup of the group G/K_1. Then $L/K_1 \in \mathfrak{F}$. In addition, in view of the isomorphism $L/L \cap K_2 \cong LK_2/K_2$, the group $L/L \cap K_2 \in \mathfrak{F}$. Hence, by induction, we get

$$L/K_1 \cap L \cap K_2 = L \in \mathfrak{F}.$$

If another maximal normal subgroup L_1/K_1 exists in G/K_1, then, by using the same reasoning, we conclude that $L_1 \in \mathfrak{F}$. In this case, $G = L_1L_2 \in \mathfrak{F}$. The obtained contradiction proves that the group G/K_1 is comonolithic. Similarly, we conclude that G/K_2 is a comonolithic group.

Let H be the minimal subnormal complement to K_1K_2 in G. Since the group G/K_i, $i = 1, 2$, is comonolithic, by the assertion (2) of Lemma 2.1, the group H is comonolithic and $H/H \cap K_i \cong G/K_i$. Since $|K_1K_2| < |G|$ and $K_1K_2/K_i \in \mathfrak{F}$, $i = 1, 2$, by induction, we get $K_1K_2/K_i \in \mathfrak{F}$. In this case, the condition $G \notin \mathfrak{F}$ implies that $H \notin \mathfrak{F}$. Since $H/H \cap K_i \in \mathfrak{F}$, in view of the minimality of the choice of the group G, we prove that $H = G$ and G is a comonolithic group with comonolith $G_{\mathfrak{F}}$ of index p in G.

In view of the assertion (3) of Lemma 2.1, for the comonolithic groups G/K_1 and G/K_2, there exists a comonolithic group M with two maximal normal subgroups M_1 and M_2 such that $M_1 \cap M_2 = 1$, M/M_1M_2 is a nontrivial cyclic p-group, and $M/M_i \cong G/K_i$ for $i = 1, 2$. Since $G/K_i \in \mathfrak{F}$, by the quasi-R_0-lemma (see [1], Theorem IX, 1.13), we conclude that $M \in \mathfrak{F}$. Moreover, in view of Lemma 2.2, we prove that $\tau_m M$ is a Lockett class. Hence, according to Lemma 2.3 (improved version of the quasi-R_0-lemma), we find

$$G/K_i \cong M/M_i \in \tau_m M, \qquad i = 1, 2.$$

Since $G \in \tau_m M$, the inclusion

$$\tau_m G \subseteq \tau_m M \tag{3.8}$$

is true.

Thus, we have proved that G is a comonolithic group with comonolith $G_{\mathfrak{F}}$ of index p in G and M is a comonolithic $\mathfrak{F}$-group.

Further, following the proof of Q-closure of the class $\mathfrak{F}$ with obvious changes and replacements of (3.1) with (3.8) and the groups $\overline{G}$ with G, G with M, we arrive at a contradiction with the τ_m-saturation of the class $\mathfrak{F}$.

Assume that $G = K_1K_2$. In this case, $K_2 \cong G/K_1$ and $K_2 \cong G/K_2$. Hence, $G = G/K_1 \cap K_2 \in \mathfrak{F}$. The obtained contradiction completes the proof of the theorem.

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