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On Injectors of a Hartley Set of a Finite Group*

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Abstract. Let G be a finite group and $\mathcal{H}$ be a *Hartley set* of G. In this paper, we prove the existence and conjugacy of $\mathcal{H}$-injectors of G and describe the characterization of injectors via radicals. As applications, some known results are directly followed.

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1 Introduction

Throughout this paper all groups are finite. In the theory of classes of finite soluble groups, a basic result which generalizes the fundamental theorems of Sylow and Hall is the theorem of Fischer, Gaschütz and Hartley [5] on the existence and conjugacy of $\mathfrak{F}$-injectors in soluble groups for every Fitting class $\mathfrak{F}$.

Recall that a class $\mathfrak{F}$ is called a *Fitting class* if $\mathfrak{F}$ is closed under taking normal subgroups and products of normal $\mathfrak{F}$-subgroups. For any nonempty class $\mathfrak{F}$ of groups, a subgroup V of a group G is said to be $\mathfrak{F}$-*maximal* if $V \in \mathfrak{F}$ and $U = V$ whenever $V \leq U \leq G$ and $U \in \mathfrak{F}$. From the definition of Fitting class $\mathfrak{F}$, every group G has the largest normal $\mathfrak{F}$-subgroup $G_{\mathfrak{F}}$, the so-called $\mathfrak{F}$-*radical* of G, which is the product of all normal $\mathfrak{F}$-subgroups. In particular, if $\mathfrak{F} = \mathfrak{N}$ is the Fitting class of all nilpotent groups, then $G_{\mathfrak{N}} = F(G)$ is the Fitting subgroup of G. A subgroup V of

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a group G is said to be an $\mathfrak{F}$-*injector* of G if $V \cap N$ is an $\mathfrak{F}$-maximal subgroup of N for every subnormal subgroup N of G. Note that if $\mathfrak{F} = \mathfrak{N}_p$ is the Fitting class of all p-groups, then the $\mathfrak{F}$-injectors of a group G are Sylow p-subgroups of G; if $\mathfrak{F}$ is the Fitting class of all groups with soluble Hall π-subgroups (i.e., G is an E_π^s-group [6, p. 81]), where π is a set of prime numbers, then the $\mathfrak{F}$-injectors of G are Hall π-subgroups of G.

As a development of the theorem of Fischer, Gaschütz and Hartley [5], Shemetkov [14] (resp., Anderson [1]) proved that if G is a π-soluble group (resp., soluble group) and $\mathcal{F}$ is a Fitting set of G, then G posseses exactly one conjugacy class of $\mathcal{F}$-injectors, where π is the set of all primes dividing orders of all subgroups of G in $\mathcal{F}$.

Recall that a nonempty set $\mathcal{F}$ of subgroups of a group G is called a *Fitting set* of G [5, 14] if the following three conditions hold: (i) If $T \trianglelefteq S \in \mathcal{F}$, then $T \in \mathcal{F}$; (ii) If $S \in \mathcal{F}$, $T \in \mathcal{F}$, $S \trianglelefteq ST$ and $T \trianglelefteq ST$, then $ST \in \mathcal{F}$; (iii) If $S \in \mathcal{F}$ and $x \in G$, then $S^x \in \mathcal{F}$. Hence from the definition of Fitting set $\mathcal{F}$, the $\mathcal{F}$-radical $G_{\mathcal{F}}$ of a group G can also be defined as the product of all its normal $\mathcal{F}$-subgroups. For a Fitting set $\mathcal{F}$ of G, the $\mathcal{F}$-injector of G is similarly defined as the $\mathfrak{F}$-injector for Fitting class $\mathfrak{F}$ (see [1, Definition VIII(2.5)]).

If $\mathfrak{F}$ is a Fitting class and G is a group, then the set $\{H \leq G : H \in \mathfrak{F}\}$ is a Fitting set, which is denoted by $\mathrm{Tr}_{\mathfrak{F}}(G)$ and called the *trace* of $\mathfrak{F}$ in G (see [3, VIII, 2.2(a)]). Note that for a Fitting class $\mathfrak{F}$, the $\mathfrak{F}$-injectors and $\mathrm{Tr}_{\mathfrak{F}}(G)$-injectors of G coincide, but not every Fitting set of G is the trace of a Fitting class (see [3, VIII, Examples (2.2)(c)]). Hence, if $\mathcal{F} = \mathrm{Tr}_{\mathfrak{F}}(G)$, then the theorem of Anderson [1] and the theorem of Fischer, Gaschütz and Hartley [5] are corollaries of the theorem of Shemetkov [14]. Vorob'ev and Semenov [16] proved that for every set π of primes and every Fitting set $\mathcal{F}$ of a π-soluble group G, G possesses an $\mathcal{F}$-injector and any two $\mathcal{F}$-injectors are conjugate if $\mathcal{F}$ is π-saturated, i.e., $\mathcal{F} = \{H \leq G : H/H_{\mathcal{F}} \in \mathfrak{E}_{\pi'}\}$. In connection with these theorems, the following question naturally arise:

Question 1.1. For an arbitrary Fitting set $\mathcal{F}$ of a group G (in the case where G is a non-π-soluble group), when does G possess an $\mathcal{F}$-injector and any two $\mathcal{F}$-injectors are conjugate?

There has been substantial research on the characterizations of $\mathfrak{F}$-injector for various types of soluble Fitting classes $\mathfrak{F}$ (see [4, 7–10, 12–14]). It is well known that the product of any two Fitting classes is also a Fitting class and the multiplication of Fitting classes satisfies the associative law (see [3, Theorem IX.(1.12)(a) and (c)]). Hartley [8] proved that for the Fitting class of type $\mathfrak{X}\mathfrak{N}$ (where $\mathfrak{X}$ is a nonempty Fitting class and $\mathfrak{N}$ is the Fitting class of all nilpotent groups), a subgroup V of a soluble group G is an $\mathfrak{X}\mathfrak{N}$-injector of G if and only if $V/G_{\mathfrak{X}}$ is a nilpotent subgroup of G. As a further improvement, Guo and Vorob'ev [7] proved that for a Hartley class $\mathfrak{H}$, the set of all $\mathfrak{H}$-injectors of a soluble group G coincides with the set of all $\mathfrak{H}$-maximal subgroups of G containing the $\mathfrak{H}$-radical of G. Let $\mathbb{P}$ be the set of all prime numbers. Following [8, p. 201], a function $h : \mathbb{P} \to \{\text{nonempty Fitting classes}\}$ is a *Hartley function* (or for brevity, *H-function*). Let $LH(h) = \bigcap_{p\in\mathbb{P}} h(p)\mathfrak{S}_{p'}\mathfrak{N}_p$, where $\mathfrak{N}_p$ is the class of all p-groups and $\mathfrak{S}_{p'}$ is the class of all soluble p'-groups. A Fitting class $\mathfrak{H}$ is called a *Hartley class* if $\mathfrak{H} = LH(h)$ for some H-function h.

We need to develop and extend the local method of Hartley [8] (for soluble Fitting classes) for Fitting sets of groups (not necessary in the universe of soluble groups). For a Fitting set $\mathcal{H}$ of a group G and a nonempty Fitting class $\mathfrak{F}$, we call the set $\{H \leq G : H/H_{\mathcal{H}} \in \mathfrak{F}\}$ of subgroups of G the *product* of $\mathcal{H}$ and $\mathfrak{F}$, and denote it by $\mathcal{H} \circ \mathfrak{F}$, which is a Fitting set of G (see Lemma 2.1).

Following [15], a function $h : \mathbb{P} \to \{\text{Fitting sets of } G\}$ is called a Hartley function of G (or for brevity, an H-function of G).

Definition 1.2. Let h be an H-function of a group G and

$$HS(h) = \bigcap_{p\in\mathbb{P}} h(p) \circ (\mathfrak{E}_{p'}\mathfrak{N}_p),$$

where $\mathfrak{E}_{p'}$ is the class of all p'-groups. A Fitting set $\mathcal{H}$ of G is called a *Hartley set* of G if $\mathcal{H} = HS(h)$ for some H-function h.

Definition 1.3. Let $\mathcal{H} = HS(h)$ be a Hartley set of a group G. Then h is said to be

(a) *integrated* if $h(p) \subseteq \mathcal{H}$ for all p;
(b) *full* if $h(p) \subseteq h(q) \circ \mathfrak{E}_{q'}$ for all different primes p and q;
(c) *full integrated* if h is full and integrated as well.

It is easy to see that every Hartley set of a group G can be defined by an integrated H-function. Moreover, we prove that every Hartley set of G can be defined by a full integrated H-function in Lemma 3.4.

In connection with the above, the following question naturally arise.

Question 1.4. Let G be a group (in particular, G can be a soluble group) and $\mathcal{H}$ be a Hartley set of G. What is the structure of $\mathcal{H}$-injectors of G?

For a Hartley set $\mathcal{H} = HS(h)$ of G, where h is a full integrated H-function of $\mathcal{H}$, we call the subgroup $G_h = \prod_{p\in\mathbb{P}} G_{h(p)}$ the *h-radical* of G. A group G is said to be $\mathfrak{N}$-*constrained* if $C_G(F(G)) \leq F(G)$. It is well known that if G is soluble, then G is constrained (in general, the converse is not true [11]).

The following theorem resolves Questions 1.1 and 1.4.

Theorem 1.5. *Let $\mathcal{H}$ be a Hartley set of a group G defined by a full integrated H-function h, and G_h the h-radical of G. If G/G_h is $\mathfrak{N}$-constrained, then the following statements hold:*

(a) *A subgroup V of G is an $\mathcal{H}$-injector of G if and only if V/G_h is a nilpotent injector of G/G_h.*
(b) *G possesses an $\mathcal{H}$-injector and any two $\mathcal{H}$-injectors are conjugate in G.*
(c) *A subgroup V of G is an $\mathcal{H}$-injector of G if and only if V is an $\mathcal{H}$-maximal subgroup of G and $G_{\mathcal{H}} \leq V$.*

Theorem 1.5 gives a new theory of $\mathcal{F}$-injectors for Fitting sets of non-soluble groups. From Theorem 1.5, a series of famous results can be directly generalized, for example Fischer [4, Corollary IX.4.13], Hartley [8, Section 4.1], Mann [12, Theorem IX.4.12], and Guo and Vorob'ev [6, Theorem 5.6.8].

All unexplained notion and terminology are standard. The reader is referred to [2, 3, 6].

2 Preliminaries

Note that if all groups in a class $\mathfrak{X}$ are soluble groups (that is, $\mathfrak{X} \subseteq \mathfrak{S}$), then $\mathfrak{X}$ is said to be a soluble class.

Lemma 2.1. [17, Proposition 3.1] *Let $\mathcal{F}$ be a Fitting set of a group G and $\mathfrak{X}$ be a nonempty Fitting class. Then the product $\mathcal{F} \circ \mathfrak{X}$ is a Fitting set of G.*

Lemma 2.2. *Let $\mathcal{F}$ and $\mathcal{H}$ be Fitting sets of G, and $\mathfrak{X}$ and $\mathfrak{Y}$ be nonempty Fitting formations. Then the following statements hold:*

(a) [17, Proposition 3.4(3)] $\mathcal{F} \circ (\mathfrak{X} \cap \mathfrak{Y}) = \mathcal{F} \circ \mathfrak{X} \cap \mathcal{F} \circ \mathfrak{Y}$.

(b) [17, Proposition 3.2(1)] *If $\mathfrak{M}$ is a nonempty Fitting class, then $\mathcal{F} \subseteq \mathcal{F} \circ \mathfrak{M}$.*

(c) [17, Proposition 3.4(2)] $(\mathcal{F} \cap \mathcal{H}) \circ \mathfrak{X} = \mathcal{F} \circ \mathfrak{X} \cap \mathcal{H} \circ \mathfrak{X}$.

(d) [17, Proposition 3.4(1)] *If $\mathcal{F} \subseteq \mathcal{H}$, then $\mathcal{F} \circ \mathfrak{X} \subseteq \mathcal{H} \circ \mathfrak{X}$.*

Lemma 2.3. [17, Proposition 3.3] *Let $\mathcal{F}$ be a Fitting set of a group G, and $\mathfrak{X}$ and $\mathfrak{Y}$ be Fitting formations. Then $(\mathcal{F} \circ \mathfrak{X}) \circ \mathfrak{Y} = \mathcal{F} \circ (\mathfrak{X}\mathfrak{Y})$.*

Lemma 2.4. [3, Theorem IV.(1.8)] *Let $\mathfrak{F}$ and $\mathfrak{H}$ be nonempty formations. If $\mathfrak{F} \subseteq \mathfrak{H}$, then $G^{\mathfrak{H}} \leq G^{\mathfrak{F}}$ for every group G.*

Lemma 2.5. [3, Proposition VIII.(2.4)(d)] *Let $\mathcal{F}$ be a Fitting set of a group G. If $N \trianglelefteq\trianglelefteq G$, then $N_{\mathcal{F}} = N \cap G_{\mathcal{F}}$.*

Let $\mathfrak{F}$ be a nonempty Fitting class. A group G is said to be $\mathfrak{F}$-constrained if $C_G(G_{\mathfrak{F}}) \leq G_{\mathfrak{F}}$.

Lemma 2.6. [3, Remark, p. 624] or [11] *The class of all $\mathfrak{N}$-constrained groups is a Fitting class strictly larger than $\mathfrak{S}$.*

Lemma 2.7. [3, Theorem IX.(4.12)(c)–(d)] *Let G be a group. If G is $\mathfrak{N}$-constrained, then G possesses exactly one conjugacy class of nilpotent injectors.*

The following properties follow at once from the definition of an $\mathcal{F}$-injector of a group G and [3, Remarks IX.(1.3) and VIII.(2.6)–(2.7)].

Lemma 2.8. *Let $\mathcal{F}$ be a Fitting set of a group G and $\mathfrak{F}$ a class of finite groups. Then the following statements hold*

(a) *If $K \trianglelefteq\trianglelefteq G$ and V is an $\mathcal{F}$-injector of G, then $V \cap K$ is an $\mathcal{F}$-injector (or $\mathcal{F}_K$-injector) of K.*

(b) *If V is an $\mathcal{F}$-injector of G, then $G_{\mathcal{F}} \leq V$ and V is an $\mathcal{F}$-maximal subgroup of G.*

(c) *If V is an $\mathcal{F}$-maximal subgroup of G and $V \cap M$ is an $\mathcal{F}$-injector for any maximal normal subgroup M of G, then V is an $\mathcal{F}$-injector of G.*

(d) *If $V \in \mathrm{Inj}_{\mathfrak{F}}(G)$ and $\alpha : G \to G_\alpha$ is an isomorphism, then $V_\alpha \in \mathrm{Inj}_{\mathfrak{F}}(G_\alpha)$; in particular, $\mathrm{Inj}_{\mathfrak{F}}(G)$ is a union of G-conjugacy classes.*

Lemma 2.9. [12] or [3, Theorem IX.(4.12)] *If G is a $\mathfrak{N}$-constrained group, then a subgroup V of G is a nilpotent injector of G if and only if $F(G) \le V$ and V is an $\mathfrak{N}$-maximal subgroup of G.*

3 Hartley Set and *h*-Radical

In this section we give some results about Hartley sets and the h-radical of a group G, which are also main steps in the proof of Theorem 1.5. Recall that for a Fitting set $\mathcal{H}$ of G and a nonempty Fitting class $\mathfrak{F}$, the set $\mathcal{H} \circ \mathfrak{F} = \{H \le G : H/H_{\mathcal{H}} \in \mathfrak{F}\}$ is a Fitting set of G by Lemma 2.1. Firstly, we give some examples of Hartley sets.

Example 3.1. (a) Let $\mathcal{N}$ be the trace of the Fitting class $\mathfrak{N}$ in a group G, and let h be an H-function defined as follows: $h(p) = \{1\}$ for all $p \in \mathbb{P}$, where 1 is an identity subgroup of G. Then by Lemma 2.2(a), we have

$$HS(h) = \bigcap_{p\in\mathbb{P}} \{1\} \circ (\mathfrak{E}_{p'}\mathfrak{N}_p) = \{1\} \circ \Big(\bigcap_{p\in\mathbb{P}} \mathfrak{E}_{p'}\mathfrak{N}_p \Big) = \{1\} \circ \mathfrak{N} = \mathcal{N}.$$

Hence, the set of all nilpotent subgroups of G is a Hartley set of G.

(b) Let $\mathcal{F}$ be a Fitting set of G and define $\mathcal{H} = \mathcal{F} \circ \mathfrak{N}$. Suppose that h is an H-function such that $h(p) = \mathcal{F}$ for all $p \in \mathbb{P}$. Thus, by Lemma 2.2(a) we obtain $HS(h) = \mathcal{F} \circ \big(\bigcap_{p\in\mathbb{P}} \mathfrak{E}_{p'}\mathfrak{N}_p\big) = \mathcal{F} \circ \mathfrak{N}$, and so $\mathcal{H}$ is a Hartley set of G (the last equality follows from [3, Lemma II.(2.7)(a)]).

(c) If $k \in \mathbb{N}$, then let $\mathcal{N}^k$ $(k \ge 1)$ be the set of all subgroups of a soluble group G of nilpotent length at most k. If $k \ge 1$, then we take the H-function h such that $h(p) = \mathrm{Tr}_{\mathfrak{N}^{k-1}}(G)$ for all $p \in \mathbb{P}$. Thus by the example (b) above, we know that $HS(h) = \mathcal{N}^k$ is a Hartley set of G.

(d) Let $\mathcal{H}$ be the trace of Fitting class $\mathfrak{E}_{p'}\mathfrak{N}_p$ in a group G, i.e., $\mathcal{H}$ is the set of all p-nilpotent subgroups of G. Let h be an H-function defined as $h(p) = \{1\}$ and $h(q) = \mathcal{H}$ for all primes $q \neq p$. Then by Lemma 2.2(a), we have

$$HS(h) = (\{1\} \circ \mathfrak{E}_{p'}\mathfrak{N}_p) \cap \Big(\mathcal{H} \circ \Big(\bigcap_{p\neq q} \mathfrak{E}_{p'}\mathfrak{N}_p\Big)\Big) = \mathcal{H} \cap \mathcal{H} \circ (\mathfrak{N}_p\mathfrak{N}_{p'}).$$

Now by Lemma 2.2(b), $HS(h) = \mathcal{H}$ and $\mathcal{H}$ is a Hartley set.

Lemma 3.2. *Every Hartley set can be defined by an integrated H-function.*

Proof. Let $\mathcal{H}$ be a Hartley set of a group G. Then $\mathcal{H} = HS(h_1)$ for some H-function h_1. Let h be an H-function defined as follows: $h(p) = h_1(p) \cap \mathcal{H}$ for all $p \in \mathbb{P}$. By Lemma 2.2(c), we have

$$HS(h) = \bigcap_{p\in\mathbb{P}} (h_1(p) \cap \mathcal{H}) \circ \mathfrak{E}_{p'}\mathfrak{N}_p = \Big(\bigcap_{p\in\mathbb{P}} h_1(p) \circ \mathfrak{E}_{p'}\mathfrak{N}_p\Big) \cap \Big(\bigcap_{p\in\mathbb{P}} \mathcal{H} \circ (\mathfrak{E}_{p'}\mathfrak{N}_p)\Big).$$

Hence by Lemma 2.2(a)–(b), $HS(h) = \mathcal{H} \cap \mathcal{H} \circ \big(\bigcap_{p\in\mathbb{P}} \mathfrak{E}_{p'}\mathfrak{N}_p\big) = \mathcal{H} \cap \mathcal{H} \circ \mathfrak{N} = \mathcal{H}$. The lemma is proved. □

Let $\mathcal{H}$ be a set of subgroups of a group G. For $\mathcal{H}$ and a nonempty Fitting class $\mathfrak{F}$, we call the set $\{H \le G : H$ has a normal subgroup $L \in \mathcal{H}$ with $H/L \in \mathfrak{F}\}$ of subgroups of G the product of $\mathcal{H}$ and $\mathfrak{F}$, and denote it by $\mathcal{H}\mathfrak{F}$.

Remark 3.3. Let $\mathcal{H}$ be a Fitting set of a group G. Then it is clear that $\mathcal{H} \circ \mathfrak{F} \subseteq \mathcal{H}\mathfrak{F}$.

Assume that $\mathfrak{F}$ is the Fitting class and $\mathfrak{F}$ is a homomorph, i.e., $\mathfrak{F}$ is quotient closed. Put $H \leq G$ and $H \in \mathcal{H}\mathfrak{F}$. Then $H/L \in \mathfrak{F}$ for some normal subgroup $L \in \mathcal{H}$ of H. Since $L \leq H_{\mathcal{H}}$ and $H/L/H_{\mathcal{H}}/L \cong H/H_{\mathcal{H}}$, we have $H \in \mathcal{H} \circ \mathfrak{F}$. Thus, $\mathcal{H}\mathfrak{F} = \mathcal{H} \circ \mathfrak{F}$.

Let G be a group and $\mathcal{Y}$ be a set of subgroups of a group G. Then $\mathrm{Fitset}\,\mathcal{Y}$ will denote the intersection of all Fitting sets of G which contain $\mathcal{Y}$ (see [3, Definition VIII.3.1(b)]).

Lemma 3.4. *Every Hartley set can be defined by a full integrated H-function.*

Proof. Let $\mathcal{H}$ be a Hartley set of a group G. By Lemma 3.2, $\mathcal{H} = HS(h_1)$ for some integrated H-function h_1. We define a set of subgroups of G by

$$\overline{h_1}(p) = \{H \leq G : H \text{ is conjugate with } R^{\mathfrak{S}_{p'}} \text{ in } G \text{ for some } R \in h_1(p)\}$$

for all $p \in \mathbb{P}$. Note that if $H \in \overline{h_1}(p)$, then $H \in h_1(p)$ and so $\overline{h_1}(p) \subseteq h_1(p)$ for all $p \in \mathbb{P}$.

Assume that $X \in \overline{h_1}(p)\mathfrak{E}_{p'}$. Then X has a normal subgroup $K \in \overline{h_1}(p)$ with $X/K \in \mathfrak{E}_{p'}$. Since $\overline{h_1}(p) \subseteq h_1(p)$, we have $K \leq X_{h_1(p)}$. Hence by the isomorphism $X/K/X_{h_1(p)}/K \cong X/X_{h_1(p)}$, we obtain $X/X_{h_1(p)} \in \mathfrak{E}_{p'}$ and so $X \in h_1(p) \circ \mathfrak{E}_{p'}$. Thus, $\overline{h_1}(p)\mathfrak{E}_{p'} \subseteq h_1(p)\mathfrak{E}_{p'}$.

On the other hand, let $Y \in h_1(p) \circ \mathfrak{E}_{p'}$. Then $Y/Y_{h_1(p)} \in \mathfrak{E}_{p'}$ and $Y^{\mathfrak{E}_{p'}} \leq Y_{h_1(p)}$. Hence, $Y^{\mathfrak{E}_{p'}} \in h_1(p)$. Since $(Y^{\mathfrak{E}_{p'}})^{\mathfrak{E}_{p'}} = Y^{\mathfrak{E}_{p'}}$, and obviously, $Y^{\mathfrak{E}_{p'}}$ is a subgroup conjugate with $Y^{\mathfrak{E}_{p'}}$ in G, we have $Y^{\mathfrak{E}_{p'}} \in \overline{h_1}(p)$ and so $Y \in \overline{h_1}(p)\mathfrak{E}_{p'}$. Thus, we obtain the following equation:

$$\overline{h_1}(p)\mathfrak{E}_{p'} = h_1(p) \circ \mathfrak{E}_{p'}. \tag{$*$}$$

Let h be a function such that $h(p) = \mathrm{Fitset}(\overline{h_1}(p))$ for all $p \in \mathbb{P}$. We prove $HS(h) = \mathcal{H}$. Since $\overline{h_1}(p) \subseteq h_1(p)$, $\mathrm{Fitset}(\overline{h_1}(p)) \subseteq \mathrm{Fitset}(h_1(p)) = h_1(p)$ and so $h(p) \subseteq h_1(p)$. Hence by Lemma 2.2(d), $h(p) \circ \mathfrak{E}_{p'} \subseteq h_1(p) \circ \mathfrak{E}_{p'}$. By Lemma 2.3, $(h(p) \circ \mathfrak{E}_{p'}) \circ \mathfrak{N}_p = h(p) \circ (\mathfrak{E}_{p'}\mathfrak{N}_p)$ and $(h_1(p) \circ \mathfrak{E}_{p'}) \circ \mathfrak{N}_p = h_1(p) \circ (\mathfrak{E}_{p'}\mathfrak{N}_p)$. Therefore by Lemma 2.2(d), $h(p) \circ (\mathfrak{E}_{p'}\mathfrak{N}_p) \subseteq h_1(p) \circ (\mathfrak{E}_{p'}\mathfrak{N}_p)$ for all $p \in \mathbb{P}$. Consequently, $HS(h) \subseteq \mathcal{H}$.

Furthermore, by $(*)$ we have $h_1(p) \circ \mathfrak{E}_{p'} = \mathrm{Fitset}(\overline{h_1}(p)\mathfrak{E}_{p'})$. Since we have $\overline{h_1}(p) \subseteq \mathrm{Fitset}(\overline{h_1}(p))$, by Lemma 2.2(d) $\overline{h_1}(p)\mathfrak{E}_{p'} \subseteq (\mathrm{Fitset}(\overline{h_1}(p))) \circ \mathfrak{E}_{p'}$. Hence, $\mathrm{Fitset}(\overline{h_1}(p)\mathfrak{E}_{p'}) \subseteq (\mathrm{Fitset}(\overline{h_1}(p))) \circ \mathfrak{E}_{p'}$. Thus, for every $p \in \mathbb{P}$ we get the inclusion

$$\overline{h_1}(p)\mathfrak{E}_{p'} \subseteq (\mathrm{Fitset}(\overline{h_1}(p))) \circ \mathfrak{E}_{p'} = h(p) \circ \mathfrak{E}_{p'}. \tag{$**$}$$

By Lemmas 2.3 and 2.2, we have $h_1(p) \circ (\mathfrak{E}_{p'}\mathfrak{N}_p) \subseteq h(p) \circ (\mathfrak{E}_{p'} \circ \mathfrak{N}_p)$. Hence, $\mathcal{H} \subseteq HS(h)$ and $\mathcal{H} = HS(h)$.

Since $\overline{h_1}(p) \subseteq h_1(p)$ for all $p \in \mathbb{P}$ and h_1 is an integrated H-function of $\mathcal{H}$, $h(p) \subseteq \mathcal{H}$ for all $p \in \mathbb{P}$ and so h is an integrated H-function of $\mathcal{H}$.

Now, we show $h(p) \subseteq h(q) \circ \mathfrak{E}_{q'}$ for all $p \neq q$.

Let H be an arbitrary subgroup in $h_1(p)$ and $p \neq q$. Since h_1 is an integrated H-function, it follows that $H \in \mathcal{H}$ and so $H \in h_1(q) \circ (\mathfrak{E}_{q'}\mathfrak{N}_q)$. By Lemma 2.3, $h_1(q) \circ (\mathfrak{E}_{q'}\mathfrak{N}_q) = (h_1(q) \circ \mathfrak{E}_{q'}) \circ \mathfrak{N}_q$. Hence, $H^{\mathfrak{N}_q} \in h_1(q) \circ \mathfrak{E}_{q'}$. Since $p \neq q$, $H^{\mathfrak{E}_{p'}} \leq H^{\mathfrak{N}_q}$ by Lemma 2.4. Consequently, $H^{\mathfrak{E}_{p'}} \in h_1(q)\mathfrak{E}_{q'}$. Therefore by $(**)$, $H^{\mathfrak{E}_{p'}} \in h(q) \circ \mathfrak{E}_{q'}$ for every $H \in h_1(p)$ and all primes $p \neq q$.

Let $R \in \overline{h_1}(p)$. By the definition of the set $\overline{h_1}(p)$, R is a conjugate subgroup of G with $S^{\mathfrak{E}_{p'}}$ for some subgroup $S \in h_1(p)$. Therefore $R \in h(q)\mathfrak{E}_{q'}$, and so $h_1(p) \subseteq h(q) \circ \mathfrak{E}_{q'}$. Thus, $h(p) = \mathrm{Fitset}(\overline{h_1}(p)) \subseteq \mathrm{Fitset}(h(q) \circ \mathfrak{E}_{q'}) = h(q) \circ \mathfrak{E}_{q'}$ for all primes $q \neq p$. This completes the proof. □

Recall that $G_h = \prod_{p\in\mathbb{P}} G_{h(p)}$, where h is a full integrated H-function of a Hartley set of a group G, i.e., h is an h-radical of G.

Lemma 3.5. *Let $\mathcal{H}$ be a Hartley set of a group G, and h be a full integrated H-function of $\mathcal{H}$. If H is a subgroup of G such that $G_h \leq H$ and H/G_h is a nilpotent subgroup of G/G_h, then $H \in \mathcal{H}$.*

Proof. Let q be an arbitrary prime number. Since $G_{h(q)} \trianglelefteq H$, we have $G_{h(q)} \leq H_{h(q)}$ for all $q \in \mathbb{P}$. Let $p \in \mathbb{P}$ and $p \neq q$. Note that

$$G_{h(q)}G_{h(p)}/G_{h(q)} \cong G_{h(p)}/G_{h(q)} \cap G_{h(p)} = G_{h(p)}/(G_{h(p)})_{h(q)}.$$

Since h is a full integrated H-function of $\mathcal{H}$, it follows that $h(p) \subseteq h(q)\mathfrak{E}_{q'}$. Hence, $G_{h(p)} \in h(q)\mathfrak{E}_{q'}$. Therefore, $G_{h(q)}G_{h(p)}/G_{h(q)} \in \mathfrak{E}_{q'}$ for all primes p and q. Consequently, $G_h/G_{h(q)} \in \mathfrak{E}_{q'}$, and by using the isomorphisms

$$H_{h(q)}G_h/H_{h(q)} \cong G_h/H_{h(q)} \cap G_h \cong (G_h/G_{h(q)})/((H_{h(q)} \cap G_h)/G_{h(q)}),$$

we see that $H_{h(q)}G_h/H_{h(q)}$ is a q'-group. As H/G_h is a nilpotent subgroup of G/G_h, $H/H_{h(q)}G_h$ is also a nilpotent subgroup of G/G_h, so $H/H_{h(q)}G_h \in \bigcap_{p\in\mathbb{P}} \mathfrak{E}_{q'}\mathfrak{N}_q$. Therefore, by the isomorphism $H/H_{h(q)}G_h \cong (H/H_{h(q)})/(H_{h(q)}G_h/H_{h(q)})$ we have $H/H_{h(q)} \in \mathfrak{E}_{q'}\mathfrak{N}_q$ for all $q \in \mathbb{P}$. Hence, $H \in \bigcap_{q\in\mathbb{P}} h(q)\mathfrak{E}_{q'}\mathfrak{N}_q = \mathcal{H}$. The lemma is proved. □

Lemma 3.6. *Let $\mathcal{H}$ be a Hartley set of a group G. If h is a full integrated H-function of $\mathcal{H}$, then $G_{\mathcal{H}}/G_h = F(G/G_h)$.*

Proof. Let $F(G/G_h) = R/G_h$. Since h is a integrated H-function of $\mathcal{H}$, we obtain $(G_{\mathcal{H}})_{h(p)} = G_{h(p)}$. Hence, $G_{\mathcal{H}}/G_{h(p)} \in \mathfrak{E}_{p'}\mathfrak{N}_p$ for all $p \in \mathbb{P}$ and so $G_{\mathcal{H}}/G_h$ is a nilpotent subgroup of G/G_h. Thus, $G_{\mathcal{H}}/G_h \leq F(G/G_h)$ and we get $G_{\mathcal{H}} \leq R$.

On the other hand, since R/G_h is a nilpotent subgroup of G/G_h, by Lemma 3.5 $R \in \mathcal{H}$. Hence, $R \leq G_{\mathcal{H}}$. So $F(G/G_h) = G_{\mathcal{H}}/G_h$. The lemma is proved. □

Lemma 3.7. *Let h be a full integrated H-function of a Hartley set $\mathcal{H}$ of a group G. If G/G_h is $\mathfrak{N}$-constrained, $G_{\mathcal{H}} \leq H \leq G$ and $H \in \mathcal{H}$, then H/G_h is nilpotent.*

Proof. Since $G_{\mathcal{H}} \trianglelefteq H$ and h is an integrated H-function of $\mathcal{H}$, by Lemma 2.5, $G_{h(p)} = (G_{\mathcal{H}})_{h(p)} = H_{h(p)} \cap G_{\mathcal{H}}$. Hence, $[H_{h(p)}, G_{\mathcal{H}}] \leq H_{h(p)} \cap G_{\mathcal{H}} = G_{h(p)}$ and so $H_{h(p)} \leq C_G(G_{\mathcal{H}}/G_{h(p)}) \leq C_G(G_{\mathcal{H}}/G_h)$. Since G/G_h is constrained and $G_{\mathcal{H}}/G_h = F(G/G_h)$ by Lemma 3.6, $C_G(G_{\mathcal{H}}/G_h) \leq G_{\mathcal{H}}$. Thus, $H_{h(p)} \leq G_{\mathcal{H}}$. Therefore, $G_{h(p)} = (G_{\mathcal{H}})_{h(p)} = H_{h(p)} \cap G_{\mathcal{H}} = H_{h(p)}$ for all $p \in \mathbb{P}$. Hence, we have $H/G_h = H/H_h \in \mathfrak{E}_{p'}\mathfrak{N}_p$ for all $p \in \mathbb{P}$, so H/G_h is a nilpotent group. This completes the proof. □

Corollary 3.8. *Let h be a full integrated H-function of a Hartley set of a group G. Let G/G_h be an $\mathfrak{N}$-constrained group and $G_{\mathcal{H}} \leq H \leq G$. Then $H \in \mathcal{H}$ if and only if H/G_h is nilpotent.*

4 Proof and Some Applications of Theorem 1.5

Proof of Theorem 1.5. (a) We first prove that if V is a subgroup of a group G such that V/G_h is a nilpotent injector of G/G_h, then V is an $\mathcal{H}$-injector of G. For this, we use induction on the order of G.

Assume that M is an arbitrary maximal normal subgroup of G and M_h is an h-radical of M. Since h is a full integrated H-function of Hartley set $\mathcal{H}$, we can obtain $h(p) \subseteq h(q) \circ \mathfrak{E}_{q'}$ and $h(p) \subseteq \mathcal{H}$ for all different $p, q \in \mathbb{P}$. Then by the isomorphism $G_{h(q)}G_{h(p)}/G_{h(q)} \cong G_{h(p)}/G_{h(p)} \cap G_{h(q)} = G_{h(p)}/(G_{h(p)})_{h(q)}$, we see that $G_{h(q)}G_{h(p)}/G_{h(q)}$ is a q'-group of $G/G_{h(q)}$ for any prime q. Hence, $G_h/G_{h(q)}$ is also a q'-group. Since $(G_h \cap M)G_{h(q)}/G_{h(q)} \le G_h/G_{h(q)}$, by the isomorphism $(G_h \cap M)G_{h(q)}/G_{h(q)} \cong (G_h \cap M)/(G_h \cap M) \cap G_{h(q)} = (G_h \cap M)/G_{h(q)} \cap M$ and Lemma 2.5, we see that $(G_h \cap M)/M_{h(q)}$ is a q'-group for all $q \in \mathbb{P}$. Now note that $(G_h \cap M)/M_{h(q)}/M_h/M_{h(q)} \cong (G_h \cap M)/M_h$. Hence $(G_h \cap M)/M_h \in \bigcap_{q \in \mathbb{P}} \mathfrak{E}_{q'} = (1)$, and so $G_h \cap M = M_h$. If $G_h \le M$. Then $G_h = M_h$.

Since V/G_h is a nilpotent injector of G/G_h, $(V \cap M)/G_h$ is a nilpotent injector of M/G_h, and consequently $(V \cap M)/M_h$ is a nilpotent injector of M/M_h. Hence by induction, $V \cap M$ is an $\mathcal{H}$-injector of M.

Now, to complete the proof of the statement, we only need to prove that V is an $\mathcal{H}$-maximal subgroup of G. Since V/G_h is nilpotent and $G_{\mathcal{H}} \le V$, by Lemma 3.5 $V \in \mathcal{H}$. Assume that $V < V_1$, where V_1 is an $\mathcal{H}$-maximal subgroup of G. Since $V \cap M$ is an $\mathcal{H}$-maximal subgroup of M, $V \cap M = V_1 \cap M$. Hence V_1 is an $\mathcal{H}$-maximal subgroup of G, and $V \cap M$ is an $\mathcal{H}$-injector of M for any maximal normal subgroup M of G. Consequently, by Lemma 2.8(b)–(c), V_1 is an $\mathcal{H}$-injector of G and $G_{\mathcal{H}} \le V_1$. Thus, by Corollary 3.8 we know that V_1/G_h is a nilpotent subgroup of G/G_h, contrary to the fact that V/G_h is $\mathfrak{N}$-maximal in G/G_h. Hence $V = V_1$, and so by Lemma 2.8 (c), V is an $\mathcal{H}$-injector of G.

Assume that $G_h \not\le M$. In this case, by the maximality of M we have $G = G_hM$. Since $G/G_h \cong M/G_h \cap M = M/M_h$ by Lemma 2.8(d), $(V \cap M)/M_h$ is a nilpotent injector of M/M_h. Then by induction, we see that $V \cap M$ is an nilpotent injector of M. By Lemma 3.5, $V \in \mathcal{H}$. If $V < F_1$, where F_1 is an $\mathcal{H}$-maximal subgroup of G, then $V \cap M = F_1 \cap M$. Since $G_h \le V$, we get $VM = G$. Consequently, $F_1 = F_1 \cap VM = V(F_1 \cap M) = V(V \cap M) = V$ and so V is an $\mathcal{H}$-maximal subgroup of G. Therefore, V is an $\mathcal{H}$-injector of G.

Conversely, let V be an $\mathcal{H}$-injector of G. We prove that V/G_h is a nilpotent injector of G/G_h. By Lemma 2.8(b)–(c), $G_{\mathcal{H}} \le V$ and V is $\mathcal{H}$-maximal in G. Hence by Lemma 3.5, V/G_h is nilpotent. Since V is $\mathcal{H}$-maximal in G, we see that V/G_h is an $\mathfrak{N}$-maximal subgroup of G/G_h containing the nilpotent radical of G/G_h. Consequently by Lemma 2.9, V/G_h is a nilpotent injector of G/G_h. Thus, statement (a) holds.

(b) The existence of $\mathcal{H}$-injectors has been proved in (a). Let F/G_h and V/G_h be nilpotent injectors of G/G_h. Then by Lemma 2.7, F/G_h and V/G_h are conjugate in G/G_h. Hence, F and V are conjugate in G.

(c) Let V be an $\mathcal{H}$-injector of G. Then by Lemma 2.8(b)–(c), $V \ge G_{\mathcal{H}}$ and V is $\mathcal{H}$-maximal in G.

Conversely, let V be an $\mathcal{H}$-maximal subgroup of G and $V \ge G_h$. We prove that

V is an $\mathcal{H}$-injector of G. Clearly, $G_h \leq V$. Then by Lemma 3.5, V/G_h is nilpotent. Since V is $\mathcal{H}$-maximal in G, V/G_h is $\mathfrak{N}$-maximal in G/G_h. Now by Lemma 3.6, $V/G_h \geq F(G/G_h)$. Hence by Lemma 2.9, V/G_h is a nilpotent $\mathcal{H}$-injector of G/G_h. This completes the proof of the theorem. □

Let $\mathcal{X}$ be a Fitting set of a group G and $\mathcal{X} \circ \mathfrak{S} = \{H \leq G : H/H_{\mathcal{X}} \in \mathfrak{S}\}$, where $\mathfrak{S}$ is the class of all soluble groups. Note that the set $\mathcal{X} \circ \mathfrak{S}$ is a Fitting set by Lemma 2.1.

We give some applications of our main result:

Corollary 4.1. *Let $G \in \mathcal{X} \circ \mathfrak{S}$ and $\mathcal{H} = \mathcal{X} \circ \mathfrak{N}$ be a Fitting set of G. Then*

(1) *A subgroup V of G is an $\mathcal{H}$-injector of G if and only if $V/G_{\mathcal{X}}$ is an nilpotent injector of $G/G_{\mathcal{X}}$.*

(2) *The set of all $\mathcal{H}$-injectors of G is exactly the set of subgroups V of G such that $V \geq G_{\mathcal{H}}$ and V is $\mathcal{H}$-maximal in G.*

Proof. By Example 3.1(b), $\mathcal{H}$ is a Hartley set of G, which can be defined by a full integrated H-function h such that $h(p) = \mathcal{X}$ for all $p \in \mathbb{P}$. Since $G/G_{\mathcal{X}}$ is soluble, $G/G_{\mathcal{X}}$ is $\mathfrak{N}$-constrained. □

Let $\mathfrak{X}$ be a nonempty Fitting class and $\mathfrak{H} = \mathfrak{X}\mathfrak{N}$ be a Fitting product of $\mathfrak{X}$ and $\mathfrak{N}$. Then we have the following result immediately from our theorem.

Corollary 4.2. (a) *A subgroup V of a group $G \in \mathfrak{F}\mathfrak{S}$ is an $\mathfrak{H}$-injector of G if and only if $V/G_{\mathfrak{X}}$ is a nilpotent injector of $G/G_{\mathfrak{X}}$.*

(b) *A subgroup V of a group $G \in \mathfrak{F}\mathfrak{S}$ is an $\mathfrak{H}$-injector if and only if $V \geq G_{\mathfrak{H}}$ and V is $\mathfrak{H}$-maximal in G.*

Proof. Since the set of all $\mathfrak{X}\mathfrak{N}$-injectors of G and the set of all $\mathrm{Tr}_{\mathfrak{X}\circ\mathfrak{N}}(G)$-injectors of G coincide, Corollary 4.2 holds from Corollary 4.1. □

Corollary 4.3. [8] *Let $\mathfrak{X}$ be a nonempty soluble Fitting class and $\mathfrak{H} = \mathfrak{X}\mathfrak{N}$. A subgroup V of soluble group G is an $\mathfrak{H}$-injector if and only if $V/G_{\mathfrak{X}}$ is a nilpotent injector of $G/G_{\mathfrak{X}}$.*

Corollary 4.4. [4] *A subgroup V of soluble group G is a nilpotent injector of G if and only if $F(G) \leq V$ and V is $\mathfrak{N}$-maximal in G.*

Corollary 4.5. [7] *Let $\mathfrak{H}$ be a soluble Hartley class and G be a soluble group. Then a subgroup V of G is an $\mathfrak{H}$-injector of G if and only if V/G_h is a nilpotent injector of G/G_h.*

Corollary 4.6. *Let $\mathcal{N}^k$ $(k \geq 1)$ be the Fitting set of all subgroups of a soluble group G with nilpotent length at most k. Then the set of all $\mathcal{N}^k$-injectors of G is exactly the set of all subgroups V of G such that $V/G_{\mathcal{N}^{k-1}}$ is a nilpotent injector of $G/G_{\mathcal{N}^{k-1}}$. In particular, a subgroup V is an $\mathcal{N}^2$-injector of G if and only if $V/F(G)$ is a nilpotent injector of $G/F(G)$.*

Proof. By Example 3.1(c), $\mathcal{N}^k$ is a Hartley set of G and the function h such that $h(p) = \mathrm{Tr}_{\mathfrak{N}^{k-1}}(G)$ for all $p \in \mathbb{P}$ is a full integrated H-function of $\mathcal{N}^k$. Since the group G is soluble, G/G_h is $\mathfrak{N}$-constrained and so the assertion holds from our theorem. □

Corollary 4.7. [7] *Let $\mathfrak{N}^k$ $(k \geq 1)$ be the class of all groups with nilpotent length at most k and G a soluble group. Then the set of all $\mathfrak{N}^k$-injectors of G is exactly the set of all subgroups V of G such that $V/G_{\mathfrak{N}^{k-1}}$ is a nilpotent injector of $G/G_{\mathfrak{N}^{k-1}}$. In particular, a subgroup V of a soluble group G is a metanilpotent injector if and only if $V/F(G)$ is a nilpotent injector of $G/F(G)$.*

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