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ON THE PROBLEM OF EXISTENCE AND CONJUGACY OF INJECTORS OF PARTIALLY π-SOLUBLE GROUPS

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Abstract: We prove the existence and conjugacy of injectors of partially π-soluble groups for the Hartley class defined by an invariable Hartley function, and give description of the structure of the injectors.

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1. Introduction

Throughout this paper, all groups are finite and p is a prime. Also, G always denotes a group and $|G|$ is the order of G, while $\sigma(G)$ is the set of all primes dividing $|G|$ and π stands for some set of primes. Let $\mathbb{P}$ be the set of all primes, and $\pi' = \mathbb{P} \setminus \pi$. Let G_π denote a Hall π-subgroup of G.

Recall that a class $\mathfrak{F}$ of groups is called a *Fitting class* if $\mathfrak{F}$ is closed under normal subgroups and products of normal $\mathfrak{F}$-subgroups. As usual, we denote by $\mathfrak{E}$, $\mathfrak{S}$, and $\mathfrak{N}$ the classes of all groups, all soluble groups, and all nilpotent groups; $\mathfrak{E}_\pi$, $\mathfrak{S}_\pi$, and $\mathfrak{N}_\pi$ denote the classes of all π-groups, all soluble π-groups, and all nilpotent π-groups; and $\mathfrak{S}^\pi$ and $\mathfrak{N}^\pi$ denote the classes of all π-soluble groups and all π-nilpotent groups. It is well known that the above classes of groups are Fitting classes.

From the definition of Fitting class, we see that for every nonempty Fitting class $\mathfrak{F}$ each group G has a unique maximal normal $\mathfrak{F}$-subgroup called the $\mathfrak{F}$-*radical* of G and denoted by $G_\mathfrak{F}$.

Given a nonempty Fitting class $\mathfrak{F}$ of groups, a subgroup V of G is said to be

(1) $\mathfrak{F}$-*maximal* in G if $V \in \mathfrak{F}$ and $U = V$ whenever $V \leq U \leq G$ and $U \in \mathfrak{F}$;

(2) a $\mathfrak{F}$-*injector* of G if $V \cap K$ is a $\mathfrak{F}$-maximal subgroup of K for every subnormal subgroup K of G.

The importance of the theory of Fitting classes can firstly be seen in the following theorem by Fischer, Gaschütz, and Hartley [1] which is in fact a generalization of the classical Sylow and Hall Theorems.

Theorem 1.1 (see [1] or [2, Theorem VIII.2.8]). *Let $\mathfrak{F}$ be a nonempty Fitting class. Then a soluble group possesses exactly one conjugacy class of $\mathfrak{F}$-injectors.*

Note that if $\mathfrak{F} = \mathfrak{N}_p$ is the Fitting class of all p-groups, then the $\mathfrak{F}$-injectors of G are Sylow p-subgroups of G. If $\mathfrak{F} = \mathfrak{E}_\pi$ and G has a Hall π-subgroup, then the $\mathfrak{F}$-injectors of G are Hall π-subgroups of G (see [3, Example 1, p. 68] or [4, p. 238]).

About the existence of $\mathfrak{X}$-injectors of G, Shemetkov posed the following

Problem 1.2 [5, Problem 11.117]. *Let $\mathfrak{X}$ be a Fitting class of soluble groups. Is it true that each finite nonsoluble group possesses a $\mathfrak{X}$-injector?*

This problem was solved in [6, 7] for the Fitting classes $\mathfrak{X} \in \{\mathfrak{S}, \mathfrak{S}_\pi, \mathfrak{N}\}$.

So, the next problem is interesting: Find the conjugate class of $\mathfrak{X}$-injectors in π-nonsoluble (in particular, π-soluble) groups. The first result in this direction is the famous Chunikhin Theorem [8]: *A π-soluble group has Hall π-subgroups* (*i.e., $\mathfrak{E}_\pi$-injectors*) *and every two Hall π-subgroups are conjugate.*

Let $\mathfrak{F}$ be a class of groups. We will denote the set of all distinct prime divisors of all groups of $\mathfrak{F}$ by $\sigma(\mathfrak{F})$. As a development of Chunikhin's Theorem, Shemetkov and Guo proved the following

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Theorem 1.3 [9, Theorem 2.2; 10]. *Let $\mathfrak{F}$ be a Fitting class, and $\pi = \sigma(\mathfrak{F})$. If $G/G_{\mathfrak{F}}$ is π-soluble, then G has a $\mathfrak{F}$-injector and every two $\mathfrak{F}$-injectors of G are conjugate in G.*

The *product* $\mathfrak{F}\mathfrak{H}$ of two Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ is the class $(G : G/G_{\mathfrak{F}} \in \mathfrak{H})$. It is well known that the product of every two Fitting classes is also a Fitting class and the multiplication of Fitting classes satisfies the associative law (see [2, Theorem IX, 1.12(a),(c)]).

Following [11, 12], $f : \mathbb{P} \longrightarrow \{\text{Fitting classes}\}$ is called a *Hartley function* (or in brevity *H-function*). A Fitting class $\mathfrak{F}$ is *local* if

$$\mathfrak{F} = \mathfrak{E}_{\sigma(\mathfrak{F})} \cap \Big(\bigcap_{p\in\sigma(\mathfrak{F})} f(p)\mathfrak{N}_p\mathfrak{E}_{p'} \Big)$$

for some H-function f.

Given an H-function h, put $\pi = \mathrm{Supp}(h) := \{p \in \mathbb{P} : h(p) \neq \varnothing\}$ and call π the *support* of h, and $LH(h) = \bigcap_{p\in\pi} h(p)\mathfrak{E}_{p'}\mathfrak{N}_p$. Then a Fitting class $\mathfrak{F}$ is called the *Hartley class* if there exists an H-function h such that $\mathfrak{H} = LH(h)$. In this case, $\mathfrak{H}$ is said to be *defined by the H-function h or h is an H-function of* $\mathfrak{H}$.

Clearly, $\mathfrak{N} \subseteq LH(h)$. Hence, if $\mathfrak{F} = LH(h)$, then $\sigma(\mathfrak{F}) = \mathbb{P}$. Each Hartley class is a local Fitting class (see [13, p. 31]). But the converse is not true in general (see [11, p. 207, 4.2]).

In [14] (see also [11]) there was formulated

Problem 1.4. *Let $\mathfrak{F}$ be a local Fitting class of soluble groups. What is the structure of $\mathfrak{F}$-injectors of a soluble group?*

In connection with Problem 1.2 and Theorem 1.3, the following generalized variant of Problem 1.4 arises naturally:

Problem 1.5. *For a local Fitting class $\mathfrak{F}$ and a nonsoluble group G (in particular, a π-soluble group G), whether G possesses a $\mathfrak{F}$-injector and every two $\mathfrak{F}$-injectors are conjugate?*

Note that there exist nonsoluble groups G and nonlocal Fitting classes $\mathfrak{F}$ such that G has no $\mathfrak{F}$-injector (see, for example, [4, 7.1.3–7.1.4]).

By developing the local method of Hartley [11] in this paper, we will resolve Problem 1.5 for partially π-soluble groups G (in particular, G is π-soluble) and $\mathfrak{F}$ is the Hartley class defined by an invariable H-function. In fact, we will prove

Theorem 1.6. *Assume that $\mathfrak{X}$ is a nonempty Fitting class and h is an H-function with support π such that $h(p) = \mathfrak{X}$ for all $p \in \pi$. If $\mathfrak{H} = LH(h)$ and $G \in \mathfrak{X}\mathfrak{S}^{\pi}$ (in particular, $G \in \mathfrak{S}^{\pi}$), then*

(1) *G possesses a $\mathfrak{H}$-injector and every two $\mathfrak{H}$-injectors are conjugate in G;*

(2) *each $\mathfrak{H}$-injector V of G is a subgroup of type $G_{\mathfrak{X}\mathfrak{E}_{\pi'}}L$, where L is the subgroup of G such that $L/G_{\mathfrak{X}}$ is a $\mathfrak{N}_{\pi}$-injector of some Hall π-subgroup of $G/G_{\mathfrak{X}}$.*

Theorem 1.6 implies that a series of new classical conjugate injectors in any π-soluble group are obtained and the structure of injectors of some groups are described. For example, the following are straightforward from Theorem 1.6.

Corollary 1.6.1. *If $\mathfrak{X}$ is a nonempty Fitting class and $\varnothing \neq \pi \subseteq \mathbb{P}$, then each π-soluble group possesses exactly one conjugacy class of $\mathfrak{X}\mathfrak{N}^{\pi}$-injectors.*

Corollary 1.6.2. *If $\varnothing \neq \pi \subseteq \mathbb{P}$ and $k \in \mathbb{N}$, then each π-soluble group has a $(\mathfrak{N}^{\pi})^k$-injector and every two $(\mathfrak{N}^{\pi})^k$-injectors are conjugate.*

Corollary 1.6.3. *Let $\mathfrak{X}$ be a nonempty Fitting class and $G \in \mathfrak{X}\mathfrak{S}^{p}$. If $\mathfrak{H} = \mathfrak{X}\mathfrak{E}_{p'}\mathfrak{N}_p$, then G has $\mathfrak{H}$-injectors and every two of them are conjugate in G.*

Recall that G is p-nilpotent, if G has a normal Hall p'-subgroup. Obviously, $\mathfrak{N}^p = \mathfrak{E}_{p'}\mathfrak{N}_p$ is the Hartley class. Using the Iranzo–Toress Theorem of [15], we have

Corollary 1.6.4. *Each p-soluble group G possesses exactly one conjugacy class of $\mathfrak{N}^p$-injectors and each $\mathfrak{N}^p$-injector is a $\mathfrak{N}^p$-maximal subgroup of G that includes the $\mathfrak{N}^p$-radical of G.*

All unexplained notations and terms are standard. The reader is referred to [2, 4] if need be.

2. Preliminaries

Assume that $\mathfrak{F}$ is a nonempty Fitting class. If $C_G(G_{\mathfrak{F}}) \subseteq G_{\mathfrak{F}}$, then G is said to be $\mathfrak{F}$-*constrained.* Note that if $\mathfrak{F} = \mathfrak{N}$ ($\mathfrak{F} = \mathfrak{N}^\pi$), then the $\mathfrak{F}$-radical of G is the Fitting subgroup of G (the π-Fitting subgroup of G) and denoted by $G_{\mathfrak{N}}$ or $F(G)$ ($G_{\mathfrak{N}^\pi}$ or $F_\pi(G)$). The maximal normal π-subgroup (the maximal normal π'-subgroup) of G is said to be the π-*radical* of G and denoted by $G_{\mathfrak{E}_\pi}$ or $O_\pi(G)$ (called the π'-*radical* of G, and denoted by $G_{\mathfrak{E}_{\pi'}}$ or $O_{\pi'}(G)$).

Lemma 2.1 (see [3, Theorems 1.8.18 and 1.8.19] or [16, Corollary 4.1.2]). *Suppose that $G \in \mathfrak{S}^\pi$. Then G is $\mathfrak{N}^\pi$-constrained. In particular, if $G_{\mathfrak{E}_{\pi'}} = 1$, then G is π-constrained; i.e., $C_G(O_\pi(G) \leq O_\pi(G)$.*

The next results follow from the definition of $\mathfrak{F}$-injector (see, for example, [2, Remarks IX.(1.3)]).

Lemma 2.2. *Let $\mathfrak{F}$ be a nonempty class of groups.*

(1) *If V is a $\mathfrak{F}$-injector of G and $K \trianglelefteq G$, then $V \cap K$ is a $\mathfrak{F}$-injector of K.*

(2) *If V is a $\mathfrak{F}$-injector of G and $\alpha : G \to \overline{G}$ is an isomorphism, then $\alpha(V)$ is a $\mathfrak{F}$-injector of $\overline{G}$.*

(3) *If V is a $\mathfrak{F}$-maximal subgroup of G and $V \cap M$ is a $\mathfrak{F}$-injector of M for any maximal normal subgroup M of G, then V is a $\mathfrak{F}$-injector of G.*

(4) *If V is a $\mathfrak{F}$-injector of G, then $G_{\mathfrak{F}} \leq V$ and V is a $\mathfrak{F}$-maximal subgroup of G.*

Lemma 2.3 [2, Lemma IX.1.1(a) and Theorem IX.1.12(b)]. *Let $\mathfrak{F}$ be a nonempty Fitting class.*

(1) *If N is a subnormal subgroup of G, then $N_{\mathfrak{F}} = N \cap G_{\mathfrak{F}}$;*

(2) *If $\mathfrak{H}$ is a nonempty Fitting class, then the $\mathfrak{H}$-radical of $G/G_{\mathfrak{F}}$ is $G_{\mathfrak{F}\mathfrak{H}}/G_{\mathfrak{F}}$.*

Lemma 2.4 (see [8] or [17, Chapter 5, Theorem 3.7]). *If $G \in \mathfrak{S}^\pi$, then every π-subgroup of G lies in some Hall π-subgroup of G and every two Hall π-subgroups are conjugate.*

Definition 2.5. Let $\pi = \mathrm{Supp}(h)$, where h is the support of some H-function h of the Hartley class $\mathfrak{H}$. Then h is said to be

(1) *integrated* if $h(p) \subseteq \mathfrak{H}$ for all $p \in \pi$;

(2) *full* if $h(p) \subseteq h(q)\mathfrak{E}_{p'}$ for all different primes $p, q \in \pi$;

(3) *full integrated* if h is full and integrated as well;

(4) *invariable* if $f(p) = f(q)$ for all $p, q \in \pi$.

It is easy to see that each Hartley class $\mathfrak{H}$ can be defined by an integrated H-function h, and an invariable H-function h is full integrated (in fact, since $h(p) = h(q)$ for all $p, q \in \pi$, we have $h(p) \subseteq h(p)\mathfrak{E}_{q'} = h(q)\mathfrak{E}_{q'}$ and so $h(p) \subseteq \bigcap_{q\in\pi} h(q)\mathfrak{E}_{q'}\mathfrak{N}_q = \mathfrak{H}$).

3. Preliminary Results

The proof of Theorem 1.6 consists of many steps. The following five lemmas are the main.

Lemma 3.1. *Each π-soluble group G possesses exactly one conjugacy class of $\mathfrak{N}^\pi$-injectors, and each $\mathfrak{N}^\pi$-injector of G is the product of the $\mathfrak{E}_{\pi'}$-radical of G and a $\mathfrak{N}_\pi$-injector of some Hall π-subgroup of G.*

Proof. Proceed by induction on $|G|$. Let M be any maximal normal subgroup of G. The two cases are possible:

Case 1. $G_{\mathfrak{E}_{\pi'}} = 1$.

In this case $M_{\mathfrak{E}_{\pi'}} = 1$. By induction, M possesses exactly one conjugacy class of $\mathfrak{N}^\pi$-injectors and each $\mathfrak{N}^\pi$-injector of M is a $\mathfrak{N}_\pi$-injector of some Hall π-subgroup of M.

Let F_1 be a $\mathfrak{N}_\pi$-injector of some Hall π-subgroup M_π of M. Since every Hall π-subgroup G_π of G is soluble, G_π has a $\mathfrak{N}_\pi$-injector V and every two $\mathfrak{N}_\pi$-injectors of G_π are conjugate in G_π. Since $M_\pi =$

$M \cap G_\pi \lhd G_\pi$; therefore, $V \cap M_\pi$ is a $\mathfrak{N}_\pi$-injector of M by Lemma 2.2(1). In view of the conjugacy of $\mathfrak{N}_\pi$-injectors of M, we may assume that $F_1 = V \cap M_\pi$. Since a Hall π-subgroup of each π-nilpotent group is a nilpotent π-subgroup, each $\mathfrak{N}^\pi$-injector of G_π is a $\mathfrak{N}_\pi$-injector of G_π. Hence, if V is a $\mathfrak{N}^\pi$-maximal subgroup of G, then V is a $\mathfrak{N}_\pi$-injector of G by Lemma 2.2(3).

Suppose that $V < V_1$, where V_1 is a $\mathfrak{N}^\pi$-maximal subgroup of G. Since $G_{\mathfrak{N}_\pi}$ and $(V_1)_{\mathfrak{E}_{\pi'}}$ are normal in V_1, we have $[(V_1)_{\mathfrak{E}_{\pi'}}, G_{\mathfrak{N}_\pi}] \le (V_1)_{\mathfrak{E}_{\pi'}} \cap G_{\mathfrak{N}_\pi} = 1$. Hence, $(V_1)_{\mathfrak{E}_{\pi'}} \le C_G(G_{\mathfrak{N}_\pi})$.

Since $F_\pi(G) = G_{\mathfrak{N}^\pi} = G_{\mathfrak{N}_\pi}$ and $C_G(G_{\mathfrak{N}^\pi}) \le G_{\mathfrak{N}^\pi}$ by Lemma 2.1, $(V_1)_{\mathfrak{E}_{\pi'}} = 1$. This means that $V_1 \in \mathfrak{N}_\pi$ and so $V = V_1$ is a $\mathfrak{N}^\pi$-maximal subgroup of G. The statement of the lemma holds in Case 1.

Case 2. $G_{\mathfrak{E}_{\pi'}} \neq 1$.

Let $G_1 = G/G_{\mathfrak{E}_{\pi'}}$. By Lemma 2.3(2),

$$(G_1)_{\mathfrak{E}_{\pi'}} = G_{\mathfrak{E}_{\pi'}} G_{\mathfrak{E}_{\pi'}}/G_{\mathfrak{E}_{\pi'}} = 1.$$

Hence by Case 1 we have that G_1 possesses exactly one conjugate class of $\mathfrak{N}^\pi$-injectors of type $(G_1)_{\mathfrak{E}_{\pi'}} V_1$, where V_1 is a $\mathfrak{N}_\pi$-injector of some Hall π-subgroup of G_1. Moreover, the set of $\mathfrak{N}^\pi$-injectors of G_1 coincides with the set of $\mathfrak{N}_\pi$-injectors of a Hall π-subgroup $G_\pi G_{\mathfrak{E}_{\pi'}}/G_{\mathfrak{E}_{\pi'}}$. But since G_π is soluble, by Theorem 1.1 G_π has a $\mathfrak{N}_\pi$-injector of V. Then by Lemma 2.2(2) the subgroup $VG_{\mathfrak{E}_{\pi'}}/G_{\mathfrak{E}_{\pi'}}$ is a $\mathfrak{N}_\pi$-injector of $G_\pi G_{\mathfrak{E}_{\pi'}}/G_{\mathfrak{E}_{\pi'}}$. It follows that $VG_{\mathfrak{E}_{\pi'}}$ is a π-nilpotent subgroup of G.

We prove now that $VG_{\mathfrak{E}_{\pi'}}$ is a $\mathfrak{N}^\pi$-maximal subgroup of G. Assume that $VG_{\mathfrak{E}_{\pi'}} < F$ and F is a $\mathfrak{N}^\pi$-maximal subgroup of G. Then $F = F_{\mathfrak{E}_{\pi'}} F_\pi$ and $F_\pi \in \mathfrak{N}_\pi$. Without loss of generality, we may assume that $F_\pi \le G_\pi$ by Lemma 2.4. Hence, $V \le F_\pi \le G_\pi$. But since the $\mathfrak{N}_\pi$-injector V is $\mathfrak{N}_\pi$-maximal subgroup of G_π, we have $V = F_\pi$. It follows from Lemma 2.3 that $(F/G_{\mathfrak{E}_{\pi'}})_{\mathfrak{E}_{\pi'}} = F_{\mathfrak{E}_{\pi'}}/G_{\mathfrak{E}_{\pi'}}$. Therefore,

$$(F/G_{\mathfrak{E}_{\pi'}})/(F/G_{\mathfrak{E}_{\pi'}})_{\mathfrak{E}_{\pi'}} \cong F/F_{\mathfrak{E}_{\pi'}} \cong F_\pi = V \in \mathfrak{N}_\pi.$$

This shows that $F/G_{\mathfrak{E}_{\pi'}}$ is π-nilpotent and $VG_{\mathfrak{E}_{\pi'}}/G_{\mathfrak{E}_{\pi'}} \le F/G_{\mathfrak{E}_{\pi'}}$. Thus, $G_{\mathfrak{E}_{\pi'}} V = F$, and $G_{\mathfrak{E}_{\pi'}} V$ is a $\mathfrak{N}^\pi$-maximal subgroup of G.

In order to prove that $G_{\mathfrak{E}_{\pi'}} V$ is a $\mathfrak{N}^\pi$-injector of G, by Lemma 2.2(3) we should prove that $G_{\mathfrak{E}_{\pi'}} V \cap M$ is a $\mathfrak{N}^\pi$-injector of M.

By induction, M has a $\mathfrak{N}^\pi$-injector of type $M_{\mathfrak{E}_{\pi'}} L$, where L is a $\mathfrak{N}_\pi$-injector of some Hall π-subgroup M_π of M. Since $M_\pi = M \cap G_\pi \trianglelefteq G_\pi$ and every two $\mathfrak{N}_\pi$-injectors of M_π are conjugate by Theorem 1.1, we may assume without loss of generality that $L = V \cap G_\pi$. Since $G_{\mathfrak{E}_{\pi'}} V \cap M \trianglelefteq G_{\mathfrak{E}_{\pi'}} V$ and $G_{\mathfrak{E}_{\pi'}} V$ is π-nilpotent, $G_{\mathfrak{E}_{\pi'}} V \cap M \in \mathfrak{N}^\pi$. But if $M_{\mathfrak{E}_{\pi'}} L \le G_{\mathfrak{E}_{\pi'}} V \cap M$ and $M_{\mathfrak{E}_{\pi'}} L$ is a $\mathfrak{N}^\pi$-injector of M, then $M_{\mathfrak{E}_{\pi'}} L = G_{\mathfrak{E}_{\pi'}} V \cap M$. Therefore, $G_{\mathfrak{E}_{\pi'}} V \cap M$ is a $\mathfrak{N}^\pi$-injector of M. The existence of a $\mathfrak{N}^\pi$-injector in a π-soluble group was proved.

The conjugacy of $\mathfrak{N}^\pi$-injectors follows from the conjugacy of $\mathfrak{N}_\pi$-injectors of a Hall π-subgroup of every π-soluble group.

The lemma is proved.

Corollary 3.2. *Each p-soluble group G possesses exactly one conjugate class of $\mathfrak{N}^p$-injectors, and each $\mathfrak{N}^p$-injector of G is a subgroup of type $G_{\mathfrak{E}_{p'}} P$, where P is a Sylow p-subgroup of G.*

Recall that h is a full integrated H-function of the Hartley class $\mathfrak{H}$, if $h(p) \subseteq \mathfrak{H}$ and $h(p) \subseteq h(q)\mathfrak{E}_{q'}$ for all different primes p and q from $\mathrm{Supp}(h)$.

Lemma 3.3. *Each Hartley class can be defined by a full integrated H-function.*

The proof of the lemma is carried out by direct verification.

Let h be the H-function and $\pi = \mathrm{Supp}(h)$. If h if a full integrated H-function of the Hartley class $\mathfrak{H}$, then $G_h = \prod_{p \in \pi} G_{h(p)}$ of G is called the *h-radical* of G.

Lemma 3.4. *Let π be the support of a full integrated H-function h of the Hartley class $\mathfrak{H}$. If G is such that G/G_h is $\mathfrak{N}^\pi$-constrained (in particular, G/G_h is π-soluble), then V, including $G_\mathfrak{H}$, belongs to $\mathfrak{H}$ if and only if $V/G_h \in \mathfrak{N}^\pi$.*

Proof. Assume that $V \in \mathfrak{H}$ and $G_\mathfrak{H} \le V$. Then $V_{h(p)} \cap G_\mathfrak{H} = (G_\mathfrak{H})_{h(p)} = G_{h(p)}$, and therefore $[V_{h(p)}, G_\mathfrak{H}] \le G_{h(p)}$. This implies that $V_{h(p)} \le C_G(G_\mathfrak{H}/G_{h(p)})$ for all $p \in \pi$.

Prove first that $G_{\mathfrak{H}}/G_h = F_\pi(G/G_h)$. Put $F_\pi(G/G_h) = L/G_h$. Since $G_{\mathfrak{H}} \in \mathfrak{H} = \bigcap_{p\in\pi} h(p)\mathfrak{E}_{p'}\mathfrak{N}_p$ and $(G_{\mathfrak{H}})_{h(p)} = G_{h(p)}$; therefore, $G_{\mathfrak{H}}/G_{h(p)}$ is p-nilpotent for all $p \in \pi$. Consequently, $G_{\mathfrak{H}}/G_h \in \mathfrak{N}^\pi$, $G_{\mathfrak{H}}/G_h \leq L/G_h$, and $G_{\mathfrak{H}} \leq L$.

Demonstrate that $L \leq G_{\mathfrak{H}}$. It suffices to show that $L \in \mathfrak{H}$. Since $L/G_h \in \mathfrak{N}^\pi$, by the isomorphism $L/L_{h(p)}G_h \cong (L/G_h)/(L_{h(p)}G_h/G_h)$ we have $L/L_{h(p)}G_h \in \mathfrak{E}_{p'}\mathfrak{N}_p$ and $(L/G_{h(p)})/(L_{h(p)}G_h/L_{h(p)}) \in \mathfrak{E}_{p'}\mathfrak{N}_p$ for all $p \in \pi$.

It remains to prove that $L_{h(p)}G_h/L_{h(p)}$ is a p'-group for all $p \in \pi$. Since $G_h \trianglelefteq L$, by Lemma 2.3 $G_{h(p)} = (G_h)_{h(p)} = G_h \cap L_{h(p)} \leq L_{h(p)}$.

Let q be an arbitrary prime in π different from $p \in \pi$. Since $G_{h(p)}G_{h(q)}/G_{h(p)} \cong G_{h(q)}/G_{h(q)} \cap G_{h(p)}$, we obtain $G_{h(p)}G_{h(q)}/G_{h(p)} \cong G_{h(q)}/(G_{h(q)})_{h(p)}$. In view of the completeness of the H -function h, we have $h(q) \subseteq h(p)\mathfrak{E}_{p'}$. Hence, $G_{h(q)} \in h(p)\mathfrak{E}_{p'}$, and so

$$G_{h(q)}/(G_{h(q)})_{h(p)} \in \mathfrak{E}_{p'}$$

for all $p \in \pi$. Thus, $G_{h(p)}G_{h(q)}/G_{h(p)}$ is a p' -group for all different simple $p, q \in \pi$. Consequently, $G_h/G_{h(p)} \in \mathfrak{E}_{p'}$ for all $p \in \pi$. Considering the isomorphisms

$$L_{h(p)}G_h/L_{h(p)} \cong G_h/G_h \cap L_{h(p)} \cong (G_h/G_{h(p)})/(G_h \cap L_{h(p)}/G_{h(p)}),$$

we conclude that $L_{h(p)}G_h/L_{h(p)}$ is a p'-group for all $p \in \pi$ and so $L \in \mathfrak{H}$. This proves that $G_{\mathfrak{H}}/G_h = F_\pi(G/G_h)$.

Since G/G_h is $\mathfrak{N}^\pi$-constrained and $G_{\mathfrak{H}}/G_h = F_\pi(G/G_h)$, we have $C_{G/G_h}(G_{\mathfrak{H}}/G_h) \leq G_{\mathfrak{H}}/G_h$. But, clearly, $C_G(G_{\mathfrak{H}}/G_h) \leq G_{\mathfrak{H}}$. Since $V_{h(p)} \leq C_G(G_{\mathfrak{H}}/G_{h(p)}) \leq C_G(G_{\mathfrak{H}}/G_h)$, we get $V_{h(p)} \leq G_{\mathfrak{H}}$. Furthermore, $V_{h(p)} = G_{h(p)}$ for all $p \in \pi$. From $V \in \mathfrak{H} = \bigcap_{p\in\pi} h(p)\mathfrak{E}_{p'}\mathfrak{N}_p$ it follows that $V/G_{h(p)} = V/V_{h(p)}$, and $V/G_{h(p)}$ is p-nilpotent for all $p \in \pi$. Thereby $V/G_h \in \mathfrak{N}^\pi$.

Conversely, if $V/G_h \in \mathfrak{N}^\pi$ by a similar argument (as the above proof of $L \leq G_{\mathfrak{H}}$), we can see that $V \in \mathfrak{H}$. This completes the proof.

Lemma 3.5. *Let G be a group, let π be the support of the full integrated H-function h of the Hartley class $\mathfrak{H}$ and $\sigma(G_h) \subseteq \pi$. If the quotient G/G_h is $\mathfrak{N}^\pi$-constrained and V/G_h is the $\mathfrak{N}^\pi$-injector of G/G_h, then V is a $\mathfrak{H}$-injector of G.*

Proof. Proceed by induction on $|G|$. Let M be an arbitrary maximal normal subgroup of G, while p and q are different primes in π.

We first prove that $G_h/G_{h(q)}$ is a q'-group for all $q \in \pi$. Indeed, since $h(p) \subseteq h(q)\mathfrak{E}_{q'}$ for all different $p, q \in \pi$, by the isomorphism $G_{h(q)}G_{h(p)}/G_{h(q)} \cong G_{h(p)}/(G_{h(p)})_{h(q)}$ we have $G_{h(q)}G_{h(p)}/G_{h(q)} \in \mathfrak{E}_{q'}$. This implies that $G_h/G_{h(q)} \in \mathfrak{E}_{q'}$ for all $q \in \pi$.

Now let $M_h = \prod_{p\in\pi} M_{h(p)}$. Since

$$(G_h \cap M)G_{h(q)}/G_{h(q)} \cong G_h \cap M/M_{h(q)};$$

therefore, $G_h \cap M/M_{h(q)}$ is a q'-group for all $q \in \pi$. Hence,

$$G_h \cap M/M_h \in \bigcap_{q\in\pi} \mathfrak{E}_{q'} = \mathfrak{E}_{\pi'}.$$

It follows from $\sigma(G_h) \subseteq \pi$ that $G_h \cap M/M_h \in \mathfrak{E}_\pi \cap \mathfrak{E}_{\pi'} = (1)$. Thus, $G_h \cap M = M_h$.

We consider the two possible cases:

Case 1. $G_h \leq M$.

In this case we have $G_h = M_h$. Since V/G_h is a $\mathfrak{N}^\pi$-injector of G/G_h by Lemma 2.2(1), $V \cap M/M_h$ is a $\mathfrak{N}^\pi$-injector of M/M_h. Since G/G_h is $\mathfrak{N}^\pi$-constrained and the class of $\mathfrak{N}^\pi$-constrained groups is a Fitting class by Theorem B(b) of [18], M/M_h is $\mathfrak{N}^\pi$-constrained too. Hence, by induction, $V \cap M$ is a $\mathfrak{H}$-injector of M.

Assume that $V < V_1$ and V_1 is a $\mathfrak{H}$-maximal subgroup of G. Since $V \cap M$ is a $\mathfrak{H}$-maximal subgroup of M; therefore, $V \cap M = V_1 \cap M$. Hence $V_1 \cap M$ is a $\mathfrak{H}$-injector of M for any maximal subgroup M of G. It follows from Lemma 2.2(3) that V_1 is a $\mathfrak{H}$-injector of G. But then $G_{\mathfrak{H}} \leq V_1$, and so V_1/G_h is π-nilpotent by Lemma 3.4. As V/G_h is a $\mathfrak{N}^\pi$-injector of G/G_h, V/G_h is a maximal $\mathfrak{N}^\pi$-subgroup of G/G_h, which contradicts $V/G_h < V_1/G_h$. Thus $V = V_1$ and V is a $\mathfrak{H}$-maximal subgroup of G. Hence by Lemma 2.2(3), V is a $\mathfrak{H}$-injector.

Case 2. $G_h \not\leq M$.

In this case $G = G_h M$. Since V/G_h is a $\mathfrak{N}^\pi$-injector of G/G_h and

$$G/G_h \cong M/G_h \cap M = M/M_h,$$

$V \cap M/M_h$ is a $\mathfrak{N}^\pi$-injector of M/M_h by Lemma 2.2(2). Then by induction $V \cap M$ is a $\mathfrak{H}$-injector of M. By analogy to Case 1, we see that V is a $\mathfrak{H}$-injector of G.

The lemma is proved.

4. Proof of Theorem 1.6

(1) Let M be a maximal normal subgroup of G. Since h is an invariable H-function of the Hartley class $\mathfrak{H}$; therefore, h is a full integrated H-function of $\mathfrak{H}$. Since $G/G_{\mathfrak{X}}$ is π-soluble, by Lemma 2.1 $G/G_{\mathfrak{X}}$ is $\mathfrak{N}^\pi$-constrained and by Lemma 3.1 has a $\mathfrak{N}^\pi$-injector of $V/G_{\mathfrak{X}}$. Clearly, $G_h = G_{\mathfrak{X}}$, $M_h = M_{\mathfrak{X}}$, and $G_h \cap M = M_h$. Then, with the same arguments as in the proof of Lemma 3.5, we see that V is a $\mathfrak{H}$-injector of G.

Prove now that if V is a $\mathfrak{H}$-injector of G, then $V/G_{\mathfrak{X}}$ is a $\mathfrak{N}^\pi$-injector of $G/G_{\mathfrak{X}}$.

In fact, assume that V is a $\mathfrak{H}$-injector of G. Then $V \cap S$ is a $\mathfrak{H}$-maximal subgroup of G for each subnormal subgroup S of G. In order to prove that V/G_h is a $\mathfrak{N}^\pi$-injector of G/G_h, it suffices to show that $V/G_{\mathfrak{X}} \cap S/G_{\mathfrak{X}} = (V \cap S)/G_{\mathfrak{X}}$ is a $\mathfrak{N}^\pi$-maximal subgroup of $G/G_{\mathfrak{X}}$. Since $\mathfrak{H} = LH(h) = \bigcap_{p\in\pi} \mathfrak{X}\mathfrak{E}_{p'}\mathfrak{N}_p$ for all $p \in \pi$ and $V \cap S \in \mathfrak{H}$, we have $(V \cap S)/G_{\mathfrak{X}} \in \mathfrak{N}^\pi$. Assume that $(V \cap S)/G_{\mathfrak{X}}$ is not a $\mathfrak{N}^\pi$-maximal subgroup of $G/G_{\mathfrak{X}}$. Let $(V \cap M)/G_{\mathfrak{X}} < D/G_{\mathfrak{X}}$, where $D/G_{\mathfrak{X}}$ is a $\mathfrak{N}^\pi$-maximal subgroup of $G/G_{\mathfrak{X}}$. Clearly,

$$D \in \bigcap_{p\in\pi} \mathfrak{X}\mathfrak{E}_{p'}\mathfrak{N}_p = LH(h) = \mathfrak{H}.$$

But as $V \cap S$ is a $\mathfrak{H}$-maximal subgroup of G, we have $V \cap S = D$. This contradiction shows that $V/G_{\mathfrak{X}}$ is a $\mathfrak{N}^\pi$-injector of $G/G_{\mathfrak{X}}$.

Assume that F is another $\mathfrak{H}$-injector of G. By the above $F/G_{\mathfrak{X}}$ is a $\mathfrak{N}^\pi$-injector of $G/G_{\mathfrak{X}}$. Hence by Lemma 3.1 $F/G_{\mathfrak{X}}$ and $V/G_{\mathfrak{X}}$ are conjugate in $G/G_{\mathfrak{X}}$. This implies that the $\mathfrak{H}$-injectors V and F are conjugate in G. Hence we have (1).

(2) Let V be a $\mathfrak{H}$-injector of G. Then $V/G_{\mathfrak{X}}$ is a $\mathfrak{N}^\pi$-injector of $G/G_{\mathfrak{X}}$. Hence, by Lemma 3.1

$$V/G_{\mathfrak{X}} = (G/G_{\mathfrak{X}})_{\mathfrak{E}_{\pi'}}(L/G_{\mathfrak{X}}),$$

where $L/G_{\mathfrak{X}}$ is a $\mathfrak{N}_\pi$-injector of some Hall π-subgroup of $G/G_{\mathfrak{X}}$. But by Lemma 2.3

$$(G/G_{\mathfrak{X}})_{\mathfrak{E}_{\pi'}} = G_{\mathfrak{X}\mathfrak{E}_{\pi'}}/G_{\mathfrak{X}},$$

and so $V = G_{\mathfrak{X}\mathfrak{E}_{\pi'}}L$. Thus (2) holds.

The theorem is proved.

By Lemma 3.1, each π-soluble group possesses exactly one conjugacy class of $\mathfrak{N}^\pi$-injectors. Note also that every π-soluble group is $\mathfrak{N}^\pi$-constrained. In connection with this, we raise the following

Question 4.1. *Suppose that G is a $\mathfrak{N}^\pi$-constrained group. Is it true that G possesses exactly one conjugacy class of $\mathfrak{N}^\pi$-injectors?*

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