Algebra Colloquium **22** : 2 (2015) 211–214
DOI: 10.1142/S1005386715000188

Algebra Colloquium

On the Cover-Avoid Property of Injectors for Hartley Classes

Yufeng Liu[†]
School of Mathematics and Informational Science
Shandong Institute of Business and Technology, Yantai, Shandong 264005, China
E-mail: yfliu@sdibt.edn.cn

Xiaolan Yi
Department of Mathematics, Zhejiang Sci-Tech University
Hangzhou, Zhejiang 310018, China
E-mail: yixiaolan@126.com

N.T. Vorob'ev
Department of Mathematics, Masherov Vitebsk State University
Vitebsk 210038, Belarus
E-mail: nicholas@vsu.by

Received 4 January 2011
Revised 13 July 2011

Communicated by K.P. Shum

Abstract. In this paper, we study the cover-avoid property of $\mathfrak{F}$-injectors on chief factors of a group G for a Fitting class $\mathfrak{F}$ in some universe $\mathfrak{U}$ with partial solubility.

2010 Mathematics Subject Classification: 20D10

Keywords: Fitting class, Hartley class, $\mathfrak{F}$-injector, cover-avoid property, chief factor

1 Introduction

Throughout this paper, all groups are finite. The notations and terminologies are standard as in [1, 4].

The famous Sylow theorem asserts that if G is a finite group and p is a prime divisor of $|G|$, then G has a Sylow p-subgroup, and any two Sylow p-subgroups of G are conjugate in G. In 1928, Hall [7] proved that a finite soluble group G has a Hall π-subgroup, and any two Hall π-subgroups are conjugate in G. In 1967, Fischer, Gaschütz and Hartley [2] developed further the Sylow theorem and Hall theorem, and proved the following bright result:

Theorem 1.1. [2] *For any Fitting class $\mathfrak{F}$, every finite soluble group G has an $\mathfrak{F}$-injector, and any two $\mathfrak{F}$-injectors are conjugate in G.*

[†]Research of the first author is supported by the Natural Science Foundation of Sandong Province (No. ZR2014AL001), China.

Recall that a class $\mathfrak{F}$ of groups is called a Fitting class provided (i) if $G \in \mathfrak{F}$ and $N \trianglelefteq G$, then $N \in \mathfrak{F}$; and (ii) if $N_1, N_2 \trianglelefteq G$ and $N_1, N_2 \in \mathfrak{F}$, then $N_1 N_2 \in \mathfrak{F}$.

From the condition (ii), we see that for a Fitting class $\mathfrak{F}$, every group G has a largest normal $\mathfrak{F}$-subgroup, which is called the $\mathfrak{F}$-radical of G and denoted by $G_{\mathfrak{F}}$.

Let $\mathfrak{F}$ be a Fitting class. A subgroup V of G is called an $\mathfrak{F}$-injector of G if $V \cap N$ is an $\mathfrak{F}$-maximal subgroup of N for any subnormal subgroup N of G.

Hartley [8] proved that for any soluble Fitting class $\mathfrak{F}$ (that is, all groups in $\mathfrak{F}$ are soluble), every $\mathfrak{F}$-injector V of a soluble group G either covers or avoids every chief factor H/K of G, that is, either $(V \cap H)K = H$ or $(V \cap H)K = K$.

In this connection, the following problem arises:

Problem 1.2. In the class of non-soluble groups, describe the cover-avoid property of $\mathfrak{F}$-injectors of a group G on its chief factors.

In [3], Guo proved that if $\mathfrak{F}$ is a Fitting class and $G \in \mathfrak{F}\mathfrak{S}^{\pi(\mathfrak{F})}$ (that is, $G/G_{\mathfrak{F}}$ is a π-soluble group, where $\pi = \pi(\mathfrak{F}) := \{\text{prime divisors } p \text{ of } G : G \in \mathfrak{F}\}$, then G has a unique conjugate class of $\mathfrak{F}$-injectors (see also [4, Theorem 2.5.3]). The result in [3] allows us in some universe $\mathfrak{U}$ contained in $\mathfrak{F}\mathfrak{S}^{\pi(\mathfrak{F})}$ to further study the cover-avoid property of $\mathfrak{F}$-injectors on chief factors of a group G. Obviously, we may choose, for example, the universe $\mathfrak{U}$ as the class $\mathfrak{S}$ of all finite soluble groups or the class $\mathfrak{S}_\pi$ of all finite π-soluble groups.

2 Preliminaries

Suppose that $\mathfrak{F}$ and $\mathfrak{H}$ be two Fitting classes. Then the product $\mathfrak{F}\mathfrak{H}$ of $\mathfrak{F}$ and $\mathfrak{H}$ is the class $(G : G/G_{\mathfrak{F}} \in \mathfrak{H})$. It is well known that the product of any two Fitting classes is also a Fitting class and the multiplication of Fitting classes satisfies the associative law (see [1, X.1.12]).

We use $\mathbb{P}$ to denote the set of all prime numbers. If $\pi \subseteq \mathbb{P}$ and $\mathfrak{H}$ is a Fitting class, then put $\mathfrak{H}_\pi = \mathfrak{H} \cap \mathfrak{E}_\pi$, where $\mathfrak{E}_\pi$ is the class of all finite π-groups. In particular, if $\pi = \{p\}$, then $\mathfrak{H}_p$ is the class of all finite p-groups in $\mathfrak{H}$. For a group G, let $\pi(G)$ be the set of prime divisors of G. If $\mathfrak{X}$ is a class, then let $\pi(\mathfrak{X}) = \bigcup\{\pi(G) : G \in \mathfrak{X}\}$.

Recall that a subgroup L of G covers (resp., avoids) H/K if $(L \cap H)K = H$ (resp., $(L \cap H)K = K$).

A function $f : \mathbb{P} \to \{\text{Fitting classes}\}$ is called a Hartley function, or in brevity, an H-function (see, for example, [11]), and $\text{Supp}(f) = \{p \in \mathbb{P} : f(p) \neq \emptyset\}$ is called the support of f. Let $\pi = \text{Supp}(f)$ and $LR(f) = \mathfrak{S}_\pi \cap \big(\bigcap_{p \in \pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'}\big)$. A Fitting class $\mathfrak{F}$ is said to be local if there exists an H-function f such that $\mathfrak{F} = LR(f)$.

For a class $\mathfrak{X}$ of groups, let $\text{Char}(\mathfrak{X}) = \{p \in \mathbb{P} : Z_p \in \mathfrak{X}\}$, called the characteristic of $\mathfrak{X}$. By [5, Lemma 2.3], we know that if $\mathfrak{F}$ is a local Fitting class, then $\text{Supp}(f) = \pi(\mathfrak{F}) = \text{Char}(\mathfrak{F})$.

For an H-function f with $\pi = \text{Supp}(f) = \mathbb{P}$, put $LH(h) = \bigcap_{p \in \mathbb{P}} h(p)\mathfrak{U}_{p'}\mathfrak{U}_p$. Following [6, 8], a Fitting class $\mathfrak{H}$ is called a Hartley class (or in brevity, an H-class) if there exists an H-function h such that $\mathfrak{H} = LH(h)$.

Proposition 2.1. *Every H-class $\mathfrak{H}$ is a local Fitting class, but in general, the converse is not true.*

Proof. Since $\mathfrak{H}$ is an H-class, $\mathfrak{H} = \bigcap_{p\in\mathbb{P}} h(p)\mathfrak{U}_{p'}\mathfrak{U}_p$ for some H-function h. It is easy to see that the intersection of local Fitting classes is also a local Fitting class (see [10, Theorem]). Hence, we only need to prove that $\mathfrak{X}_p := h(p)\mathfrak{U}_{p'}\mathfrak{U}_p$ is a local Fitting class for any $p \in \mathbb{P}$. In fact, by [9, Corollary], we see that $\mathfrak{X}_p = LR(f)$ for the H-function f such that $f(p) = h(p)\mathfrak{U}_{p'}$ and $f(q) = \mathfrak{X}_p$ for all $q \neq p$.

Now we prove that the converse is not true in general. Let $\mathfrak{F} = \mathfrak{U}_p\mathfrak{U}_{p'}$. It is well known that $\mathfrak{U}_p$ and $\mathfrak{U}_{p'}$ are local. Hence, the product $\mathfrak{F}$ is local by [10, Theorem]. Moreover, by [9, Corollary], we have $\mathfrak{F} = LR(f)$ for the H-function f such that $f(p) = \mathfrak{N}_p$ and $f(q) = \mathfrak{F}$ for all $p \neq q$. But by [8, 4.2, p. 207], $\mathfrak{F} \neq LH(h)$ for any H-function h. Thus, $\mathfrak{F}$ is not an H-class. This completes the proof. □

Lemma 2.2. [6] *Let $\mathfrak{H}$ be an H-class. Then $\mathfrak{H} = LH(h)$ for any function h such that $h(p) \subseteq \mathfrak{H}$ for any $p \in \mathbb{P}$ and $h(p) \subseteq h(q)\mathfrak{U}_{q'}$ for any $q \in \mathbb{P}$ with $q \neq p$.*

Recall [6] that an H-function h of $\mathfrak{H}$ is called integrated if $h(p) \subseteq \mathfrak{H}$ for all $p \in \mathbb{P}$.

3 Main Results

Theorem 3.1. *Let $\mathfrak{F}$ be a Fitting class and $\mathfrak{U} = \mathfrak{S}^\pi$ be the class of all π-soluble groups, where $\pi = \sigma(\mathfrak{F})$. Then every $\mathfrak{F}$-injector of $G \in \mathfrak{U}$ either covers or avoids every chief factor of G.*

Proof. Since $G \in \mathfrak{U}$, $G/G_{\mathfrak{F}} \in \mathfrak{U}$. By [4, Theorem 2.5.3], there exist $\mathfrak{F}$-injectors of G, and any two $\mathfrak{F}$-injectors of G are conjugated in G. Let V be an arbitrary $\mathfrak{F}$-injector of G and H/K be a chief factor of G. Then by the definition of $\mathfrak{F}$-injectors, we know that $V \cap H$ is an $\mathfrak{F}$-injector of H, and by [4, Theorem 2.5.3], any two $\mathfrak{F}$-injectors of H are conjugated in H. Hence, $V \cap H$ and $(V \cap H)^g$ are conjugated in H for any $g \in G$. Then by the Frattini argument, $G = N_G(V \cap H)H$.

Since G is π-soluble, the chief factor H/K is either an elementary abelian p-group for some $p \in \pi$ or a π'-group.

Assume that H/K is an elementary abelian p-group, then obviously, $K(V \cap H) \trianglelefteq H$. Hence, $K(V \cap H) = K$ or $K(V \cap H) = H$. This shows that the $\mathfrak{F}$-injector V either covers H/K or avoids H/K.

Assume that H/K is a π'-group, then $(V \cap H)K/K \leq H/K \in \mathfrak{E}_{\pi'}$, that is, $(V \cap H)K/K$ is a π'-group. On the other hand, since G is a π-soluble group and $\pi = \pi(\mathfrak{F})$, the $\mathfrak{F}$-injector V is a π-group and so $(V \cap H)K/K \simeq (V \cap H)/(V \cap K) \leq V/(V \cap K) \in \mathfrak{E}_\pi$, that is, $(V \cap H)K/K$ is a π-group. Hence, $(V \cap H)K/K = 1$ and so $(V \cap H)K = K$. This shows that the $\mathfrak{F}$-injector V avoids H/K. □

Theorem 3.2. *Let $\mathfrak{H} = LH(h)$ be an H-class and $G \in \mathfrak{U} \subseteq \mathfrak{H}\mathfrak{S}$. Then any $\mathfrak{H}$-injector of G covers all such G-chief factors which are covered by the $h(p)$-radical of G for all $p \in \mathbb{P}$.*

Proof. Let V be an $\mathfrak{H}$-injector of G. Since $G \in \mathfrak{H}\mathfrak{S}$, by [4, Theorem 2.5.3], all $\mathfrak{H}$-injectors of G are conjugated in G. Hence, the cover-avoid property of $\mathfrak{H}$-injectors on chief factors of G does not depend on the choice of $\mathfrak{H}$-injectors. Obviously, V covers all such G-chief factors that covered by $V_{f(p)}$ for all $p \in \mathbb{P}$. Therefore, we only need to prove $V_{h(p)} = G_{h(p)}$ for all $p \in \mathbb{P}$.

By the definition of $\mathfrak{H}$-injectors, we know that $G_{\mathfrak{H}} \subseteq V$. Hence, by Lemma 2.2, we may without loss of generality assume that the H-function h is integrated. Then

$$V_{h(p)} \cap G_{\mathfrak{H}} = (G_{\mathfrak{H}})_{h(p)} = G_{h(p)\cap\mathfrak{H}} = G_{h(p)}. \qquad (*)$$

Hence, $[V_{h(p)}, G_{\mathfrak{H}}] \leq G_{h(p)}$ and thereby $V_{h(p)} \subseteq C_G(G_{\mathfrak{H}}/G_{h(p)})$. By using $(*)$, we only need to prove $C := C_G(G_{\mathfrak{H}}/G_{h(p)}) \subseteq G_{\mathfrak{H}}$.

Assume that it is not true. Since $C \cap G_{\mathfrak{H}} \trianglelefteq G$, we may construct a normal series $1 \trianglelefteq C \cap G_{\mathfrak{H}} \triangleleft K \trianglelefteq C \trianglelefteq G$ such that $K/(C \cap G_{\mathfrak{H}})$ is a non-trivial chief factor of G. Obviously, $K \cap G_{\mathfrak{H}} = C \cap G_{\mathfrak{H}}$. Then $K/(C \cap G_{\mathfrak{H}}) = K/(K \cap G_{\mathfrak{H}}) \simeq KG_{\mathfrak{H}}/G_{\mathfrak{H}}$. By hypothesis, $G/G_{\mathfrak{H}}$ is a soluble group. Hence, $KG_{\mathfrak{H}}/G_{\mathfrak{H}} \simeq K/(K \cap G_{\mathfrak{H}})$ is a non-trivial abelian p-group. It follows that the $\mathfrak{A}$-residual $(K/(K \cap G_{\mathfrak{H}}))^{\mathfrak{A}} = 1$, where $\mathfrak{A}$ is the class of all abelian groups. By [4, Lemma 2.1.3], $K^{\mathfrak{A}}(K \cap G_{\mathfrak{H}})/(K \cap G_{\mathfrak{H}}) = 1$. Hence, $K^{\mathfrak{A}} \subseteq K \cap G_{\mathfrak{H}}$. Since $K \subseteq C_G(G_{\mathfrak{H}}/G_{h(p)})$, we have $K \subseteq C_G((K \cap G_{\mathfrak{H}})/G_{h(p)})$ and so $[K^{\mathfrak{A}}, K] \subseteq [K \cap G_{\mathfrak{H}}, K] \subseteq G_{h(p)}$. This shows that $K/G_{h(p)}$ is a nilpotent group with nilpotent class at most 2. Let $P/G_{h(p)}$ be a non-trivial normal Sylow p-subgroup of $K/G_{h(p)}$. By [4, Theorems 2.6.7 and 2.6.14], P covers the p-chief factor $K/(K \cap G_{\mathfrak{H}})$, that is, $P(K \cap G_{\mathfrak{H}}) \supseteq K$. Hence, $PG_{\mathfrak{H}} = KG_{\mathfrak{H}}$.

Now we prove $P \in \mathfrak{H}$. Since $P/G_{h(p)} \in \mathfrak{U}_p$, $P \in h(p)\mathfrak{U}_p$. But by Lemma 2.2, $h(p) \subseteq h(q)\mathfrak{U}_{q'}$ for all $q \neq p$. Thus, $h(p)\mathfrak{U}_p \subseteq h(q)\mathfrak{U}_{q'}\mathfrak{U}_p = h(q)\mathfrak{U}_{q'} \subseteq h(q)\mathfrak{U}_{q'}\mathfrak{U}_q$ for all $q \neq p$. This shows $P \in h(q)\mathfrak{U}_{p'}\mathfrak{U}_q$ for all $q \neq p$. On the other hand, $P \in h(p)\mathfrak{U}_p \subseteq h(p)\mathfrak{U}_{p'}\mathfrak{U}_p$. Thus, $P \in \bigcap_{p\in\mathbb{P}} h(q)\mathfrak{U}_{p'}\mathfrak{U}_p = \mathfrak{H}$.

Since $P \trianglelefteq\trianglelefteq G$, we have $PG_{\mathfrak{H}} = G_{\mathfrak{H}}$ and consequently $KG_{\mathfrak{H}} = G_{\mathfrak{H}}$. It follows that the chief factor $KG_{\mathfrak{H}}/G_{\mathfrak{H}} \simeq K/(K \cap G_{\mathfrak{H}}) = 1$. This contradiction completes the proof. □

References

[1] K. Doerk, T.O. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin-New York, 1992.

[2] B. Fischer, W. Gaschütz, B. Hartley, Injektoren endlicher auflösbarer Gruppen, *Math. Z.* **102** (1967) 337–339.

[3] W. Guo, Injectors of finite groups, *Chin. Ann. Math.* **18A** (1997) 145–148.

[4] W. Guo, *The Theory of Classes of Groups*, Science Press-Kluwer Academic Publishers, Beijing-New York-Dordrecht-Boston-London, 2000.

[5] W. Guo, X. Liu, B. Li, On $\mathfrak{F}$-radicals of finite π-soluble groups, *Algebra Discrete Math.* 2006, No.3, 49–54.

[6] W. Guo, N.T. Vorob'ev, On injectors of finite soluble groups, *Comm. Algebra* **36** (2008) 3200–3208.

[7] P. Hall, A note on soluble groups, *J. London Math. Soc.* **3** (1928) 98–105.

[8] B. Hartley, On Fisher's dualization of formation theory, *Proc. London Math. Soc.* (Ser. 3) **19** (1969) 193–207.

[9] N.T. Vorob'ev, On radical classes of finite groups with the Lockett condition, *Math. Zametki* **43** (1988) 161–168; translated in: *Math. Notes* **43** (1988) 91–94.

[10] N.T. Vorob'ev, Local products of Fitting classes, *Vesti AN BSSR* (Ser. Fiz. Math. Navuk) **6** (1991) 28–32.

[11] N.T. Vorob'ev, On the Hawkes conjecture for radical classes, *Siberian Math. J.* **37** (5) (1996) 1296–1302.