

FINITE π -GROUPS WITH NORMAL INJECTORS

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Abstract: Denote by \mathbb{P} the set of all primes and take a nonempty set $\pi \subseteq \mathbb{P}$. A Fitting class $\mathfrak{F} \neq (1)$ is called *normal in the class* \mathfrak{S}_π of all finite soluble π -groups or π -*normal*, whenever $\mathfrak{F} \subseteq \mathfrak{S}_\pi$ and for every $G \in \mathfrak{S}_\pi$ its \mathfrak{F} -injectors constitute a normal subgroup of G .

We study the properties of π -normal Fitting classes. Using Lockett operators, we prove a criterion for the π -normality of products of Fitting classes. A π -normal Fitting class is *normal* in the case $\pi = \mathbb{P}$. The lattice of all solvable normal Fitting classes is a sublattice of the lattice of all solvable Fitting classes; but the question of modularity of the lattice of all solvable Fitting classes is open (see Question 14.47 in [1]). We obtain a positive answer to a similar question in the case of π -normal Fitting classes.

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1. Introduction

All groups in this article are finite and soluble, unless stated otherwise. A class \mathfrak{F} of groups is called a *Fitting class* whenever \mathfrak{F} is closed under taking normal subgroups and products of normal \mathfrak{F} -subgroups. Given a nonempty Fitting class \mathfrak{F} , the \mathfrak{F} -radical of a group G is the greatest normal subgroup of G belonging to \mathfrak{F} , denoted by $G_{\mathfrak{F}}$.

A subgroup V of a group G is called an \mathfrak{F} -*injector* of G whenever $V \cap N$ is an \mathfrak{F} -maximal subgroup of N for every subnormal subgroup N of G . The Fischer–Gaschütz–Hartley theorem [2] asserts that, given a Fitting class \mathfrak{F} , every group admits \mathfrak{F} -injectors and each pair of them are conjugate. If $\mathfrak{F} = \mathfrak{S}_\pi$ is the class of all π -groups (in particular, $\mathfrak{F} = \mathfrak{N}_p$ is the class of all p -groups) then the \mathfrak{F} -injector of a group G is precisely its Hall π -subgroup (Sylow p -subgroup) of the group G ; thus, the Fischer–Gaschütz–Hartley theorem implies Hall's fundamental theorem [3] (in particular, Sylow's theorem [4] in the solvable case).

Many results concerning the structure of Fitting classes and characterization of \mathfrak{F} -injectors and \mathfrak{F} -radicals of groups address the properties of normal Fitting classes (see [5, X–XI]). The article of Blessenohl and Gaschütz [6] laid foundation for the studies in this direction by constructing a series of nontrivial examples of normal Fitting classes each of which is not a formation and establishing that the intersection of an arbitrary set of nontrivial normal Fitting classes is a nontrivial normal Fitting class. Moreover, Gaschütz later applied [7] normal Fitting classes to describe the properties of solvable radicals of arbitrary finite groups. Recall [6] that a Fitting class \mathfrak{F} is called *normal* whenever the \mathfrak{F} -injectors of every group G constitute a normal subgroup of G .

In the 1970s the algebra of solvable normal Fitting classes was studied by Cossey [8], Beidleman [9], Hauck [10], Cusack [11], and Lausch [12]. In particular, Cossey [8] established that the product of two arbitrary normal Fitting classes is a normal Fitting class, while the results of Cusack [11] imply that the Fitting class generated by the union of two normal Fitting classes is normal. Moreover, Lausch proved [12] that the lattice of all normal Fitting classes is modular.

Some natural generalization of the concept of normal Fitting class is proposed in [5, X.3.7] (also see [6, 9, 13]). Denote by \mathbb{P} the set of all primes and take a nonempty set $\pi \subseteq \mathbb{P}$. A Fitting class $\mathfrak{F} \neq (1)$ is called *normal in the class* \mathfrak{S}_π of all π -groups or simply π -*normal*, whenever $\mathfrak{F} \subseteq \mathfrak{S}_\pi$ and the \mathfrak{F} -injectors of each π -group G constitute a normal subgroup of G . We express this as $\mathfrak{F} \trianglelefteq \mathfrak{S}_\pi$. Observe that every π -normal Fitting class is normal in case $\pi = \mathbb{P}$.

In this regard, it is an interesting problem to study the properties of π -normal Fitting classes, in particular, their products and lattices. In this article we use Lockett operators [14] * and $*$ to characterize products of π -normal Fitting classes. Recall [14] that given a nonempty Fitting class \mathfrak{F} , we denote by \mathfrak{F}^* the smallest Fitting class including \mathfrak{F} such that all groups G and H satisfy $(G \times H)_{\mathfrak{F}^*} = G_{\mathfrak{F}^*} \times H_{\mathfrak{F}^*}$. The class \mathfrak{F}_* is defined as the intersection of all Fitting classes \mathfrak{X} with $\mathfrak{F}^* = \mathfrak{X}^*$. In particular, if $\mathfrak{F} = \mathfrak{S}$ is the class of all solvable groups then \mathfrak{F}_* is the smallest normal Fitting class, denoted by \mathfrak{S}_* . Refer as the *product* of Fitting classes \mathfrak{F} and \mathfrak{H} to the class of groups $\mathfrak{FH} = (G : G/G_{\mathfrak{F}} \in \mathfrak{H})$. The product of Fitting classes is a Fitting class, and the multiplication of Fitting classes is an associative operation [5, IX.1.12(a),(c)]. In Section 3 we extend the results of Hauck [10] characterizing the normal product of Fitting classes to the case of their π -normality. In particular, Theorem 3.1 shows that if \mathfrak{F} and \mathfrak{H} are Fitting classes of π -groups then their product \mathfrak{FH} is a π -normal Fitting class exactly when $\mathfrak{F}^* \mathfrak{H}^* = \mathfrak{S}_\pi$.

The main result of this article appears in Section 4, where we study the lattice of all π -normal Fitting classes pursuing the question whether the lattice of all solvable Fitting classes of finite groups is modular (see Question 14.47 in [1]). The positive answer to this question in the case of solvable normal Fitting classes follows from Lausch's result [12] that the lattice of all solvable normal Fitting classes is isomorphic to the lattice of subgroups of a certain infinite abelian group, called the Lausch group. For its definition, see [5, Definition X.4.2(a)]). We prove that the lattice of all π -normal Fitting classes is modular. Note in addition that our proof of the modularity of this lattice is an alternative to Lausch's proof [12] and rests only on description for the structure of the lattice join of Fitting classes in terms of the groups factorized by radicals.

Our definitions and notation follow [5].

2. Preliminaries

In order to characterize the products of π -normal Fitting classes, we use the available properties of the \mathfrak{F} -radical of the group, presented as lemmas.

Lemma 2.1 [5, IX.1.1(a)]. *Given a nonempty Fitting class \mathfrak{F} and a group G , if $N \trianglelefteq \trianglelefteq G$ then $N_{\mathfrak{F}} = N \cap G_{\mathfrak{F}}$.*

Lemma 2.2 [5, IX.1.12(b)]. *Given nonempty Fitting classes \mathfrak{F} and \mathfrak{H} , every group G satisfies $(G/G_{\mathfrak{F}})_{\mathfrak{H}} = G_{\mathfrak{FH}}/G_{\mathfrak{F}}$.*

Given nonempty Fitting classes \mathfrak{X} and \mathfrak{F} , denote by $\mathfrak{X}/\mathfrak{F}$ the class of groups $(G/G_{\mathfrak{F}} : G \in \mathfrak{X})$; see [5, X.3.4(b)].

Denote by S the mapping that associates to each class \mathfrak{X} of groups the class

$$S\mathfrak{X} = (G : G \leq H \text{ for some group } H \in \mathfrak{X}).$$

Lemma 2.3 [15]. *Take a nonempty set $\pi \subseteq \mathbb{P}$ and consider a nontrivial Fitting class \mathfrak{F} . In order for $\mathfrak{F} \trianglelefteq \mathfrak{S}_\pi$ to hold, it is necessary and sufficient that $S(\mathfrak{S}_\pi/\mathfrak{F}) \neq \mathfrak{S}_\pi$.*

We also use the properties of π -normal Fitting classes and Lockett operators obtained in [16].

Lemma 2.4 [16, Theorem 4.2(d)]. *Take a nonempty set $\pi \subseteq \mathbb{P}$. If at least one of the two Fitting classes \mathfrak{F} and \mathfrak{H} is normal in \mathfrak{S}_π then their product is a π -normal Fitting class.*

Lemma 2.5 [16, Theorem 4.2(a),(b)]. *Take a nonempty set $\pi \subseteq \mathbb{P}$. Given two Fitting classes \mathfrak{F} and \mathfrak{H} of π -groups, the following hold:*

- (a) $\mathfrak{FH} \trianglelefteq \mathfrak{S}_\pi$ if and only if $\mathfrak{FH}^* \trianglelefteq \mathfrak{S}_\pi$;
- (b) $\mathfrak{F}^* \mathfrak{H} \trianglelefteq \mathfrak{S}_\pi$ if and only if $\mathfrak{F}^* \mathfrak{H}^* = \mathfrak{S}_\pi$.

A class of groups is called a *formation* whenever it is closed under homomorphic images and finite subdirect products. If \mathfrak{F} is a nonempty formation then refer as the \mathfrak{F} -coradical of a group G to the intersection $G^{\mathfrak{F}}$ of all normal subgroups of G the quotients by which belong to \mathfrak{F} . Observe that if $\mathfrak{F} = \mathfrak{A}$ is the formation of all abelian groups then $G^{\mathfrak{A}} = G'$ is the commutant of the group G .

In the next lemma we recall the necessary properties of the Lockett operators [14] * and $*$.

Lemma 2.6 [5]. Given two nonempty Fitting classes \mathfrak{F} and \mathfrak{H} , the following hold:

- (1) [5, X.1.8(b)] if $\mathfrak{F} \subseteq \mathfrak{H}$ then $\mathfrak{F}^* \subseteq \mathfrak{H}^*$;
- (2) [5, X.1.15] $(\mathfrak{F}_*)_* = \mathfrak{F}_* = (\mathfrak{F}^*)_* \subseteq \mathfrak{F} \subseteq \mathfrak{F}^* = (\mathfrak{F}_*)^* \subseteq \mathfrak{F}_*\mathfrak{A}$;
- (3) [5, X.1.18] if $\mathfrak{F} \subseteq \mathfrak{H}$ then $\mathfrak{F}_* \subseteq \mathfrak{F} \cap \mathfrak{H}_*$;
- (4) [5, X.3.12] if π is a nonempty set of primes with $\mathfrak{F}\mathfrak{S}_\pi = \mathfrak{F}$ then $\mathfrak{F} \cap \mathfrak{S}_* \subseteq \mathfrak{F}_*\mathfrak{S}_{\pi'}$.

Denote by $G \wr H$ the regular wreath product of groups G and H and by G^\natural , the base of $G \wr H$. In the next lemmas we recall the necessary properties of wreath products.

Lemma 2.7 [5, A.18.8(a),(b)]. Given nontrivial groups G and X , put $W = X \wr G$. Then

- (1) if $H \leq G$ then $X^\natural H \cong (X^{|G:H|}) \wr H$;
- (2) if $N \trianglelefteq W$ and $N \cap X^\natural = 1$ then $N = 1$.

Lemma 2.8 [5, X.2.1(a)]. Take a Lockett class \mathfrak{F} and a group G . If $G \notin \mathfrak{F}$ then $(G \wr H)_{\mathfrak{F}} = (G_{\mathfrak{F}})^\natural$ for all groups H .

Lemma 2.9 [5, X.2.4]. Take a Fitting class \mathfrak{F} , a group G , and a nilpotent group H . If $G^m \wr H \in \mathfrak{F}$ for some positive integer m then $G^n \wr H \in \mathfrak{F}^*$ for every $n \in \mathbb{N}$.

To characterize the products of π -normal Fitting classes, we use the criterion for π -normality of a Fitting class:

Lemma 2.10 [5, X.3.7]. Take a nonempty set π of primes. Given a nonempty Fitting class \mathfrak{F} with $\mathfrak{F} \subseteq \mathfrak{S}_\pi$, the following are equivalent:

- (a) $\mathfrak{F} \trianglelefteq \mathfrak{S}_\pi$;
- (b) for every prime $p \in \pi$ and every group $G \in \mathfrak{F}$ we have $G^n \wr Z_p \in \mathfrak{F}$ for some positive integer n ;
- (c) $\mathfrak{F}^* = \mathfrak{S}_\pi$;
- (d) $G/G_{\mathfrak{F}}$ is an abelian group for all groups G in \mathfrak{S}_π .

3. π -Normal Products

Recall that a group G is called *comonolithic* whenever G has a normal subgroup M , the *comonolith* of G , such that G/M is a simple group and $N \leq M$ for every proper normal subgroup N of G . Refer to the product $\mathfrak{F}\mathfrak{H}$ of Fitting classes \mathfrak{F} and \mathfrak{H} as *π -normal* whenever $\mathfrak{F}\mathfrak{H}$ is a π -normal Fitting class. The following theorem characterizes π -normal products of Fitting classes.

Theorem 3.1. Take a nonempty set $\pi \subseteq \mathbb{P}$. Given two Fitting classes \mathfrak{F} and \mathfrak{H} of π -groups, the following are equivalent:

- (a) $\mathfrak{F}\mathfrak{H} \trianglelefteq \mathfrak{S}_\pi$;
- (b) $\mathfrak{F}\mathfrak{H}^* \trianglelefteq \mathfrak{S}_\pi$;
- (c) $\mathfrak{F}^*\mathfrak{H} \trianglelefteq \mathfrak{S}_\pi$;
- (d) $\mathfrak{F}^*\mathfrak{H}^* = \mathfrak{S}_\pi$;
- (e) there exists a set of primes $\sigma \subseteq \pi$ such that $\mathfrak{F}^*\mathfrak{S}_\sigma = \mathfrak{F}^*$ and $\mathfrak{S}_\sigma\mathfrak{H}^* = \mathfrak{S}_\pi$.

PROOF. Lemma 2.5 implies the equivalence of (a) and (b), as well as (c) and (d). Let us establish the equivalence of (b) and (e).

(b) \Rightarrow (e) On assuming that $\mathfrak{F}\mathfrak{H}^* \trianglelefteq \mathfrak{S}_\pi$ and $\sigma = \{p \in \pi : \mathfrak{F}^*\mathfrak{N}_p = \mathfrak{F}^*\}$, verify that $\mathfrak{F}^*\mathfrak{S}_\sigma = \mathfrak{F}^*$ and $\mathfrak{S}_\sigma\mathfrak{H}^* = \mathfrak{S}_\pi$. It is obvious that $\mathfrak{F}^* \subseteq \mathfrak{F}^*\mathfrak{S}_\sigma$. To verify the inverse inclusion, take a group G of minimal order in the class $\mathfrak{F}^*\mathfrak{S}_\sigma \setminus \mathfrak{F}^*$. Then by induction G is a comonolithic group with the comonolith $M = G_{\mathfrak{F}^*}$. By the definition of σ , in the case $p \in \sigma$ we would obviously have $G \in \mathfrak{F}^*\mathfrak{N}_p = \mathfrak{F}^*$, which is impossible.

Assume that $p \notin \sigma$. Then $G/M \in \mathfrak{N}_p \cap \mathfrak{S}_\sigma = (1)$, where (1) stands for the class of trivial groups. In this case $G = M$ and $G \in \mathfrak{F}^*$, which contradicts the choice of G .

Let us now establish that $\mathfrak{S}_\sigma\mathfrak{H}^* = \mathfrak{S}_\pi$. Suppose that $\sigma = \pi$. Then by (1) of Lemma 2.6, the condition $\mathfrak{H} \subseteq \mathfrak{S}_\pi$ implies that $\mathfrak{H}^* \subseteq (\mathfrak{S}_\pi)^*$. Since \mathfrak{S}_π is a Lockett class, we have $(\mathfrak{S}_\pi)^* = \mathfrak{S}_\pi$ and $\mathfrak{H}^* \subseteq \mathfrak{S}_\pi$. Consequently, $\mathfrak{S}_\sigma\mathfrak{H}^* = \mathfrak{S}_\pi\mathfrak{H}^* = \mathfrak{S}_\pi$.

Suppose that $\sigma \neq \pi$. Clearly, $\mathfrak{S}_\sigma \mathfrak{H}^* \subseteq \mathfrak{S}_\pi$. To prove the inverse inclusion $\mathfrak{S}_\pi \subseteq \mathfrak{S}_\sigma \mathfrak{H}^*$, take a group H of minimal order in $\mathfrak{S}_\pi \setminus \mathfrak{S}_\sigma \mathfrak{H}^*$. If $O_\sigma(H) \neq 1$ then $H/O_\sigma(H) \in \mathfrak{S}_\sigma \mathfrak{H}^*$ by induction. Therefore, $H \in \mathfrak{S}_\sigma(\mathfrak{S}_\sigma \mathfrak{H}^*)$. Since the multiplication of Fitting classes is an associative operation, we infer that $H \in \mathfrak{S}_\sigma \mathfrak{H}^*$, which contradicts the choice of H .

Assume that $O_\sigma(H) = 1$ and verify that in this case $H \notin \mathfrak{H}^*$. Indeed, if $H \in \mathfrak{H}^*$ then $H = H/H_{\mathfrak{S}_\sigma} \in \mathfrak{H}^*$ and $H \in \mathfrak{S}_\sigma \mathfrak{H}^*$, which contradicts the choice of H .

Take a nontrivial π -group R , put $W = H \wr R$, and verify that $O_\sigma(W) = 1$. Suppose that $O_\sigma(W) \neq 1$. Take the base H^\natural of W . Since $O_\sigma(H) = 1$ and \mathfrak{S}_σ is a Lockett class, it follows that

$$O_\sigma(H^\natural) = H_{\mathfrak{S}_\sigma}^\natural \times H_{\mathfrak{S}_\sigma}^\natural \times \cdots \times H_{\mathfrak{S}_\sigma}^\natural.$$

Hence, Lemma 2.1 yields $O_\sigma(W) \cap H^\natural = O_\sigma(H^\natural) = 1$. Consequently, (2) of Lemma 2.7 implies that $O_\sigma(W) = 1$. Since $H \notin \mathfrak{H}^*$, by Lemma 2.8 we obtain $W_{\mathfrak{H}^*} = (H \wr R)_{\mathfrak{H}^*} = (H_{\mathfrak{H}^*})^\natural$ and the quotient group $W/W_{\mathfrak{H}^*}$ is not abelian.

Consider the socle

$$\text{Soc}(W) = \prod_{N_i \triangleleft W} N_i$$

of W , with $\sigma(\text{Soc}(W)) = \{p_1, \dots, p_m\}$. Since $O_\sigma(W)$ is the greatest normal σ -subgroup of the group W , it follows that $O_\sigma(W) > N_i$ for all $i \in \{1, 2, \dots, m\}$. Since $N_i \neq 1$, we infer that $O_\sigma(W) \neq 1$. The resulting contradiction shows that $\{p_1, \dots, p_m\} \subseteq \sigma^* = \pi \setminus \sigma$. Consequently, $\sigma^* = \{p \in \pi : \mathfrak{F}^* \mathfrak{N}_p \neq \mathfrak{F}^*\}$ and $\mathfrak{F}^* \mathfrak{N}_{p_i} \neq \mathfrak{F}^*$ for every $i \in \{1, 2, \dots, m\}$. By [5, X.2.14], there exists $G \in \mathfrak{F}$ with $G \wr P_i \notin \mathfrak{F}^*$ for all $i \in \{1, 2, \dots, m\}$ and all p_i -groups $P_i \neq 1$.

Put $W_1 = G \wr W$. Then $W_1 = G^\natural \wr W$. Since $G \in \mathfrak{F}$ and \mathfrak{F} is a Fitting class, it follows that $G^\natural \in \mathfrak{F}$. Therefore, $G^\natural \leq (W_1)_{\mathfrak{F}}$.

Consider the two cases:

CASE 1. $G^\natural < (W_1)_{\mathfrak{F}}$.

By the definition of \mathfrak{F} -radical of W and the property $G^\natural < (W_1)_{\mathfrak{F}}$, there exists a normal subgroup $N \in \mathfrak{F}$ with $G^\natural N \leq (W_1)_{\mathfrak{F}}$. Take as N the minimal normal subgroup of the group W_1 . Claim (1) of Lemma 2.7 yields $G^\natural N \cong G^{|W:N|} \wr N$. Consequently, $G^{|W:N|} \wr N \in \mathfrak{F}$ and N is a nilpotent group. Hence, Lemma 2.9 yields $G \wr N \in \mathfrak{F}^*$. This contradicts the assumption that $G \wr P_i \notin \mathfrak{F}^*$ for all $i \in \{1, 2, \dots, m\}$ and all p_i -groups $P_i \neq 1$. Thus, in this case (b) \Rightarrow (e).

It remains to settle

CASE 2. $G^\natural = (W_1)_{\mathfrak{F}}$.

Since $W_1 = G^\natural \wr W$, it follows that $W_1/G^\natural \cong W$ and $G^\natural \cap W = 1$. Consequently, in view of the isomorphisms $W_1/(W_1)_{\mathfrak{F}} \cong W$ and $(W_1/(W_1)_{\mathfrak{F}})/((W_1)_{\mathfrak{FH}^*}/(W_1)_{\mathfrak{F}}) \cong W_1/(W_1)_{\mathfrak{FH}^*}$, taking into account Lemma 2.2 and the isomorphism $(W_1)_{\mathfrak{FH}^*}/(W_1)_{\mathfrak{F}} \cong (W_1/(W_1)_{\mathfrak{F}})_{\mathfrak{H}^*} \cong W_{\mathfrak{H}^*}$, we infer that

$$(W_1/(W_1)_{\mathfrak{F}})/((W_1)_{\mathfrak{FH}^*}/(W_1)_{\mathfrak{F}}) \cong W_1/(W_1)_{\mathfrak{FH}^*} \cong W/(W)_{\mathfrak{H}^*}.$$

As established above, $W/W_{\mathfrak{H}^*} \notin \mathfrak{A}$. Therefore, as (d) \Rightarrow (a) in Lemma 2.10 the product \mathfrak{FH}^* is not π -normal. This completes the proof of the equality $\mathfrak{S}_\sigma \mathfrak{H}^* = \mathfrak{S}_\pi$ and the implication (b) \Rightarrow (e).

(e) \Rightarrow (b) On assuming that there is a nonempty subset σ of π such that $\mathfrak{F}^* \mathfrak{S}_\sigma = \mathfrak{F}^*$ and $\mathfrak{S}_\sigma \mathfrak{H}^* = \mathfrak{S}_\pi$, verify that $\mathfrak{FH}^* \trianglelefteq \mathfrak{S}_\pi$.

If $\sigma^* = \pi \setminus \sigma$ is empty then $\sigma = \pi \setminus \sigma^* = \pi$. Consequently, $\mathfrak{S}_\sigma = \mathfrak{S}_\pi$ and $\mathfrak{F}^* \mathfrak{S}_\pi = \mathfrak{F}^* = \mathfrak{S}_\pi$. Using the implication (c) \Rightarrow (a) in Lemma 2.10, we infer that \mathfrak{F} is a π -normal Fitting class. Hence, Lemma 2.4 yields $\mathfrak{FH}^* \trianglelefteq \mathfrak{S}_\pi$.

If $\sigma^* = \pi \setminus \sigma \neq \emptyset$ then $\mathfrak{S}_\sigma \neq \mathfrak{S}_\pi$. To verify that $(\mathfrak{FH}^*) \mathfrak{S}_\sigma = \mathfrak{S}_\pi$, assume on the contrary that the class $\mathfrak{S}_\pi \setminus (\mathfrak{FH}^*) \mathfrak{S}_\sigma$ is nonempty and take in it a group G of minimal order. Then G is a comonolithic group with comonolith $M = G_{(\mathfrak{FH}^*) \mathfrak{S}_\sigma}$, and $G/M \cong Z_p$.

Take $p \in \sigma$. Then $G/M = G/G_{(\mathfrak{F}\mathfrak{H}^*)\mathfrak{S}_\sigma} \in \mathfrak{N}_p \subseteq \mathfrak{S}_\sigma$. Since the multiplication of Fitting classes is associative, $G \in (\mathfrak{F}\mathfrak{H}^*)\mathfrak{S}_\sigma\mathfrak{S}_\sigma = (\mathfrak{F}\mathfrak{H}^*)\mathfrak{S}_\sigma$. The resulting contradiction shows that $(\mathfrak{F}\mathfrak{H}^*)\mathfrak{S}_\sigma = \mathfrak{S}_\pi$.

Now take $p \in \sigma^* = \pi \setminus \sigma \subseteq \sigma'$. Observe that $G/G' \cong Z_{p^n}$ for some $p \in \sigma'$. Since (2) of Lemma 2.6 implies that $\mathfrak{S}_\pi \subseteq (\mathfrak{S}_\pi)_*\mathfrak{A}$, we infer that $G/G_{(\mathfrak{S}_\pi)_*}$ is abelian. Consequently, $G' \leq G_{(\mathfrak{S}_\pi)_*}$ for every group G .

Verify that

$$G_{\mathfrak{F}^*}/(G_{\mathfrak{F}^*} \cap G_{(\mathfrak{S}_\pi)_*}) = G_{\mathfrak{F}^*}/G_{\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_*} \in \mathfrak{S}_{\sigma'}.$$

Since $G_{\mathfrak{F}^*}/G_{\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_*} \cong G_{(\mathfrak{S}_\pi)_*}G_{\mathfrak{F}^*}/G_{(\mathfrak{S}_\pi)_*}$ and $G_{(\mathfrak{S}_\pi)_*}G_{\mathfrak{F}^*}/G' \leq G/G' \cong Z_{p^n} \in \mathfrak{N}_p \subseteq \mathfrak{S}_{\sigma'}$, while $\mathfrak{S}_{\sigma'}$ is a hereditary formation, it follows that $G_{(\mathfrak{S}_\pi)_*}G_{\mathfrak{F}^*}/G' \in \mathfrak{S}_{\sigma'}$. Therefore, $G_{\mathfrak{F}^*}/G_{\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_*} \in \mathfrak{S}_{\sigma'}$ by the isomorphism $(G_{(\mathfrak{S}_\pi)_*}G_{\mathfrak{F}^*}/G')/(G_{(\mathfrak{S}_\pi)_*}/G') \cong G_{\mathfrak{F}^*}/G_{\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_*}$.

Claims (2) and (3) of Lemma 2.6 yield $\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_* \subseteq (\mathfrak{F}^*)_*\mathfrak{S}_{\sigma'} = \mathfrak{F}_*\mathfrak{S}_{\sigma'}$. Hence, $G_{\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_*} \in \mathfrak{F}_*\mathfrak{S}_{\sigma'}$, and so $G_{\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_*}/(G_{\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_*})_{\mathfrak{F}*} \in \mathfrak{S}_{\sigma'}$. Since $\mathfrak{F} \subseteq \mathfrak{S}_\pi$, claim (3) of Lemma 2.6 yields $\mathfrak{F}_* \subseteq (\mathfrak{S}_\pi)_*$. Therefore, $G_{\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_*} \cap \mathfrak{F}_* = G_{\mathfrak{F}^*}$ and $G_{\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_*}/G_{\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_*} \cap \mathfrak{F}_* = G_{\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_*}/G_{\mathfrak{F}^*} \in \mathfrak{S}_{\sigma'}$.

By the isomorphism $(G_{\mathfrak{F}^*}/G_{\mathfrak{F}^*})/(G_{\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_*}/G_{\mathfrak{F}^*}) \cong G_{\mathfrak{F}^*}/G_{\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_*}$, the group $G_{\mathfrak{F}^*}/G_{\mathfrak{F}^* \cap (\mathfrak{S}_\pi)_*}$ belongs to $\mathfrak{S}_{\sigma'}$. Consequently, $(G_{\mathfrak{F}^*}/G_{\mathfrak{F}^*}) \in (\mathfrak{S}_{\sigma'})^2 = \mathfrak{S}_{\sigma'}$. Since $\mathfrak{S}_{\sigma'}$ is a formation, we have

$$(G_{\mathfrak{F}^*}/G_{\mathfrak{F}^*})/(G_{\mathfrak{F}}/G_{\mathfrak{F}^*}) \cong G_{\mathfrak{F}^*}/G_{\mathfrak{F}} \in \mathfrak{S}_{\sigma'}.$$

From the condition $\mathfrak{F}^*\mathfrak{S}_\sigma = \mathfrak{F}^*$ and Lemma 2.2 we obtain $G_{\mathfrak{F}^*\mathfrak{S}_\sigma}/G_{\mathfrak{F}^*} = (G/G_{\mathfrak{F}^*})_{\mathfrak{S}_\sigma}$ and conclude that the σ -radical of $G/G_{\mathfrak{F}^*}$ is trivial. Claim (2) of Lemma 2.6 implies that $\mathfrak{F}\mathfrak{S}_\sigma \subseteq \mathfrak{F}^*\mathfrak{S}_\sigma = \mathfrak{F}^*$. Consequently, $G_{\mathfrak{F}\mathfrak{S}_\sigma} \leq G_{\mathfrak{F}^*}$. Then $G_{\mathfrak{F}\mathfrak{S}_\sigma}/G_{\mathfrak{F}} \leq G_{\mathfrak{F}^*}/G_{\mathfrak{F}} \in \mathfrak{S}_{\sigma'}$, and since $\mathfrak{S}_{\sigma'}$ is a hereditary formation, the σ -radical of $G/G_{\mathfrak{F}}$ is a σ' -group. Therefore, $(G/G_{\mathfrak{F}})_{\mathfrak{S}_\sigma} = 1$.

Since $G \in \mathfrak{S}_\pi$, it follows that $G/G_{\mathfrak{F}} \in \mathfrak{S}_\pi = \mathfrak{S}_\sigma\mathfrak{H}^*$ and $(G/G_{\mathfrak{F}})/(G/G_{\mathfrak{F}})_{\mathfrak{S}_\sigma} \in \mathfrak{H}^*$. Hence, $G/G_{\mathfrak{F}} \in \mathfrak{H}^*$ and $G \in \mathfrak{F}\mathfrak{H}^* \subseteq (\mathfrak{F}\mathfrak{H}^*)\mathfrak{S}_\sigma$. The resulting contradiction justifies that $(\mathfrak{F}\mathfrak{H}^*)\mathfrak{S}_\sigma = \mathfrak{S}_\pi$. Thus, $G/G_{\mathfrak{F}\mathfrak{H}^*} \in \mathfrak{S}_\sigma$ and $G \in \mathfrak{S}_\pi$. Consequently,

$$\mathfrak{S}_\pi/\mathfrak{F}\mathfrak{H}^* = (G/G_{\mathfrak{F}\mathfrak{H}^*} : G \in \mathfrak{S}_\pi) \subseteq \mathfrak{S}_\sigma.$$

Since $S(\mathfrak{S}_\pi/\mathfrak{F}\mathfrak{H}^*) \subseteq S(\mathfrak{S}_\sigma) = \mathfrak{S}_\sigma \neq \mathfrak{S}_\pi$, by Lemma 2.3 the class $\mathfrak{F}\mathfrak{H}^*$ is normal in \mathfrak{S}_π . The proof of (e) \Rightarrow (b) is complete.

(e) \Rightarrow (d) Assume that there exists a nonempty set $\sigma \subseteq \pi$ of primes such that $\mathfrak{F}^*\mathfrak{S}_\sigma = \mathfrak{F}^*$ and $\mathfrak{S}_\sigma\mathfrak{H}^* = \mathfrak{S}_\pi$. Using (1) of Lemma 2.6, infer from $\mathfrak{F} \subseteq \mathfrak{S}_\pi$ that $\mathfrak{F}^* \subseteq (\mathfrak{S}_\pi)^* = \mathfrak{S}_\pi$. Thus,

$$\mathfrak{F}^*\mathfrak{H}^* = (\mathfrak{F}^*\mathfrak{S}_\sigma)\mathfrak{H}^* = \mathfrak{F}^*(\mathfrak{S}_\sigma\mathfrak{H}^*) = \mathfrak{F}^*\mathfrak{S}_\pi = \mathfrak{S}_\pi$$

and the implication (e) \Rightarrow (d) is established.

Observe that $(\mathfrak{F}^*)^* = \mathfrak{F}^*$ and $(\mathfrak{H}^*)^* = \mathfrak{H}^*$. For this reason, we state (e) and (d) for the Lockett classes \mathfrak{F}^* and \mathfrak{H}^* . As established above, (e) \Leftrightarrow (b). Consequently, (b) holds for Lockett classes and yields (d). Thus, (d) \Rightarrow (e).

The proof of the theorem is complete.

4. Modularity of the Lattice of \mathfrak{E}_π -Normal Fitting Classes

In this section we do not assume that the groups under consideration are soluble.

Recall that the *lattice join* $\mathfrak{F} \vee \mathfrak{H}$ of two Fitting classes \mathfrak{F} and \mathfrak{H} [11] is the Fitting class generated by the union of \mathfrak{F} and \mathfrak{H} .

Denote by Sn the operator that associates to each class \mathfrak{X} of groups the class

$$\text{Sn } \mathfrak{X} = (G : G \trianglelefteq \trianglelefteq H \text{ for some group } H \in \mathfrak{X}).$$

Lemma 4.1. Given two Fitting classes \mathfrak{X} and \mathfrak{Y} , if $\mathfrak{X} \subseteq \mathfrak{Y}^*$ then $\mathfrak{X} \vee \mathfrak{Y} = \text{Sn}(G : G = G_{\mathfrak{X}}G_{\mathfrak{Y}})$.

PROOF. In order to show that $\mathfrak{X} \vee \mathfrak{Y} \subseteq \text{Sn}(G : G = G_{\mathfrak{X}}G_{\mathfrak{Y}})$, take $X \in \mathfrak{X} \vee \mathfrak{Y}$ and verify that $X \in \text{Sn}(G : G = G_{\mathfrak{X}}G_{\mathfrak{Y}})$. Since $\mathfrak{X} \subseteq \mathfrak{Y}^*$ by assumption,

$$X \in \mathfrak{X} \vee \mathfrak{Y} \subseteq \mathfrak{X} \vee \mathfrak{Y}^* \subseteq \mathfrak{Y}^*.$$

By the definition of $*$, we have

$$X \in \mathfrak{Y}^* = (G : (G \times G)_{\mathfrak{Y}} \quad \text{appears subdirectly in } (G \times G)).$$

Hence, $X \trianglelefteq (X \times X)_{\mathfrak{Y}} K$, where $K \in \mathfrak{X}$. Consequently, $X \trianglelefteq X_{\mathfrak{Y}} X_{\mathfrak{X}}$ and $X \in \text{Sn}(G : G = G_{\mathfrak{X}}G_{\mathfrak{Y}})$.

To establish the inverse inclusion $\text{Sn}(G : G = G_{\mathfrak{X}}G_{\mathfrak{Y}}) \subseteq \mathfrak{X} \vee \mathfrak{Y}$, take $Y \in (G : G = G_{\mathfrak{X}}G_{\mathfrak{Y}})$ and verify that $Y \in \mathfrak{X} \vee \mathfrak{Y}$. Observe that $Y_{\mathfrak{X}} \in \mathfrak{X} \subseteq \mathfrak{X} \vee \mathfrak{Y}$ and $Y_{\mathfrak{Y}} \in \mathfrak{Y} \subseteq \mathfrak{X} \vee \mathfrak{Y}$. Therefore, $Y = Y_{\mathfrak{X}}Y_{\mathfrak{Y}} \in \mathfrak{X} \vee \mathfrak{Y}$, $(G : G = G_{\mathfrak{X}}G_{\mathfrak{Y}}) \subseteq \mathfrak{X} \vee \mathfrak{Y}$ and $\text{Sn}(G : G = G_{\mathfrak{X}}G_{\mathfrak{Y}}) \subseteq \text{Sn}(\mathfrak{X} \vee \mathfrak{Y}) = \mathfrak{X} \vee \mathfrak{Y}$.

The proof of the lemma is complete.

Using the equivalence (a) \Leftrightarrow (c) of Lemma 2.10, let us introduce

DEFINITION 4.2. Take a nonempty set $\pi \subseteq \mathbb{P}$. Refer to a Fitting class \mathfrak{F} as \mathfrak{E}_{π} -normal, or normal in the class \mathfrak{E}_{π} of all finite π -groups, whenever $\mathfrak{F}^* = \mathfrak{E}_{\pi}$.

In the case $\pi = \mathbb{P}$ refer to every \mathfrak{E}_{π} -normal Fitting class as *normal*. By Lemma 2.10, in the universe \mathfrak{S} of all finite soluble groups the condition $\mathfrak{F}^* = \mathfrak{S}_{\pi}$ is equivalent to $\mathfrak{F} \trianglelefteq \mathfrak{S}_{\pi}$. Observe also that the lattice of all normal Fitting classes is a sublattice of the lattice of all Fitting classes.

By Lausch's theorem [12], the lattice of all solvable normal Fitting classes is modular. But the question of modularity of the lattice of all solvable Fitting classes remains open (see Question 14.47 in [1]). The next theorem gives a positive answer to the analog of this question for π -normal Fitting classes.

Theorem 4.3. The lattice of all \mathfrak{E}_{π} -normal Fitting classes is modular.

PROOF. Consider π -normal Fitting classes $\mathfrak{X}, \mathfrak{Y}$, and \mathfrak{F} with $\mathfrak{X} \subseteq \mathfrak{F}$ and verify that

$$\mathfrak{X} \vee (\mathfrak{Y} \cap \mathfrak{F}) \subseteq (\mathfrak{X} \vee \mathfrak{Y}) \cap \mathfrak{F}.$$

The definition of lattice join yields $\mathfrak{X} \subseteq \mathfrak{X} \vee \mathfrak{Y}$. By the hypotheses of the theorem, $\mathfrak{X} \subseteq \mathfrak{F}$. Consequently, $\mathfrak{X} \subseteq (\mathfrak{X} \vee \mathfrak{Y}) \cap \mathfrak{F}$. Since $\mathfrak{Y} \cap \mathfrak{F} \subseteq \mathfrak{F}$ and $\mathfrak{Y} \cap \mathfrak{F} \subseteq \mathfrak{X} \vee \mathfrak{Y}$, it follows that $\mathfrak{Y} \cap \mathfrak{F} \subseteq (\mathfrak{X} \vee \mathfrak{Y}) \cap \mathfrak{F}$. Hence, the definition of lattice join yields $\mathfrak{X} \vee (\mathfrak{Y} \cap \mathfrak{F}) \subseteq (\mathfrak{X} \vee \mathfrak{Y}) \cap \mathfrak{F}$.

Let us establish the inverse inclusion $(\mathfrak{X} \vee \mathfrak{Y}) \cap \mathfrak{F} \subseteq \mathfrak{X} \vee (\mathfrak{Y} \cap \mathfrak{F})$. Since \mathfrak{X} and \mathfrak{Y} are \mathfrak{E}_{π} -normal Fitting classes, we have $\mathfrak{X} \subseteq \mathfrak{Y}^*$. Consequently, Lemma 4.1 implies that

$$\mathfrak{X} \vee \mathfrak{Y} = \text{Sn}(G : G = G_{\mathfrak{X}}G_{\mathfrak{Y}}). \tag{1}$$

Take $K \in \mathfrak{X} \vee (\mathfrak{Y} \cap \mathfrak{F})$. By (1), there exists a group $G = G_{\mathfrak{X}}G_{\mathfrak{Y}}$ with $K \trianglelefteq G$. Since $K \in \mathfrak{F}$, it follows that $K \trianglelefteq G_{\mathfrak{F}}$. The condition $\mathfrak{X} \subseteq \mathfrak{F}$ obviously implies that $G_{\mathfrak{X}} \subseteq G_{\mathfrak{F}}$. By Dedekind's identity,

$$K \trianglelefteq G_{\mathfrak{F}} = G \cap G_{\mathfrak{F}} = G_{\mathfrak{X}}G_{\mathfrak{Y}} \cap G_{\mathfrak{F}} = G_{\mathfrak{X}}(G_{\mathfrak{Y}} \cap G_{\mathfrak{F}}) = G_{\mathfrak{X}}G_{\mathfrak{Y} \cap \mathfrak{F}}.$$

Thus, $K \in \mathfrak{X} \vee (\mathfrak{Y} \cap \mathfrak{F})$ and $(\mathfrak{X} \vee \mathfrak{Y}) \cap \mathfrak{F} \subseteq \mathfrak{X} \vee (\mathfrak{Y} \cap \mathfrak{F})$.

The proof of the theorem is complete.

Corollary 4.4. The lattice of all normal Fitting classes is modular.

Corollary 4.5 [12]. The lattice of all solvable normal Fitting classes is modular.

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