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Injectors in Fitting Sets of Finite Groups

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Abstract—A set of subgroups $\mathscr{F}$ of a finite group G is referred to as a *Fitting set* if it is closed with respect to taking normal subgroups, products of normal $\mathscr{F}$-subgroups, and inner automorphisms of G. A Fitting set $\mathscr{F}$ of a group G is said to be *π-saturated* if $H \in \mathscr{F}$ for every subgroup H in G such that $O^{\pi'}(H) \in \mathscr{F}$. In the paper, it is proved that, if $\mathscr{F}$ is a π-saturated Fitting set of a π-solvable group G, then there are $\mathscr{F}$-injectors in G and every two of them are conjugate.

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1. INTRODUCTION

A basic result in the theory of classes of finite solvable groups is the generalization of fundamental Sylow and Hall theorems which was obtained by Fischer, Gaschütz, and Hartley in [1], where it was proved that, for every Fitting class $\mathfrak{F}$, every finite solvable group in G contains $\mathfrak{F}$-injectors, and every two of them are conjugate. Recall that a class of groups $\mathfrak{F}$ is said to be a *Fitting class* if $\mathfrak{F}$ is closed with respect to taking normal subgroups and products of normal $\mathfrak{F}$-subgroups. Here a subgroup V of a group G is said to be an *$\mathfrak{F}$-injector of G* if, for every subnormal subgroup N of the group G, the subgroup $V \cap N$ is maximal among the subgroups of N belonging to $\mathfrak{F}$. By a *Fitting set* $\mathscr{F}$ of a group G one means a set of subgroups of G which is closed with respect to taking normal subgroups, their products, and conjugate subgroups. The notion of *$\mathscr{F}$-injector* of a group for a Fitting set $\mathscr{F}$ of the group is defined similarly to the above definition of $\mathfrak{F}$-injector for a Fitting class $\mathfrak{F}$.

The validity of the above Fischer–Gaschütz–Hartley theorem in [1] was confirmed by Shemetkov [2] for a Fitting set of a finite partially solvable group (for the solvable case, see also [3]). As was established in [2], for every Fitting set $\mathscr{F}$ of a finite π-solvable group G (π stands for the set of all prime divisors of all groups in $\mathscr{F}$), G contains a unique class of conjugate $\mathscr{F}$-injectors.

Note that, if $\mathfrak{F}$ is a Fitting class, then the set of subgroups $\{H \le G \mid H \in \mathfrak{F}\}$ of the group G is a Fitting set of G. It is denoted by $\mathrm{Tr}_{\mathfrak{F}}(G)$ and referred to as the *trace of the Fitting class* $\mathfrak{F}$ in the group G. As is well known (see [4, Examples VIII.2.2]), to every Fitting class $\mathfrak{F}$ there corresponds its trace in the group G; however, the converse is false in general. Moreover, it is clear that the set of $\mathfrak{F}$-injectors for a Fitting class $\mathfrak{F}$ and of $\mathscr{F}$-injectors for the Fitting set $\mathscr{F} = \mathrm{Tr}_{\mathfrak{F}}(G)$ coincide and, therefore, the above-mentioned theorem of Shemetkov [2], in particular, implies the Fischer–Gaschütz–Hartley theorem [1].

Let π be an arbitrary nonempty set of primes and let π' be the complement of π in the set of all primes. The main result of the present paper is the proof of the fact that every π-solvable group G contains $\mathscr{F}$-injectors for every π-saturated Fitting set $\mathscr{F}$ in G, and every two of these injectors are conjugate (Theorem 3.10).

In the concluding section of the paper, we generalize Shemetkov's result from [2] concerning the existence and conjugacy of $\mathscr{F}$-injectors by weakening the condition that the group is π-solvable and replacing it by the condition that an appropriate quotient group of G is π-solvable. All groups considered in the paper are finite. For the definitions and notation which we do not present, see [4]–[6] if necessary.

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2. PRELIMINARIES

Let $\mathscr{X}$ be a set of subgroups of a group G, $H \leq G$, and $\mathscr{X}_H = \{S \leq H : S \in \mathscr{X}\}$. In this case, if $\mathscr{X}$ is a Fitting set of G, then $\mathscr{X}_H$ is obviously a Fitting set of the group H. Denote by the symbol $G_{\mathscr{X}}$ the largest normal $\mathscr{X}$-subgroup of G. This subgroup is referred to as the *$\mathscr{X}$-radical* of G.

We shall use the well-known property of the $\mathscr{F}$-radical of a group given by the following lemma.

Lemma 2.1 (see [4, Property VIII.2.4(a)]). *Let $\mathscr{F}$ be a Fitting set of a group G, and let N be a subnormal subgroup. Then $N_{\mathscr{F}} = G_{\mathscr{F}} \cap N$.*

Definition 2.2. Let $\mathscr{F}$ be a set of subgroups of a group G.

(a) An $\mathscr{F}$-subgroup V of G is said to be (see [4, VIII.2.5(a)]) *$\mathscr{F}$-maximal* if it follows from $V \leq W \leq G$ and $V \in \mathscr{F}$ that $V = W$.

(b) By an *$\mathscr{F}$-injector* of G one means (see [4, VIII.2.5(b)]) a subgroup V such that $V \cap K$ is an $\mathscr{F}$-maximal subgroup of K for every subnormal subgroup K of G.

We also use some known assertions concerning $\mathscr{F}$-injectors of a group for Fitting sets; we present these assertions as lemmas.

Lemma 2.3 (see [4, Theorem VIII.2.9]). *If $\mathscr{F}$ is a Fitting set of a solvable group G, then the group G contains $\mathscr{F}$-injectors, and every two of them are conjugate.*

By the symbol $\sigma(G)$ we denote the set of all prime divisors of the order of the group G and by $\sigma(\mathscr{F})$ the union of the sets $\sigma(G)$ for all groups G in a Fitting set $\mathscr{F}$.

Lemma 2.4 (see [2, Theorem 2.2]). *Let $\mathscr{F}$ be a Fitting set of a π-solvable group G, where $\pi = \sigma(\mathscr{F})$. Then the group G contains $\mathscr{F}$-injectors, and every two of them are conjugate.*

Lemma 2.5 (see [3, Property 2.2]). *Let A be a normal subgroup of a group G. Then the following assertions hold.*

(1) *If $\mathscr{F}$ is a Fitting set of G and $A \in \mathscr{F}$, then*

$$\overline{\mathscr{F}} = \{S/A : A \leq S \in \mathscr{F}\}$$

is a Fitting set of the group G/A. Moreover, if V is an $\mathscr{F}$-injector of G, then V/A is an $\overline{\mathscr{F}}$-injector of G/A.

(2) *If $\overline{\mathscr{F}}$ is a Fitting set of G/A and V/A is an $\overline{\mathscr{F}}$-injector of G/A, then*

$$\mathscr{F}_0 = \{S \leq G : (SA)/A \in \overline{\mathscr{F}}\}$$

is a Fitting set of G and V is an $\mathscr{F}_0$-injector of G.

(3) *If $\mathscr{F}$ is a Fitting set of G and $V \in \mathscr{F}$ is a subgroup of G such that $VA = G$ and $V \cap A$ is an $\mathscr{F}$-injector of A, then V is an $\mathscr{F}$-injector of G.*

(4) *If V is an $\mathscr{F}$-injector of G, then VA/A is an $\mathscr{F}$-injector of G/A.*

Recall that the symbol $F(G)$ denotes the *Fitting subgroup* of a group G, i.e., the largest normal nilpotent subgroup of G, and the symbol $F_\pi(G)$ denotes the largest normal π-nilpotent subgroup of G.

Lemma 2.6 (see [5, Corollary 4.1.2]). *For every π-solvable group G, we have the inclusion*

$$C_G(F_\pi(G)) \subseteq F_\pi(G).$$

Let $\mathbb{P}$ be the set of all primes, let $\pi \subseteq \mathbb{P}$, and let $\pi' = \mathbb{P} \setminus \pi$. Recall that the symbol $O_{\pi'}(G)$ denotes the largest normal π'-subgroup of a group G and the symbol $O^\pi(G)$ denotes the smallest normal subgroup of G for which the quotient group $G/O^\pi(G)$ is a π-group.

To prove the main result, we use also properties of Hall θ-bases.

Definition 2.7. Let θ be a finite system of pairwise disjoint subsets of the set of primes,

$$\theta = \{\pi_1, \pi_2, \pi_3, \dots, \pi_k\}.$$

The set of subgroups

$$H_1, \quad H_2, \quad H_3, \quad \dots, \quad H_k \tag{2.1}$$

is said to be (see [8]) a *Hall θ-Base of a group G* if these subgroups satisfy the following conditions:

1) H_i, $i = 1, 2, \dots, k$, is a Hall π_i-subgroup of G;

2) the subgroups (2.1) pairwise commute.

Lemma 2.8 (see [8, Theorem 1]). *Let G be a π-solvable group, let $\overline{\pi}$ be the set of prime divisors of the order of G not belonging to π, and let a set $\theta = \{\pi_1, \pi_2, \pi_3, \dots, \pi_k\}$ be an arbitrary finite system of pairwise disjoint subsets π_1, π_2, π_3, …, π_k of the set of prime numbers satisfying the following condition: either $\pi_i \cap \overline{\pi} = \varnothing$, $i = 1, 2, \dots, k$, or θ contains a π_s such that $\overline{\pi} \subseteq \pi_s$.*

Then the group G admits at least one Hall θ-base and every two Hall θ-bases are conjugate to each other.

3. INJECTORS FOR π-SATURATED FITTING SETS

The present subsection is devoted to the proof of the existence and conjugacy of injectors for a π-saturated Fitting set of a π-solvable group.

Definition 3.1. A Fitting set $\mathscr{F}$ of a group G is said to be π-*saturated* if $H \in \mathscr{F}$ for every subgroup H in G such that $O^{\pi'}(H) \in \mathscr{F}$.

We use the notion of strong π-closeness for π-subgroups of a group and some properties of this notion.

Definition 3.2. Let G be a group, let π be a set of primes, and let H_0 be a π-subgroup of G such that $H_0 \le H \in \mathrm{Hall}_\pi(G)$. A subgroup H_0 of H is said to be *strongly π-closed with respect to G* if $H_0^g \cap H \le H_0$ for any $g \in G$.

Note that the notion of strong closeness (π-closeness for $\pi = \{p\}$) was introduced in [7].

Let us prove the properties of strong π-closeness of subgroups similar to properties of strong closeness in [7] that we use below.

Lemma 3.3. *Let G be a group, let π be a set of primes, and let H_0 be a π-subgroup of G such that $H_0 \le H \in \mathrm{Hall}_\pi(G)$. If H_0 is strongly π-closed in H with respect to G, then the following assertions hold*:

1) *if $H_0 \le H^x$ for some element $x \in G$, then the subgroup H_0 is strongly π-closed in H^x with respect to G*;

2) *if N is a normal subgroup of G, then the subgroup H_0N/N is strongly π-closed in HN/N with respect to G/N*;

3) *H_0^x is strongly π-closed in H^x with respect to G.*

Proof. 1) It follows from the condition $H_0 \le H^x$ that $H_0^{x^{-1}} \le H$. Then, by the definition of strong π-closeness,

$$H_0^{x^{-1}} = H_0^{x^{-1}} \cap H \le H_0.$$

Hence $x \in N_G(H_0)$. Since, for every element $g \in G$, we have

$$H_0^g \cap H^x = (H_0^{gx^{-1}} \cap H)^x \le H_0^x = H_0,$$

it follows that H_0 is strongly π-closed in H^x with respect to G.

2) Note that, for every element $g \in G$, there is an element $x \in N$ for which

$$H_0^g \cap HN = H_0^g \cap H^x = (H_0^{gx^{-1}} \cap H)^x \le H_0^x \le H_0N.$$

This implies that H_0N/N is strongly π-closed in HN/N with respect to G/N.

3) For every element $g \in G$, the inclusion $H_0^g \cap H \le H_0$ holds. However,

$$(H_0^g \cap H)^x = H_0^{gx} \cap H^x \le H_0^x.$$

Therefore, since the choice of g is arbitrary, H_0^x is strongly π-closed in H^x with respect to G. □

Lemma 3.4. *Let G be a π-solvable group and let H_0 be strongly π-closed in $H \in \mathrm{Hall}_\pi(G)$ with respect to G. Then there is a normal subgroup N of G such that $N \cap H = H_0$.*

Proof. We carry out the proof by induction on the order of the group. Let G be a group of the least order for which the lemma is false and let M be a nonidentity normal subgroup of G. Denote by $\overline{K}$ the subgroup KM/M of the quotient group $\overline{G} = G/M$. Then, by assertion 2 of Lemma 3.3, the subgroup $\overline{H}_0$ is strongly π-closed in $\overline{H}$ with respect to $\overline{G}$. Since $|M| > 1$, it follows that $|\overline{G}| < |G|$. In this case, by induction, $\overline{G}$ contains a normal subgroup $\overline{L}$ such that $\overline{L} \cap \overline{H} = \overline{H}_0$. Hence the group G contains a normal subgroup L such that $LM \cap HM = H_0M$. Applying the Dedekind identity, we obtain

$$LM \cap H = LM \cap HM \cap H = H_0M \cap H = H_0(M \cap H).$$

This means that G contains a normal subgroup N for which $N \cap H = H_0(M \cap H)$.

Let $O_{\pi'}(G) \ne 1$ and $M = O_{\pi'}(G)$. Then

$$N \cap H = H_0(O_{\pi'}(G) \cap H).$$

Since H is a π-subgroup and $O_{\pi'}(G)$ is a π'-subgroup, it follows that $O_{\pi'}(G) \cap H = 1$. Hence $N \cap H = H_0$, and the lemma is true in this case.

Suppose that $O_\pi(G) \cap H_0 \ne 1$. Thus, we may assume that $M = O_\pi(G) \cap H_0$. Then

$$N \cap H = H_0(O_\pi(G) \cap H_0 \cap H) = H_0,$$

and the lemma holds.

Suppose now that $O_{\pi'}(G) = 1$ and $O_\pi(G) \cap H_0 = 1$. Then $H_0 \le C_G(O_\pi(G))$. Note that the inclusion $C_G(F_\pi(G)) \subseteq F_\pi(G)$ holds by Lemma 2.6. Since $O_{\pi'}(G) = 1$, we have $C_G(O_\pi(G)) \subseteq O_\pi(G)$. Hence

$$H_0 \le C_G(O_\pi(G)) \le O_\pi(G), \qquad H_0 = O_\pi(G) \cap H_0 \ne 1.$$

The contradiction thus obtained completes the proof of the lemma. □

Corollary 3.5 (see [7]). *Let G be a π-solvable group and let P_0 be strongly closed in $P \in \mathrm{Syl}_p(G)$ with respect to G for some prime p in π. Then there is a normal subgroup N in G such that $N \cap P = P_0$.*

Lemma 3.6. *Let a group G be π-solvable, let π_1 be a subset of the set π, and let H_0 be a strongly π_1-closed subgroup of $H \in \mathrm{Hall}_{\pi_1}(G)$ with respect to G. Then the following assertions hold:*

1) *if a group L_0 is strongly π'-closed in $L \in \mathrm{Hall}_{\pi'}(G)$ with respect to G, then there are elements $s \in G$ and $t \in G$ such that*

$$H_0^s L_0^t = L_0^t H_0^s;$$

2) *if π_2 is a subset of π such that either $\pi_1 \cap \pi_2 = \varnothing$ or $\pi_1 = \pi_2$ and a subgroup L_0 is strongly π_2-closed in $L \in \mathrm{Hall}_{\pi_2}(G)$ with respect to G, then there are elements $s \in G$ and $t \in G$ such that $H_0^s L_0^t = L_0^t H_0^s$.*

Proof. 1) Let a group L_0 be strongly π'-closed in $L \in \mathrm{Hall}_{\pi'}(G)$ with respect to G and let $\theta = \{\pi_1, \pi'\}$. Note that, for this choice of a set θ, G admits at least one Hall θ-base by Lemma 2.8. Since the Hall subgroups are conjugate, it follows that the group G contains elements $s \in G$ and $t \in G$ for which H^s and L^t belong to a Hall θ-base. Thus, $H^s L^t = L^t H^s$. In this case, by assertion 3 of Lemma 3.3 and by Lemma 3.4, there are normal subgroups N and M in G such that $N \cap H^s = H_0^s$ and $M \cap L^t = L_0^t$. Hence

$$H_0^s L_0^t = (N \cap H^s)(M \cap L^t) \subseteq NL^t \cap H^s M \cap H^s L^t.$$

Since $N \cap H^s = H_0^s$, we have $H_0^s \in \mathrm{Hall}_{\pi_1}(N)$. Then it follows from $L^t \in \mathfrak{E}_{\pi'} \subseteq \mathfrak{E}_{\pi_1'}$ that

$$|NL^t| = |H_0^s| \cdot l, \qquad \text{where } l \text{ is a } \pi_1'\text{-number.}$$

Similarly, $|H^s M| = |L_0^t| \cdot m$, where m is a π-number. Since $|NL^t \cap H^s M \cap H^s L^t|$ divides $|NL^t|$, $|H^s M|$, and $|H^s L^t|$, we obtain

$$|NL^t \cap H^s M \cap H^s L^t| \le |H_0^s| \cdot |L_0^t| = |H_0^s L_0^t|.$$

Hence $NL^t \cap H^s M \cap H^s L^t = H_0^s L_0^t$, and $H_0^s L_0^t$ is a subgroup of G. This fact completes the proof of assertion 1).

2) Let the group L_0 be strongly π_2-closed in $L \in \mathrm{Hall}_{\pi_2}(G)$ with respect to G and let $\pi_2 \subseteq \pi$. Consider two cases.

a) *Case* $\pi_1 \cap \pi_2 = \varnothing$. Let $\theta = \{\pi_1, \pi_2\}$. Then, by Lemma 2.8, the group G admits at least one Hall θ-base. As in the proof of assertion 1), there are elements $s \in G$ and $t \in G$ such that H^s and L^t belong to a Hall θ-base and $H^s L^t = L^t H^s$. Further, taking the equation $\pi_1 \cap \pi_2 = \varnothing$ into account and following the lines of the proof of assertion 1) of the present lemma, one can readily show that

$$NL^t \cap H^s M \cap H^s L^t = H_0^s L_0^t.$$

Thus, $H_0^s L_0^t$ is a subgroup of G and $H_0^s L_0^t = L_0^t H_0^s$.

b) *Case* $\pi_1 = \pi_2$. Since the Hall π_1-subgroups of G are conjugate, there are elements $s \in G$ and $t \in G$ such that $H^s = L^t = \overline{H}$. In this case, by assertion 3 of Lemma 3.3 and by Lemma 3.4, there are normal subgroups N and M in G such that $N \cap \overline{H} = H_0^s$ and $M \cap \overline{H} = L_0^t$. Hence, by Lemma 4.1 of [9],

$$H_0^s L_0^t = (N \cap \overline{H})(M \cap \overline{H}) = NM \cap \overline{H}.$$

Thus, $H_0^s L_0^t$ is a subgroup of G and $H_0^s L_0^t = L_0^t H_0^s$. □

Corollary 3.7 (see [7]). *Let G be a π-solvable group, let p and q be primes in π, and let a group P_0 be strongly closed in $P \in \mathrm{Syl}_p(G)$ with respect to G and a group Q_0 be strongly closed in $Q \in \mathrm{Syl}_q(G)$ with respect to G. Then there are elements $s \in G$ and $t \in G$ such that*

$$P_0^s Q_0^t = Q_0^t P_0^s.$$

Corollary 3.8. *Let G be a π-solvable group and $p \in \pi$. If a group P_0 is strongly closed in $P \in \mathrm{Syl}_p(G)$ with respect to G and H_0 is strongly π'-closed in $H \in \mathrm{Hall}_{\pi'}(G)$ with respect to G, then there are elements $s \in G$ and $t \in G$ such that*

$$P_0^s H_0^t = H_0^t P_0^s.$$

The crucial property for the proof of the main result of the paper is as follows.

Lemma 3.9. *Let G be a π-solvable group, and let $\mathscr{F}$ be a π-saturated Fitting set of G. Let N be a subgroup of G such that G/N is either a π'-group or a nilpotent π-group. If W is an $\mathscr{F}$-maximal subgroup of N and if V_1 and V_2 are $\mathscr{F}$-maximal subgroups G such that $W \le V_1 \cap V_2$, then the subgroups V_1 and V_2 are conjugate in G.*

Proof. Let G be a group of the least order for which the lemma fails. Note that $W = V_1 \cap N = V_2 \cap N$ and $V_i \le N_G(W)$ for $i \in \{1, 2\}$. It can readily be seen that the conditions of the lemma are satisfied for the group $N_G(W)$. If $N_G(W) < G$, then the lemma holds for $N_G(W)$ by induction. In this case, the lemma obviously holds for the group G. Therefore, we may assume that $N_G(W) = G$, i.e., $W \trianglelefteq G$. Thus,

$$V_i/W = V_i/V_i \cap N \cong V_iN/N \le G/N \qquad \text{for} \quad i \in \{1, 2\}.$$

Consider two cases.

Case 1: *G/N is a π'-group.* In that case, V_i/W is a π'-group. Hence there are Hall π'-subgroups H_i/W of the quotient group G/W such that $V_i/W \le H_i/W$. Since H_i/W are π'-subgroups and $W \trianglelefteq G$, it follows that $O^{\pi'}(H_i) \trianglelefteq W$. Thus, $O^{\pi'}(H_i) \in \mathscr{F}$. Since $\mathscr{F}$ is a π-saturated Fitting set of G, it follows that $H_i \in \mathscr{F}$. In this case, since V_i is $\mathscr{F}$-maximal, one can conclude that $V_i = H_i$, and the conjugacy of V_1 and V_2 follows from the conjugacy of the Hall π'-subgroups.

Case 2: *G/N is a nilpotent π-group.* In this case, since W is $\mathscr{F}$-maximal in N and $\mathscr{F}$ is π-saturated, it follows that N/W is also a π-group. Hence by the isomorphism

$$G/N \cong G/W/N/W,$$

we see that G/W is a solvable π-group, and the proof is just like that of Lemma VIII.2.8 in [4]. □

The main result of the paper is the following theorem.

Theorem 3.10. *Let G be a π-solvable group, and let $\mathscr{F}$ be a π-saturated Fitting set of G. Then the group G contains $\mathscr{F}$-injectors, and every two of them are conjugate.*

Proof. Let us prove the theorem by induction on the order of the group for all pairs $(G, \mathscr{F})$ satisfying the conditions of the theorem. Let G be a counterexample of minimal order, and let M be a maximal normal subgroup of G. Since the group G is π-solvable, it follows that the quotient group G/M is either a π'-group or an elementary Abelian p-group for some prime $p \in \pi$. Consider the following two cases.

Case 1: *G/M is a π'-group for every maximal normal subgroup M of G.* By the induction assumption, there are $\mathscr{F}$-injectors in M. Let V_1 be an $\mathscr{F}$-injector of M and let $\overline{V}_1$ be an $\mathscr{F}$-maximal subgroup of G such that $V_1 \le \overline{V}_1$. We claim that $\overline{V}_1 \cap N$ is an $\mathscr{F}$-injector of N for every maximal normal subgroup N of G.

By induction, there are $\mathscr{F}$-injectors of N, and any two of them are conjugate. Let V_2 be an $\mathscr{F}$-injector of N and let $\overline{V}_2$ be a maximal $\mathscr{F}$-subgroup of G such that $V_2 \le \overline{V}_2$. It follows from the conjugacy of injectors of M and N, and also of $M \cap N$, that

$$W = V_1 \cap M \cap N = V_2 \cap M \cap N.$$

Then $W \le \overline{V}_1 \cap \overline{V}_2$. Since $G/M \cap N$ is a π'-group in the present case, it follows that, by Lemma 3.9, there is an element $x \in G$ for which $\overline{V}_1^x = \overline{V}_2$. Hence

$$(\overline{V}_1 \cap N)^x = \overline{V}_1^x \cap N = \overline{V}_2 \cap N = V_2.$$

It follows now from the conjugacy of the $\mathscr{F}$-injectors of N that $\overline{V}_1 \cap N$ is an $\mathscr{F}$-injector of N for every maximal normal subgroup N of G. Hence the subgroup $\overline{V}_1$ is an $\mathscr{F}$-injector of G, and this proves the existence of $\mathscr{F}$-injectors of G.

Let us prove now the conjugacy of $\mathscr{F}$-injectors of G. Let V_1 and V_2 be $\mathscr{F}$-injectors of G. Then the subgroups $V_1 \cap M$ and $V_2 \cap M$ are $\mathscr{F}$-injectors of M. Hence, by induction, there is an element x of the group M such that

$$(V_1 \cap M)^x = V_1^x \cap M = V_2 \cap M.$$

Let

$$W = V_1^x \cap M = V_2 \cap M.$$

Then V_1^x and V_2 are $\mathscr{F}$-maximal subgroups of G containing W. Moreover, the subgroup W is an $\mathscr{F}$-injector of M. Hence V_1^x and V_2 are conjugate in G, and this completes the proof of the theorem in Case 1.

Case 2. *There is a maximal normal subgroup M of G such that G/M is a p-group for some number $p \in \pi$*. In this case, $O^p(G) < G$. If $O^p(G) = 1$, then the group G is solvable, and the theorem holds by Lemma 2.3. Let $O^p(G) \neq 1$. Then, by induction, there are $\mathscr{F}$-injectors in the group $O^p(G)$, and any two of them are conjugate. If a subgroup S is an $\mathscr{F}$-injector of $O^p(G)$, then the subgroup S^g is an $\mathscr{F}$-injector of $O^p(G)$ for any $g \in G$. Applying induction again, we see that $S^g = S^h$ for some element $h \in O^p(G)$. By the Frattini lemma, $G = N_G(S)O^p(G)$. Hence if P is a Sylow p-subgroup of $N_G(S)$, then $G = PO^p(G)$.

Let R be a subgroup generated by the $\mathscr{F}$-subgroups of the group PS that contain S. Since every subgroup of this kind is subnormal in PS, it follows that $R \in \mathscr{F}$.

Let T be an $\mathscr{F}$-subgroup of G such that S is contained in T. Note that $T \cap O^p(G)$ is an $\mathscr{F}$-subgroup. It follows from the $\mathscr{F}$-maximality of S in $O^p(G)$ that $S = T \cap O^p(G)$. Hence $T \le N_G(S)$. Thus, every Sylow p-subgroup of T is conjugate in $N_G(S)$ to a subgroup of P. Since the quotient group

$$T/S = T/T \cap O^p(G) \cong TO^p(G)/O^p(G)$$

is a p-group, it follows that T is conjugate to a subgroup of the form P_0S in $N_G(S)$ for some subgroup P_0 of P. Hence all extensions of S in $\mathscr{F}$ are conjugate in $N_G(S)$ to subgroups of R. In particular, if there are $\mathscr{F}$-injectors of G, then they are conjugate to R.

Thus, to complete the proof of the theorem, it remains to show that R is an $\mathscr{F}$-injector of G. Since the subgroup R is $\mathscr{F}$-maximal in G, it suffices to prove that R contains an $\mathscr{F}$-injector of a subgroup L for every maximal normal subgroup L of G.

Since the group G is π-solvable, it follows that either $|G : L| = q$ for some prime $q \in \pi$ or $|G : L|$ is a π'-number.

Let T be an $\mathscr{F}$-injector of the group L. The subgroups

$$T \cap L \cap O^p(G) = T \cap O^p(G) \qquad \text{and} \qquad S \cap L \cap O^p(G) = S \cap L$$

are $\mathscr{F}$-injectors of the normal subgroup $L \cap O^p(G)$. Hence these subgroups are conjugate in the group $L \cap O^p(G)$. Choose a group T in such a way that

$$T \cap O^p(G) = L \cap S = U.$$

Consider the following two cases separately.

Case 2.1. *The index $|G : L|$ is a π'-number*. Let $P_1 \in \mathrm{Syl}_p(T)$, and let $H_1 \in \mathrm{Hall}_{\pi'}(S)$. Note that the group

$$T/U = T/T \cap O^p(G) \cong TO^p(G)/O^p(G)$$

is a p-group, and the group

$$S/U = S/S \cap L \cong SL/L$$

is a π'-group. Hence $T = P_1U$ and $S = H_1U$. Since S and T are subgroups of $N_G(U)$, it follows that there are a Sylow subgroup P and a Hall π'-subgroup H of $N_G(U)$ for which $P_1 \le P$ and $H_1 \le H$. If $g \in N_G(U)$, then

$$(H_1^g \cap H)U \le S^g \in \mathscr{F}.$$

Since HU/U is a π'-group, it follows that $\langle H_1^g \cap H, H_1\rangle U/U$ is also a π'-group. Since the Fitting set is π-saturated and $U \trianglelefteq HU$, it follows now that the group $\langle H_1^g \cap H, H_1\rangle U$ is an $\mathscr{F}$-subgroup of HU. Thus,

$$S \le \langle H_1^g \cap H, H_1\rangle U \le \langle S^g, S\rangle \le O^p(G).$$

It follows from the $\mathscr{F}$-maximality of S in $O^p(G)$ that $H_1^g \cap H \le H_1$. Thus, H_1 is strongly π'-closed in H with respect to $N_G(U)$. It can readily be seen that $(P_1^g \cap P)U$ and $T = P_1U$ are subnormal subgroups of PU and, therefore, $\langle P_1^g \cap P, P_1\rangle U$ is an $\mathscr{F}$-subgroup of PU. In this case,

$$T \le \langle P_1^g \cap P, P_1\rangle U \le \langle T^g, T\rangle \le L,$$

and we have $P_1^g \cap P \le P_1$ because T is $\mathscr{F}$-maximal in L. Thus, P_1 is strongly closed in P with respect to $N_G(U)$. Therefore, by Corollary 3.8, we conclude that there is an element $g \in N_G(U)$ for which the product $P_1^g H_1$ is a subgroup of $N_G(U)$.

Let

$$K = P_1^g H_1 U = (P_1U)^g (H_1U) = T^g S.$$

K is a subgroup. Then

$$K \cap O^p(G) = T^g S \cap O^p(G) = (T^g \cap O^p(G))S = (T \cap O^p(G))^g S = U^g S = US = S$$

and, similarly, $K \cap L = T^g$. Hence S and T^g are normal $\mathscr{F}$-subgroups of K and, therefore, $K \in \mathscr{F}$. Since S is contained in K, it follows that R contains a subgroup conjugate to K. Hence R contains an $\mathscr{F}$-injector of the subgroup L, and the theorem is proved in Case 2.1.

It remains to consider

Case 2.2. *The index* $|G : L|$ *is equal to* q *for some prime* $q \in \pi$. Let

$$P_1 \in \mathrm{Syl}_p(T) \qquad \text{and} \qquad Q_1 \in \mathrm{Syl}_q(S).$$

Note that the group

$$T/U = T/T \cap O^p(G) \cong TO^p(G)/O^p(G)$$

is a p-group, and the group

$$S/U = S/S \cap L \cong SL/L$$

is a q-group. In this case, $T = P_1U$ and $S = Q_1U$. Since S and T are subgroups of $N_G(U)$, it follows that there are Sylow subgroups P and Q of $N_G(U)$ such that $P_1 \le P$ and $Q_1 \le Q$. If $g \in N_G(U)$, then

$$(P_1^g \cap P)U \le T^g \in \mathscr{F}.$$

As in Case 2.1, P_1 is strongly closed in P with respect to $N_G(U)$ and Q_1 is strongly closed in Q with respect to $N_G(U)$. By Corollary 3.7, there is an element $g \in N_G(U)$ such that the product $P_1^g Q_1$ is a subgroup of $N_G(U)$.

Let

$$K_2 = P_1^g Q_1 U = (P_1U)^g (Q_1U) = T^g S.$$

K_2 is a subgroup. In this case,

$$K_2 \cap O^p(G) = T^g S \cap O^p(G) = (T^g \cap O^p(G))S = (T \cap O^p(G))^g S = U^g S = US = S.$$

Similarly, one can show that $K_2 \cap L = T^g$. Hence3 S and T^g are normal $\mathscr{F}$-subgroups of K_2 and $K_2 \in \mathscr{F}$. Since S is contained in K_2, it follows that R contains a subgroup conjugate to K_2. Hence R contains an $\mathscr{F}$-injector of the subgroup L. □

Following Definition 3.1, we say that a Fitting class $\mathfrak{F}$ is π-*saturated* if $\mathfrak{F} = \mathfrak{F}\mathfrak{E}_{\pi'}$.

Corollary 3.11. *Let G be a π-solvable group, and let $\mathfrak{F}$ be a π-saturated Fitting class of G. Then there are $\mathfrak{F}$-injectors in the group G, and every two of them are conjugate.*

Proof. Let

$$\mathscr{F} = \mathrm{Tr}_{\mathfrak{F}}(G) = \{H \le G : H \in \mathfrak{F}\}.$$

Then $\mathscr{F}$ is a π-saturated Fitting set, and the sets of $\mathscr{F}$-injectors and $\mathfrak{F}$-injectors of the group G coincide. The existence and conjugacy of $\mathfrak{F}$-injectors G now immediately follows from Theorem 3.10. □

A group G is said to be (see [5, p. 251]) π-*closed* if it has a normal Hall π-subgroup and π-*special* if it has a normal nilpotent Hall π-subgroup. It can readily be seen that the class of all π-closed groups and the class of all π-special groups are π-saturated Fitting classes. Therefore, the following assertions hold.

Corollary 3.12. *Every π-solvable group contains a unique class of conjugate π-closed injectors.*

Corollary 3.13. *Every π-solvable group contains π-special injectors, and any two of them are conjugate.*

Corollary 3.14 (see [7, Theorem 3])**.** *If $\mathscr{F}$ is a Fitting set of a solvable group G, then G contains $\mathscr{F}$-injectors, and every two of them are conjugate.*

Corollary 3.15 (see [1, Theorem 1])**.** *If $\mathfrak{F}$ is a Fitting class and a group G is solvable, then G contains $\mathfrak{F}$-injectors, and every two of them are conjugate.*

4. INJECTORS OF GROUPS WITH π-SOLVABLE QUOTIENT GROUP

In this subsection, we extend known results of Shemetkov [2] and Ballester-Bolinches [6, Theorem 2.4.27] on the existence and conjugacy of $\mathscr{F}$-injectors of a group G under the assumption that the quotient group by the $\mathscr{F}$-radical is π-solvable (rather than the group G itself is π-solvable).

Theorem 4.1. *Let $\mathscr{F}$ be a Fitting set of a group G and let $G/G_{\mathscr{F}}$ be a π-solvable group, where $\pi = \sigma(\mathscr{F})$. Then the group G contains $\mathscr{F}$-injectors, and every two of them are conjugate.*

Proof. By Lemma 2.5, the set

$$\mathscr{F}^* = \{H/G_{\mathscr{F}} : H \in \mathscr{F} \wedge G_{\mathscr{F}} \le H\}$$

is a Fitting set of the group $G/G_{\mathscr{F}}$, and

$$\mathscr{F}_0 = \{S \le G : SG_{\mathscr{F}}/G_{\mathscr{F}} \in \mathscr{F}^* \wedge S \trianglelefteq SG_{\mathscr{F}}\}$$

is a Fitting set of G.

Let us show first that the equation $K_{\mathscr{F}} = K_{\mathscr{F}_0}$ holds for every subnormal subgroup K in G. Obviously, $\mathscr{F}_0 \subseteq \mathscr{F}$ and $K_{\mathscr{F}_0} \le K_{\mathscr{F}}$. Since $K_{\mathscr{F}}G_{\mathscr{F}} \in \mathscr{F}$ and $K_{\mathscr{F}}G_{\mathscr{F}}/G_{\mathscr{F}} \in \mathscr{F}^*$, it follows that $K_{\mathscr{F}} \in \mathscr{F}_0$ and $K_{\mathscr{F}} \le K_{\mathscr{F}_0}$. Thus, $K_{\mathscr{F}} = K_{\mathscr{F}_0}$.

Note that $\sigma(\mathscr{F}^*) \subseteq \sigma(\mathscr{F})$, and $G/G_{\mathscr{F}}$ is a π-solvable group for $\pi = \sigma(\mathscr{F}^*)$. Now, applying Lemma 2.4, we see that the π-solvable group $G/G_{\mathscr{F}}$ contains an $\mathscr{F}^*$-injector $V/G_{\mathscr{F}}$. Hence, by Lemma 2.5, V is an $\mathscr{F}_0$-injector of G. We claim that V is an $\mathscr{F}$-injector of G. To this end, it suffices to show that, for every subnormal subgroup K of G, the subgroup $K \cap V$ is $\mathscr{F}$-maximal in K. Let there be a subgroup $W \in \mathscr{F}$ such that $K \cap V \le W \le K$. Then

$$(K \cap V)G_{\mathscr{F}}/G_{\mathscr{F}} = (V/G_{\mathscr{F}}) \cap (KG_{\mathscr{F}}/G_{\mathscr{F}}) \le WG_{\mathscr{F}}/G_{\mathscr{F}} \le KG_{\mathscr{F}}/G_{\mathscr{F}}.$$

Note that $K \cap V$ is an $\mathscr{F}_0$-injector of K. Hence

$$K_{\mathscr{F}} = K_{\mathscr{F}_0} \le V \cap K \le W,$$

and therefore $K_{\mathscr{F}} \le W$. Now $K_{\mathscr{F}} = K \cap G_{\mathscr{F}}$ by Lemma 2.1, and thus

$$WG_{\mathscr{F}} \cap K = W(G_{\mathscr{F}} \cap K) = WK_{\mathscr{F}} = W.$$

Hence W is subnormal in $WG_{\mathscr{F}}$ and $WG_{\mathscr{F}} \in \mathscr{F}$, i.e., $WG_{\mathscr{F}}/G_{\mathscr{F}} \in \mathscr{F}^*$. Since $(V/G_{\mathscr{F}}) \cap (KG_{\mathscr{F}}/G_{\mathscr{F}})$ is $\mathscr{F}$-maximal in $KG_{\mathscr{F}}/G_{\mathscr{F}}$, we have the equation

$$(K \cap V)G_{\mathscr{F}} = WG_{\mathscr{F}}.$$

Therefore,

$$K \cap V = (K \cap V)(G_{\mathscr{F}} \cap K) = (K \cap V)G_{\mathscr{F}} \cap K = WG_{\mathscr{F}} \cap K = W$$

and V is an $\mathscr{F}$-injector of G.

Let us prove the conjugacy of injectors of G. Let V be an $\mathscr{F}$-injector of G. Then, by assertion 1 of Lemma 2.5, $V/G_{\mathscr{F}}$ is an $\mathscr{F}^*$-injector of $G/G_{\mathscr{F}}$. However, by Lemma 2.4, the $\mathscr{F}^*$-injectors of $G/G_{\mathscr{F}}$ are conjugate, and thus so are the $\mathscr{F}$-injectors of G. □

Corollary 4.2 (see [2, Theorem 2.2]). *Let $\mathscr{F}$ be a Fitting set of a π-solvable group G, where $\pi = \sigma(\mathscr{F})$. Then the group G contains $\mathscr{F}$-injectors, and any two of them are conjugate.*

Corollary 4.3 (see [6, Theorem 2.4.27]). *Let $\mathscr{F}$ be a Fitting set of a group G, and let $G/G_{\mathscr{F}}$ be a solvable group. Then the group G contains $\mathscr{F}$-injectors, and any two of them are conjugate.*

Corollary 4.4 (see [10]). *Let $\mathfrak{F}$ be a Fitting class, and let $G/G_{\mathfrak{F}}$ be a π-solvable group, where $\pi = \sigma(\mathfrak{F})$. Then the group G contains $\mathfrak{F}$-injectors, and any two of them are conjugate.*

Corollary 4.5 (see [11]). *Let $\mathfrak{F}$ be a Fitting class, and let $G/G_{\mathfrak{F}}$ be a solvable group. Then the group G contains $\mathfrak{F}$-injectors, and any two of them are conjugate.*

In conclusion, note that an interesting problem remains: To find characterizations of $\mathscr{F}$-injectors for a Fitting set $\mathscr{F}$ of a given group, by using radicals and Hall subgroups similar to the characterizations of $\mathfrak{F}$-injectors for Fitting classes $\mathfrak{F}$ that were obtained in [12] and [13].

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