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RESEARCH ARTICLE

Factorizations of Fitting classes

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Abstract In this paper, we prove that there exists a infinite set of non-trivial local Fitting classes every element in which is decomposable as a non-trivial product of Fitting classes such that every factor in the product is neither local nor a formation. In particular, this gives a positive answer to Problem 11.25 a) in The Kourovka Notebook.

Keywords Fitting class, Normal Fitting class, semilocal Fitting class, Lockett class
MSC 20D10

1 Introduction

The solutions of many problems in the theory of classes of finite groups lead to a large number of investigation on the properties of products of classes of groups. In particular, in the theory of Fitting classes, a series of the solutions of many problems related with normal Fitting classes depends strongly on the properties of product of Fitting classes in which the factors have (or do not have) a normal property (see, for example, [2,7,9,17]). In the theory of formations, the results of Shemetkov [22,23] about the structure of formations led to the theory of factorization of formations, and many serious and interesting results about local formations with given inner properties of subformations were obtained by Skiba and some other authors (see [1,12–15,21,24,27]).

In connection with this, the following problem was proposed in The Kourovka Notebook [21].

Problem 1.1 [21, Problem 11.25 a)] Do there exist local Fitting classes (different from the class $\mathfrak{S}$ of all finite soluble groups) which are decomposable into a non-trivial product of Fitting classes such that every factor in the product is neither local nor a formation?

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In [27], by using the class defined by Berger and Cossey [3], a special example has been given.

In this paper, we give a new and broad family of such non-trivial Fitting classes (without using the unwieldy structure of Berger and Cossey [3]) whose elements can be decomposed as a non-trivial product of Fitting classes such that every factor in the product is neither local nor a formation.

All groups considered in this paper are finite and soluble. All unexplained notation and terminology are standard. The reader is refereed to [10] if necessary.

2 Preliminaries

Recall that a class $\mathfrak{F}$ of groups is said to be a Fitting class if $\mathfrak{F}$ is closed with respect to taking normal subgroups and if the conditions

$$G = AB, \quad A, B \trianglelefteq G, \quad A, B \in \mathfrak{F}$$

always imply

$$G \in \mathfrak{F}.$$

Obviously, for a non-empty Fitting class $\mathfrak{F}$, every group G has a largest normal $\mathfrak{F}$-subgroup, which is denoted by $G_{\mathfrak{F}}$ and is called the $\mathfrak{F}$-radical of G.

For a set π of primes, we use $\mathfrak{S}_\pi$ to denote the class of all soluble π-groups. Clearly, every group G has an $\mathfrak{S}_\pi$-residual $G^{\mathfrak{S}_\pi}$, which is the smallest normal subgroup of G satisfying that the factor group $G/G_{\mathfrak{S}_\pi}$ belongs to $\mathfrak{S}_\pi$.

A nonempty Fitting class $\mathfrak{F}$ is said to be normal if the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ is an $\mathfrak{F}$-maximal subgroup of G for every group G.

Let $\mathfrak{F}$ and $\mathfrak{H}$ be two Fitting classes. Then the class $(G \mid G/G_{\mathfrak{F}} \in \mathfrak{H})$ is called the Fitting product of $\mathfrak{F}$ and $\mathfrak{H}$, and denoted here by $\mathfrak{F}\mathfrak{H}$. It is well known that the product of two Fitting classes is also a Fitting class and the multiplication satisfies the associative law (see [10, IX.1.12]).

If $\mathfrak{X}$ is a class of groups, then $\mathrm{Fit}(\mathfrak{X})$ denotes the Fitting class generated by $\mathfrak{X}$, that is, $\mathrm{Fit}(\mathfrak{X})$ is the smallest Fitting class containing $\mathfrak{X}$.

Let $\mathbb{P}$ be the set of all prime numbers. Then a map $f\colon \mathbb{P} \to \{\text{Fitting classes}\}$ is said to be a Hartley function (or directly, an H-function) [26]. For an H-function f, let

$$\pi = \mathrm{Supp}(f) = \{p \in \mathbb{P} \mid f(p) \neq \emptyset\},$$

which is called the support of f, and let

$$LR(f) = \mathfrak{S}_\pi \bigcap \bigg(\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'} \bigg).$$

A Fitting class $\mathfrak{F}$ is called a local Fitting class [26] if $\mathfrak{F} = LR(f)$ for some H-function f.

Let $\Omega = \{f_i \mid i \in I\}$ be some non-empty set of H-functions. Suppose $f_i, f_j \in \Omega$. Then we write $f_i \leqslant f_j$ if $f_i(p) \subseteq f_j(p)$ for all $p \in \mathbb{P}$.

In our proof, we will also need to use the operators '*' and '$_*$' which were defined by Lockett [19]. Actually, every non-empty Fitting class $\mathfrak{F}$ has two associated Fitting classes $\mathfrak{F}^*$ and $\mathfrak{F}_*$, where $\mathfrak{F}^*$ is the smallest Fitting class containing $\mathfrak{F}$ such that

$$(G \times H)_{\mathfrak{F}_*} = G_{\mathfrak{F}_*} \times H_{\mathfrak{F}_*}$$

for all groups G and H; $\mathfrak{F}_*$ is the intersection of all Fitting classes $\mathfrak{X}$ such that $\mathfrak{X}^* = \mathfrak{F}^*$. Recall that a Fitting class $\mathfrak{F}$ is called a Lockett class if $\mathfrak{F}^* = \mathfrak{F}$. It is well known that if $\mathfrak{F} = \mathfrak{S}$, then $\mathfrak{S}_*$ is the smallest normal Fitting class and $\mathfrak{S}_*$ is non-trivial, that is, $\mathfrak{S}_*$ is not equal to 1 nor $\mathfrak{S}$, and $\mathfrak{S}_*$ is not a formation (see [5]).

3 Answer to Problem 1.1 in trivial case

Remark that if a local product is trivial, that is, the local product is the class $\mathfrak{S}$ of all soluble groups, then the following example shows that the answer to Problem 1.1 is simple enough.

Example 3.1 Let $\mathfrak{X}$ be an arbitrary non-trivial (that is, $\mathfrak{X}$ is not equal to 1 nor $\mathfrak{S}$) normal Fitting class. We first prove that $\mathfrak{X}$ is non-local. In fact, if $\mathfrak{X}$ is local, then by [25, Lemma 5], $\mathfrak{X}$ is a Lockett class, and therefore,

$$\mathfrak{X}^* = \mathfrak{X} = \mathfrak{S}$$

by [10, Theorem X.3.7]. This contradiction shows that no non-trivial normal Fitting class is local.

Now, let $\mathfrak{S}_*$ be the smallest normal Fitting class. Then by [7, Theorem 4.3], the Fitting class

$$\mathfrak{F} := \mathfrak{X}\mathfrak{S}_* = \mathfrak{S}.$$

Let f be the H-function such that

$$f(p) = \mathfrak{S} \quad \forall\, p \in \mathbb{P}.$$

Then, clearly, $\mathfrak{F} = LR(f)$. Hence, $\mathfrak{F}$ is a local Fitting class and it is decomposable as a non-trivial product of Fitting classes such that every factor is non-local and is not a formation.

4 Semilocal Fitting classes and their properties

In order to describe the non-trivial local product with the property in Problem 1.1, we need to use the concept of semilocal Fitting classes (see [26] or [20]). Let

$\emptyset \neq \sigma \subseteq \mathbb{P}$. A Fitting class $\mathfrak{F}$ is said to be semilocal if there exists an H-function f such that

$$\mathfrak{F} = SLR(f) := \bigcap_{p\in\sigma} f(p)\mathfrak{S}_{p'},$$

where $\sigma = \mathrm{Supp}(f)$. In this case, f is said to be a semilocal H-function of $\mathfrak{F}$ or $\mathfrak{F}$ is semilocally defined by the H-function f.

The following lemma characterizes the semilocal Fitting classes.

Lemma 4.1 *Let $\mathfrak{F}$ be a Fitting class. Then the following statements hold.*

1) *If $\mathfrak{F}$ is a semilocal Fitting class, then $\mathfrak{F}$ can be defined by a smallest H-function f and f is a integrated, that is,*

$$f(p) \subseteq \mathscr{F}, \quad \forall\, p \in \pi = \mathrm{Supp}(f).$$

2) *If $\mathfrak{F} = SLR(f)$, where f is the smallest H-function of $\mathfrak{F}$ and $\pi =$ $\mathrm{Supp}(f)$, then*

$$f(p) = \mathrm{Fit}(G \in \mathfrak{F} \mid G \simeq H^{\mathfrak{S}_{p'}} \ for\ some\ H \in \mathfrak{F}), \quad \forall\, p \in \pi = \mathrm{Supp}(f).$$

3) *$\mathfrak{F}$ is semilocal if and only if $\mathfrak{F}\mathfrak{S}_{\pi'} = \mathfrak{F}$.*

Proof 1) Suppose that $\mathfrak{F} = SLR(h)$ for some H-function h. Let

$$\varphi(p) = h(p) \cap \mathfrak{F}, \quad \forall\, p \in \pi.$$

Then

$$\varphi \leqslant h, \quad SLR(\varphi) \subseteq \mathfrak{F}.$$

On the other hand, assume that $K \in \mathfrak{F}$. Then

$$K/K_{h(p)} \in \mathfrak{S}_{p'}, \quad \forall\, p \in \pi,$$

and therefore,

$$K^{\mathfrak{S}_{p'}} \subseteq K_{h(p)}.$$

It follows that

$$K^{\mathfrak{S}_{p'}} \in h(p) \cap \mathfrak{F} = \varphi(p), \quad \forall\, p \in \pi.$$

This shows that $K \in SLR(\varphi)$. Consequently,

$$\mathfrak{F} = SLR(\varphi), \quad \varphi(p) \subseteq \mathfrak{F}, \quad \forall\, p \in \pi.$$

Now, let Ω be the set of all semilocal H-functions of $\mathfrak{F}$, and let f be the intersection of all elements in Ω. Then, as above, we see that

$$f(p) \subseteq \mathfrak{F}, \quad \forall\, p \in \pi.$$

We prove that f is also a semilocal H-function of $\mathfrak{F}$. In fact, since $f \leqslant \varphi$ and $\mathfrak{F} = SLR(\varphi)$, we have

$$SLR(f) \subseteq \mathfrak{F}.$$

Let $L \in \mathfrak{F}$. Then
$$L/L_{h_i(p)} \in \mathfrak{S}_{p'}, \quad \forall\ h_i \in \Omega,\ \forall\ p \in \pi.$$
This implies that
$$L/\bigcap_{i\in I} L_{h_i(p)} \in \mathfrak{S}_{p'}, \quad \forall\ p \in \pi,\ \forall\ i \in I.$$
Since, obviously,
$$\bigcap_{i\in I} L_{f_i(p)} = L_{f(p)},$$
we have
$$L \in f(p)\mathfrak{S}_{p'}, \quad \forall\ p \in \pi.$$
It follows that
$$\mathfrak{F} \subseteq SLR(f).$$

2) Assume that $\mathfrak{F} = SLR(f)$, where f is the smallest H-function of $\mathfrak{F}$. Let
$$g(p) = \{G \in \mathfrak{F} \mid G \simeq H^{\mathfrak{S}_{p'}} \text{ for some } H \in \mathfrak{F}\}, \quad h(p) = \mathrm{Fit}(g(p)), \quad p \in \pi.$$
We prove that
$$f(p) = h(p), \quad \forall\ p \in \pi.$$
Put $\mathfrak{H} = SLR(h)$. If $X \in \mathfrak{F}$, then $X^{\mathfrak{S}_{p'}} \in \mathfrak{F}$. Since
$$X^{\mathfrak{S}_{p'}} = (X^{\mathfrak{S}_{p'}})^{\mathfrak{S}_{p'}},$$
we have
$$X^{\mathfrak{S}_{p'}} \in g(p).$$
Hence,
$$X^{\mathfrak{S}_{p'}} \in h(p), \quad \forall\ p \in \pi.$$
This shows that
$$X \in \bigcap_{p\in\pi} h(p)\mathfrak{S}_{p'} = \mathfrak{H}.$$
Thus, $\mathfrak{F} \subseteq \mathfrak{H}$. In order to complete the proof of 2), we only need to prove that $h(p) \subseteq f(p)$ for all primes $p \in \pi$ since f is the smallest H-function of $\mathfrak{F}$. Let Y be a group in $g(p)$ $(p \in \pi)$. Then $Y \simeq M^{\mathfrak{S}_{p'}}$ for some $M \in \mathfrak{F}$. Since $M \in \mathfrak{F}$, we have
$$M/M_{f(p)} \in \mathfrak{S}_{p'}.$$
It follows that
$$M^{\mathfrak{S}_{p'}} \subseteq M_{f(p)},$$
and therefore, $M^{\mathfrak{S}_{p'}} \in f(p)$ since $f(p)$ is normal closed. This means that $Y \in f(p)$ for all $p \in \pi$. Consequently, $g \leqslant f$. Hence,
$$h(p) = \mathrm{Fit}(g(p)) \subseteq \mathrm{Fit}(f(p)) = f(p), \quad \forall\ p \in \pi.$$

3) Let $\mathfrak{F} = SLR(f)$ be a semilocal Fitting class. We prove that

$$\mathfrak{F}\mathfrak{S}_{\pi'} = \mathfrak{F}.$$

Clearly,

$$\mathfrak{F} \subseteq \mathfrak{F}\mathfrak{S}_{\pi'}.$$

Assume that G is a group in $\mathfrak{F}\mathfrak{S}_{\pi'}$. Then

$$G/G_{\mathfrak{F}} \in \mathfrak{S}_{\pi'},$$

and therefore, $G^{\mathfrak{S}_{\pi'}}$ is an $\mathfrak{F}$-subgroup of G. This implies that

$$G^{\mathfrak{S}_{\pi'}} \in f(p)\mathfrak{S}_{p'}, \quad \forall\ p \in \pi.$$

But since $\mathfrak{S}_{\pi'} \subseteq \mathfrak{S}_{p'}$ for any $p \in \pi$, we have

$$G^{\mathfrak{S}_{p'}} \subseteq G^{\mathfrak{S}_{\pi'}}.$$

By the multiplicative associative law of Fitting classes, we obtain that

$$G \in f(p)\mathfrak{S}_{p'}, \quad \forall\ p \in \pi.$$

Hence, $G \in \mathfrak{F}$, and therefore,

$$\mathfrak{F}\mathfrak{S}_{\pi'} = \mathfrak{F}.$$

Conversely, assume that $\mathfrak{F} = \mathfrak{F}\mathfrak{S}_{\pi'}$. Let $\pi = \mathrm{Supp}(f)$, where f is the H-function such that

$$f(p) = \mathrm{Fit}\{G \in \mathfrak{F} \mid G \simeq H^{\mathfrak{S}_{p'}} \text{ for some } H \in \mathfrak{F}\}, \quad \forall\ p \in \pi,$$

and let $\mathfrak{M} = SLR(f)$. In order to prove 3), we only need to prove that $\mathfrak{F} = \mathfrak{M}$.

Assume that $G \in \mathfrak{M}$. Then

$$G/G_{f(p)} \in \mathfrak{S}_{p'},$$

and therefore,

$$G^{\mathfrak{S}_{p'}} \subseteq G_{f(p)}.$$

Let

$$f_1(p) = \{G \in \mathfrak{F} \mid G \simeq H^{\mathfrak{S}_{p'}}\ (H \in \mathfrak{F})\}, \quad \forall\ p \in \pi.$$

Since $f_1(p) \subseteq \mathfrak{F}$, we have

$$f(p) = \mathrm{Fit}(f_1(p)) \subseteq \mathrm{Fit}(\mathfrak{F}) = \mathfrak{F}, \quad \forall\ p \in \pi.$$

Hence, $G^{\mathfrak{S}_{p'}} \in \mathfrak{F}$ for every prime $p \in \pi$, and therefore,

$$R = \prod_{p\in\pi} G^{\mathfrak{S}_{p'}}$$

is a normal $\mathfrak{F}$-subgroup of G. Since

$$G/R \simeq (G/G^{\mathfrak{S}_{p'}})/(R/G^{\mathfrak{S}_{p'}}), \quad \forall\, p \in \pi,$$

we have

$$G/R \in \bigcap_{p\in\pi} \mathfrak{S}_{p'} = \mathfrak{S}_{\pi'}.$$

This implies that

$$G \in \mathfrak{F}\mathfrak{S}_{\pi'} = \mathfrak{F},$$

and therefore, $\mathfrak{M} \subseteq \mathfrak{F}$.

Now, assume that $G \in \mathfrak{F}$. Since

$$(G^{\mathfrak{S}_{p'}})^{\mathfrak{S}_{p'}} = G^{\mathfrak{S}_{p'}},$$

we have

$$G^{\mathfrak{S}_{p'}} \in f_1(p) \subseteq f(p).$$

Hence,

$$G \in \bigcap_{p\in\pi} f(p)\mathfrak{S}_{p'} = \mathfrak{M}.$$

Therefore, $\mathfrak{F} = \mathfrak{M}$. This completes the proof. □

By using Lemma 4.1, we point out some classical Fitting classes semilocally defined by H-functions.

Example 4.2 Let $\mathfrak{F}$ be a Fitting class, $\emptyset \neq \pi \subseteq \mathbb{P}$, and let $\mathfrak{R}_\pi(\mathfrak{F})$, $\mathfrak{L}_\pi(\mathfrak{F})$, $\mathfrak{L}'_\pi(\mathfrak{F})$, and $\mathfrak{K}_\pi(\mathfrak{F})$ be Fitting classes defined as follows, respectively:

(1) $G \in \mathfrak{R}_\pi(\mathfrak{F})$ if and only if the $\mathfrak{F}$-radical of G contains some Hall π-subgroup of G;

(2) $G \in \mathfrak{L}_\pi(\mathfrak{F})$ if and only if every $\mathfrak{F}$-injector of G contains some Hall π-subgroup of G;

(3) $G \in \mathfrak{L}'_\pi(\mathfrak{F})$ if and only if every Hall π-subgroup of G is a normal subgroup of some $\mathfrak{F}$-injector of G;

(4) $G \in \mathfrak{K}_\pi(\mathfrak{F})$ if and only if every Hall π-subgroup of G is an $\mathfrak{F}$-group.

By using the above classical semilocal Fitting classes, a series of profound and interesting results about Fitting class with a description of the structure of injectors and the inner structure of classes of groups were obtained (see [10, IX-X]).

From Lockett [18], Brison [6], and Gállego [11], we know that each of above classes (1)–(4) is a Fitting class with the property $\mathfrak{X}\mathfrak{S}_{\pi'} = \mathfrak{X}$, where $\mathfrak{X} \in \{\mathfrak{R}_\pi(\mathfrak{F}), \mathfrak{L}_\pi(\mathfrak{F}), \mathfrak{L}'_\pi(\mathfrak{F}), \mathfrak{K}_\pi(\mathfrak{F})\}$. Therefore, by Lemma 4.1, they are all semilocal Fitting classes, and therefore, they have an H-function as in Lemma 4.1.

The well-known class $\mathfrak{N}_\pi\mathfrak{S}_{\pi'}$ of π-special groups and the class $\mathfrak{S}_\pi\mathfrak{S}_{\pi'}$ of π-closed groups introduced by Čunihin [8] are also Fitting classes semilocally defined as in Lemma 4.1. In fact, for example, the class of π-closed groups is a special case of $\mathfrak{L}'_\pi(\mathfrak{F})$ since if $\mathfrak{F} = \mathfrak{S}$, then $\mathfrak{L}'_\pi(\mathfrak{F}) = \mathfrak{S}_\pi \mathfrak{S}_{\pi'}$.

Proposition 4.3 *The Fitting classes $\mathfrak{R}_\pi(\mathfrak{F})$, $\mathfrak{L}_\pi(\mathfrak{F})$, $\mathfrak{L}'_\pi(\mathfrak{F})$, and $\mathfrak{K}_\pi(\mathfrak{F})$ are not local, in general.*

Proof We first prove that $\mathfrak{L}_\pi(\mathfrak{F})$ is not local.

Let p and q be two primes such that $p \mid (q-1)$, and let $G = D^p_{q^n}$ be a monolitic group (see [4, p.3]) with a normal abelian Sylow q-subgroup of exponent q^n and a cyclic Hall q'-subgroup of order p. Let $\pi = \pi(G)$, and let $\mathfrak{S}_*$ be the smallest normal Fitting class. By [4, Property 3], the group G does not belong to $\mathfrak{S}_*$. Since $\mathfrak{L}_\pi(\mathfrak{S}_*)$ is the class of all groups G such that every Hall π-subgroup of G is contained in an $\mathfrak{S}_*$-injector of G, we have

$$G \notin \mathfrak{L}_\pi(\mathfrak{S}_*),$$

and therefore,

$$\mathfrak{L}_\pi(\mathfrak{S}_*) \neq \mathfrak{S}.$$

Assume that the Fitting class $\mathfrak{L}_\pi(\mathfrak{S}_*)$ is local. Then by [25, Lemma 5], $\mathfrak{L}_\pi(\mathfrak{S}_*)$ is a Lockett class, that is,

$$(\mathfrak{L}_\pi(\mathfrak{S}_*))^* = \mathfrak{L}_\pi(\mathfrak{S}_*).$$

Since $\mathfrak{S}_*$ is normal, by [10, Theorem X.1.37], we know that $\mathfrak{L}_\pi(\mathfrak{S}_*)$ is also a normal Fitting class. Hence, by [10, Theorem X.3.7]), we have

$$\mathfrak{L}_\pi(\mathfrak{S}_*) = \mathfrak{S},$$

which contradicts the fact that there exists a group $G \in \mathfrak{S} \setminus \mathfrak{L}_\pi(\mathfrak{S}_*)$. Thus, $\mathfrak{L}_\pi(\mathfrak{S}_*)$ is non-local.

Obviously,

$$\mathfrak{K}_\pi(\mathfrak{F}) = \mathfrak{L}_\pi(\mathfrak{F} \cap \mathfrak{S}_\pi)$$

(see [10, Theorem IX.1.25(a)]) and

$$\mathfrak{K}_\pi(\mathfrak{F}^*) = (\mathfrak{K}_\pi(\mathfrak{F}))^*$$

(see [10, Theorem X.1.37]. Hence, we can analogously prove that the Fitting class $\mathfrak{K}_\pi(\mathfrak{F})$ is non-local for $\mathfrak{F} = \mathfrak{S}_*$.

Note that for a non-trivial normal Fitting class $\mathfrak{F}$, by [10, Theorem X.1.37], we have

$$\mathfrak{R}_\pi(\mathfrak{F}) = \mathfrak{L}_\pi(\mathfrak{F}).$$

Hence, in general, $\mathfrak{R}_\pi(\mathfrak{F})$ is not local.

Finally, by [11, Corollary 3.3], we know that

$$\mathfrak{L}'_\pi(\mathfrak{X} \cap \mathfrak{S}_\pi \mathfrak{S}_{\pi'}) = \mathfrak{K}_\pi(\mathfrak{X}).$$

Hence, the Fitting class $\mathfrak{L}'_\pi(\mathfrak{F}) = \mathfrak{K}_\pi(\mathfrak{X}_*)$ is not local for $\mathfrak{X} = \mathfrak{S}_*$ and $\mathfrak{F} = \mathfrak{S}_* \cap \mathfrak{S}_\pi \mathfrak{S}_{\pi'}$. □

5 Main results

In this section, we give the following theorem in which we construct out a wide family of local Fitting classes in which every element is decomposable as a non-trivial product of Fitting classes such that every factor in the product is neither local nor a formation.

Theorem 5.1 *There exists an infinite set of non-trivial local Fitting classes, every element in which is decomposable into a non-trivial product of Fitting classes such that every factor in the product is neither local nor a formation, but the product is semilocal.*

Proof First, we prove that for an arbitrary set π of primes with $|\pi| \geqslant 2$, the class

$$\mathfrak{F} = (\mathfrak{S}_\pi)_*\mathfrak{S}_{\pi'}$$

is semilocally defined, but $\mathfrak{F}$ is neither local nor a formation.

If $\pi = \mathbb{P}$, then $\mathfrak{F} = \mathfrak{S}_*$, and therefore, $\mathfrak{F}$ is neither local nor a formation by Example 3.1. Let

$$SLR(f) = \bigcap_{p\in\mathbb{P}} f(p)\mathfrak{S}_{p'},$$

where f is the H-function such that $f(p) = \mathfrak{S}_*$ for all primes p. Then

$$SLR(f) = \mathfrak{S}_*\left(\bigcap_{p\in\mathbb{P}} \mathfrak{S}_{p'}\right) = \mathfrak{S}_*(1) = \mathfrak{S}_*.$$

Thus, $\mathfrak{S}_*$ is semilocal.

Now, assume that $\pi \subset \mathbb{P}$. Then $\mathfrak{F} = \mathfrak{F}\mathfrak{S}_{\pi'}$. Hence, by Lemma 4.1(3), $\mathfrak{F}$ is semilocal.

Assume that the Fitting class $\mathfrak{F}$ is local. Then by [25, Lemma 5], $\mathfrak{F}$ is a Lockett class, and therefore,

$$\mathfrak{F} = \mathfrak{F}^* = ((\mathfrak{S}_\pi)_*\mathfrak{S}_{\pi'})^*.$$

Then by [25, Lemma 3] and the property of the operator '*' (see [10, Theorem X.1.15]), we have

$$\mathfrak{F} = ((\mathfrak{S}_\pi)_*)^*\mathfrak{S}_{\pi'} = \mathfrak{S}_\pi^*\mathfrak{S}_{\pi'}.$$

But $\mathfrak{S}_\pi = LR(f)$, where f is an H-function such that

$$f(p) = \begin{cases} \mathfrak{S}_\pi, & p \in \pi, \\ \emptyset, & p \in \pi'. \end{cases}$$

Hence, by [25, Lemma 5], we have

$$\mathfrak{S}_\pi^* = \mathfrak{S}_\pi,$$

and therefore,

$$\mathfrak{F} = \mathfrak{S}_\pi \mathfrak{S}_{\pi'}.$$

On the other hand, by [16, Theorem B], the local Fitting class $\mathfrak{S}_\pi$ satisfies the Lockett conjecture. Hence,

$$(\mathfrak{S}_\pi)_* = \mathfrak{S}_\pi^* \cap \mathfrak{S}_*$$

(see [16]), and therefore,

$$(\mathfrak{S}_\pi)_* = \mathfrak{S}_\pi \cap \mathfrak{S}_*.$$

This implies that

$$\mathfrak{F} = (\mathfrak{S}_\pi \cap \mathfrak{S}_*)\mathfrak{S}_{\pi'} = \mathfrak{S}_\pi \mathfrak{S}_{\pi'} \cap \mathfrak{S}_* \mathfrak{S}_{\pi'} = \mathfrak{S}_\pi \mathfrak{S}_{\pi'}.$$

It follows that

$$\mathfrak{S}_\pi \mathfrak{S}_{\pi'} \subseteq \mathfrak{S}_* \mathfrak{S}_{\pi'},$$

and consequently,

$$\mathfrak{S}_\pi \subseteq \mathfrak{S}_* \mathfrak{S}_{\pi'} \cap \mathfrak{S}_* \mathfrak{S}_\pi = \mathfrak{S}_*(\mathfrak{S}_\pi \cap \mathfrak{S}_{\pi'}) = \mathfrak{S}_*.$$

Since $|\pi| \geqslant 2$, we have $\mathfrak{N}_p \mathfrak{N}_q \subseteq \mathfrak{S}_\pi$ for some different primes p, $q \in \pi$. But by [10, Theorem X.5.32], we have $\mathfrak{S}_\pi \nsubseteq \mathfrak{S}_*$, which contradicts $\mathfrak{S}_\pi \subseteq \mathfrak{S}_*$.

This contradiction shows that the Fitting class $\mathfrak{F}$ is not local.

We now prove that $\mathfrak{F}$ is not a formation. Assume that $\mathfrak{F}$ is a formation. Then $\mathfrak{F}$ is a Lockett class by [10, Theorem X.1.25]. As above, we may obtain that $\mathfrak{S}_\pi \nsubseteq \mathfrak{S}_*$ and a contradiction.

Let π be a set of primes such that $|\pi'| \geqslant 2$. Let $\mathfrak{F}$ and $\mathfrak{H}$ be the Fitting classes such that

$$\mathfrak{F} = (\mathfrak{S}_\pi)_* \mathfrak{S}_{\pi'}, \quad \mathfrak{H} = (\mathfrak{S}_{\pi'})_* \mathfrak{S}_\pi.$$

Then by Lemma 4.1 (3), we see that $\mathfrak{H}$ is a semilocal Fitting class. We have just proved above that each of the classes $\mathfrak{F}$ and $\mathfrak{H}$ is neither local nor a formation. Moreover, by [10, Theorem X.1.15],

$$(\mathfrak{S}_{\pi'})_* \subseteq \mathfrak{S}_{\pi'}.$$

Let $\mathfrak{X} = \mathfrak{F}\mathfrak{H}$. Then, by the multiplicative associative law of Fitting classes, we have

$$\mathfrak{X} = (\mathfrak{S}_\pi)_* \mathfrak{S}_{\pi'} \mathfrak{S}_\pi.$$

It is easy to see that $\mathfrak{X} = LR(\varphi)$, where φ is the H-function such that

$$\varphi(p) = \begin{cases} \mathfrak{X}, & p \in \pi', \\ (\mathfrak{S}_\pi)_* \mathfrak{S}_{\pi'}, & p \in \pi. \end{cases}$$

Thus, $\mathfrak{X}$ is local. By the choice of π, the set of such Fitting classes is infinite. This completes the proof. □

Now, using our main theorem and its proof, we give a concrete example (which is different from the example in [27]) which gives a positive answer to Problem 1.1.

Example 5.2 Let

$$\mathfrak{L} = \{G \mid (G/G_{\mathfrak{S}_\sigma})^{\mathfrak{S}_{2'}} \in S_n D_0(X)\} \cap \mathfrak{S}_7\mathfrak{S}_3\mathfrak{S}_2,$$

where

$$X = YP, \quad P = Z(Q_8), \quad Y = W \rtimes R,$$

W is a faithful irreducible $[RQ_8]$-module over the field $GF(7)$ of dimension 3, R is a extraspecial group (see [10, p.77]) of exponent 3 and order 27, and $\sigma = \{2, 3\}$. By [3, Theorems 4.5 and 4.7]), we know that $\mathfrak{L}$ is a Lockett class and $X \in (\mathfrak{L} \cap \mathfrak{S}_*)\backslash\mathfrak{L}_*$. Let

$$\pi = \pi(X), \quad \mathfrak{F} = \mathfrak{L}_\pi(\mathfrak{L}_*),$$

that is, $\mathfrak{F}$ is the class of all groups every Hall π-subgroup of every group G in which are contained in some $\mathfrak{L}_*$-injector of G. By Lemma 4.1 and Example 4.2, we see that $\mathfrak{F}$ is a semilocal Fitting class. Obviously,

$$X \in \mathfrak{L}_\pi(\mathfrak{L})\backslash\mathfrak{F}.$$

If $\mathfrak{F} = \mathfrak{L}_\pi(\mathfrak{L}_*)$ is a Lockett class, then, by [25, Lemma 5] and [10, Theorems X.1.37 and X.1.15], we have

$$\mathfrak{L}_\pi(\mathfrak{L}_*) = (\mathfrak{L}_\pi(\mathfrak{L}_*))^* = \mathfrak{L}_\pi((\mathfrak{L}_*)^*) = \mathfrak{L}_\pi(\mathfrak{L}),$$

which contradicts the fact that

$$\mathfrak{L}_\pi(\mathfrak{L}) \neq \mathfrak{L}_\pi(\mathfrak{L}_*).$$

Hence, $\mathfrak{L}_\pi(\mathfrak{L}_*)$ is not a Lockett class. It follows from [25, Lemma 5] and [10, Theorem X.1.25] that $\mathfrak{L}_\pi(\mathfrak{L}_*)$ is neither local nor a formation.

Now, let

$$\mathfrak{H} = (\mathfrak{S}_{\pi'})_*\mathfrak{S}_\pi.$$

In this case, $|\pi'| \geqslant 2$. Hence, by the proof of Theorem 5.1, we see that $\mathfrak{H}$ is neither local nor a formation. Put

$$\mathfrak{F}\mathfrak{H} = (\mathfrak{F}\mathfrak{S}_{\pi'})((\mathfrak{S}_{\pi'})_*\mathfrak{S}_\pi).$$

Then by the multiplicative associative law of Fitting classes, we have

$$\mathfrak{F}\mathfrak{H} = \mathfrak{F}\mathfrak{S}_{\pi'}\mathfrak{S}_\pi.$$

As in the proof of Theorem 5.1, we can see that $\mathfrak{F}\mathfrak{H}$ is a local Fitting class.

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