*Siberian Mathematical Journal, Vol.* 52, *No.* 6, *pp.* 995–1002, 2011


# FITTING FUNCTORS AND RADICALS OF FINITE GROUPS

© **E. A. Vit′ko and N. T. Vorob′ev** UDC 512.542

**Abstract:** We develop methods for recognizing the Fitting classes and radicals of finite groups by Fitting functors and the prescribed properties of Hall $\pi$-subgroups.

**Keywords:** Fitting class, radical, Fitting $\mathfrak{X}$-functor, $\pi$-normally embedded Fitting $\mathfrak{X}$-functor

## Introduction

A set of groups $\mathfrak{X}$ is a *class of groups* if, whenever $G \in \mathfrak{X}$, all groups isomorphic to $G$ also belong to $\mathfrak{X}$. A map $f$ that assigns to each group $G$ of a class $\mathfrak{X}$ a nonempty system of its subgroups $f(G)$ is said to be a *subgroup $\mathfrak{X}$-functor* [1] if

$$\alpha(f(G)) = f(\alpha(G))$$

for every isomorphism $\alpha$ of $G$. The study of subgroup functors of special kinds can be traced back to the 1960s when it was related to investigations into the subgroup structure of finite groups. The first results in this direction are due to Sudbrock [2] and Barnes and Kegel [3] who derived generalizations of Sylow's and Hall's theorems in terms of Sylow and Gaschütz functors. Later, the algebra of subgroup functors started to form, and the functors themselves became independent objects that were effectively used in the theory of classes to describe their structure and properties of canonical subgroups. In particular, A. N. Skiba in [4] developed a functor method and its applications for describing saturated formations closed under systems of subgroups, and the application of functor methods in the theory of Schunck classes [1] helped reveal a number of new properties of maximal subgroups and their intersections.

At the same time the subgroup functors and their role in class generation have been poorly investigated in the theory of Fitting classes for a long time. The first examples of the new injective properties of groups discovered by means of Fitting functors of special kinds were obtained by Anderson [5] and Schnackenberg [6]. Recall that, for a nonempty class $\mathfrak{X}$, a subgroup $\mathfrak{X}$-functor $f$ is called *Fitting* or *radical* [1] if

$$f(X) = \{X \cap H : H \in f(G)\}$$

for every $\mathfrak{X}$-group $G$ and its every normal $\mathfrak{X}$-subgroup $X$. The systematic study of the algebra of Fitting $\mathfrak{X}$-functors in the theory of soluble Fitting classes originated with a number of considerable papers by Beidleman, Brewster, and Hauck [7, 8] and Beidleman and Hauck [9].

In the above papers the scope of functor methods was however confined to the case where $\mathfrak{X}$ is the class $\mathfrak{S}$ of all finite soluble groups.

In the present paper we broaden the concept of a Fitting $\mathfrak{S}$-functor by defining a Fitting $\mathfrak{X}$-functor in the general case for any nonempty Fitting class $\mathfrak{X}$. In particular, we study Fitting $\mathfrak{X}$-functors for $\mathfrak{X} \in \{\mathfrak{E}, \mathfrak{S}^\pi\}$, where $\mathfrak{E}$ and $\mathfrak{S}^\pi$ are respectively the classes of all finite groups and all finite $\pi$-soluble groups. The first goal of our investigation is to discover some general laws for generating Fitting classes by means of Fitting $\mathfrak{X}$-functors. We prove (Theorem 3.2) that if $\mathfrak{X}$ is a nonempty Fitting class of finite groups and $f$ is a Fitting $\mathfrak{X}$-functor, then for every set of primes $\pi$ the class $L_\pi(f)$ of all groups $G \in \mathfrak{X}$ such that all subgroups lying in $f(G)$ have $\pi'$-index is a Fitting class. Note that in the case where $\mathfrak{X} = \mathfrak{S}$

The authors were supported by a joint grant of the Belarussian Republic Foundation of Fundamental Researches and the Russian Foundation for Basic Research (Grant F10R–231).

Vitebsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 52, No. 6, pp. 1253–1263, November–December, 2011. Original article submitted December 16, 2010.

0037-4466/11/5206–0995 

we can specify the $\mathfrak{S}$-functor so that the class $L_\pi(f)$ will be one of the known Fitting classes that were defined by imposing conditions on Hall subgroups (for instance, see [10, Chapters IX and X]). The study of the structure of these classes and how they can be applied to description of injectors and radicals of groups encompasses a series of papers by Lockett [11], Brison [12], Cusack [13], V. V. Shpakov and N. T. Vorob′ev [14], and others.

If $f$ and $g$ are hereditary Fitting $\mathfrak{X}$-functors, then their *product* [1] is the map $f \circ g$ that assigns to each group $G \in \mathfrak{X}$ a nonempty set of subgroups $\{X : X \in f(Y)$ for some subgroup $Y \in g(G)\}$. We write $\mathrm{Hall}_\pi$ to denote the Fitting $\mathfrak{S}^\pi$-functor defined by $\mathrm{Hall}_\pi(G) = \{G_\pi : G_\pi$ is a Hall $\pi$-subgroup of $G\}$.

The main result of the present paper is some description of the radicals of $\pi$-soluble groups (in particular, some description of radicals of a $\pi$-subgroup) in terms of Fitting $\mathfrak{X}$-functors. We prove (Theorem 4.2) that if $\mathfrak{X} = \mathfrak{S}^\pi$, $f$ is a Fitting $\mathfrak{X}$-functor, $G$ is a group, $X \in f(G)$, and $X_\pi$ and $G_\pi$ are Hall $\pi$-subgroups of $X$ and $G$ such that $X_\pi \le G_\pi$ and $|A_\pi| = |B_\pi|$ for $A_\pi, B_\pi \in (\mathrm{Hall}_\pi \circ f)(G)$, then the intersection of the Hall $\pi$-subgroup $G_\pi$ of $G$ and the radical $G_{L_\pi(f)}$ is equal to the core of the Hall $\pi$-subgroup $X_\pi$ in $G_\pi$. Theorem 4.2 provides some description of radicals of groups in terms of known Fitting classes. In particular, we establish that if $\mathfrak{K} = K_\pi(\mathfrak{F})$ [15] is the class of all $\pi$-soluble groups whose Hall $\pi$-subgroup is an $\mathfrak{F}$-group, then the $\mathfrak{F}$-radical of a Hall $\pi$-subgroup $H$ of $G$ is defined by the equality $H_\mathfrak{F} = H \cap G_\mathfrak{K}$.

In the final section, Section 5, we apply the description of group radicals to study the properties of the products of $\pi$-normally embedded conjugate Fitting $\mathfrak{S}^\pi$-functors. We prove (Theorem 5.4) that if $f$ and $g$ are $\pi$-normally embedded conjugate Fitting $\mathfrak{S}^\pi$-functors, then their product $f \circ g$ is a $\pi$-normally embedded conjugate Fitting $\mathfrak{S}^\pi$-functor.

We follow the definitions and notation of [10, 16], and we assume throughout the paper that all groups are finite.

## 1. Preliminaries

Recall that a *Fitting class* is a class of groups $\mathfrak{F}$ closed under normal subgroups and products of normal $\mathfrak{F}$-subgroups. If $\mathfrak{F}$ is a nonempty Fitting class, then a subgroup $G_\mathfrak{F}$ of $G$ is the $\mathfrak{F}$*-radical* of $G$ if it is the largest normal $\mathfrak{F}$-subgroup of $G$. A subgroup $H$ of $G$ is an $\mathfrak{F}$*-injector of* $G$ if the intersection $H \cap K$ is an $\mathfrak{F}$-maximal subgroup of $K$ for every subnormal subgroup $K$ of $G$.

The *product of Fitting classes* $\mathfrak{X}$ and $\mathfrak{Y}$ is the class of groups $\mathfrak{X}\mathfrak{Y} = (G : G/G_\mathfrak{X} \in \mathfrak{Y})$. It is known [10, Theorem IX.1.12] that if $\mathfrak{X}$ and $\mathfrak{Y}$ are Fitting classes, then their product $\mathfrak{X}\mathfrak{Y}$ is a Fitting class and that multiplication of Fitting classes is associative. Furthermore, it is obvious that $\mathfrak{X} \subseteq \mathfrak{X}\mathfrak{Y}$ for every nonempty Fitting class $\mathfrak{Y}$.

Let $\pi$ be a set of primes. Then a *Hall $\pi$-subgroup* of $G$ is a subgroup $H$ of $G$ such that $|H|$ is a $\pi$-number and its index $|G : H|$ is a $\pi'$-number.

Recall that a group $G$ is *$p$-nilpotent* if $G$ has normal Hall $p'$-subgroup. A group $G$ is *$\pi$-nilpotent* if $G$ is $p$-nilpotent for all $p$ of $\pi$. The class of all $\pi$-nilpotent groups is denoted by $\mathfrak{N}^\pi$. Notice that $\mathfrak{N}^\pi = \mathfrak{E}_{\pi'}\mathfrak{N}_\pi$.

We write $F_\pi(G)$ to denote the $\pi$-nilpotent radical of $G$. It is known [16, Corollary 4.1.2] that if $G$ is a $\pi$-soluble group, then $C_G(F_\pi(G)) \le F_\pi(G)$.

Let $f$ be a map assigning to $G$ some nonempty system of its subgroups $f(G)$. If $\alpha : G \to \alpha(G)$ is an isomorphism, then $\alpha(f(G))$ denotes the set of all images of subgroups of $f(G)$ in $\alpha(G)$: $\alpha(f(G)) = \{\alpha(X) : X \in f(G)\}$. If $N$ is a subgroup of $G$, then $f(G) \cap N$ is an abbreviation for $\{X \cap N : X \in f(G)\}$.

## 2. $\mathfrak{X}$-Functors and Their Classification

Following [1, 7], we introduce

Definition 2.1. Let $\mathfrak{X}$ be a nonempty Fitting class. A map $f$ that assigns to each group $G \in \mathfrak{X}$ a nonempty set of its subgroups $f(G)$ is a *Fitting $\mathfrak{X}$-functor* provided that the following hold:

(i) if $\alpha : G \to \alpha(G)$ is an isomorphism, then $\alpha(f(G)) = f(\alpha(G))$;

(ii) if $N \trianglelefteq G$, then $f(G) \cap N = f(N)$.

We classify Fitting $\mathfrak{X}$-functors according to the properties of $\mathfrak{X}$.

Definition 2.2. Let $\mathfrak{X}$ be a nonempty Fitting class. A Fitting $\mathfrak{X}$-functor is said to be

(1) a *π-functor* if $\mathfrak{X} = \mathfrak{E}_\pi$, and in particular *p-functor* if $\mathfrak{X} = \mathfrak{E}_p$;

(2) *soluble* if $\mathfrak{X} = \mathfrak{S}$;

(3) *π-soluble* if $\mathfrak{X} = \mathfrak{S}^\pi$;

(4) *conjugate* if for every group $G \in \mathfrak{X}$ the set $f(G)$ is a class of conjugate subgroups of $G$;

(5) *hereditary* if $\mathfrak{X}$ is hereditary.

For simplicity, a Fitting $\mathfrak{X}$-functor is said to be a *Fitting functor* if $\mathfrak{X} = \mathfrak{E}$.

Examples 2.3. (a) Let $\mathfrak{X} = \mathfrak{E}$, let $\mathfrak{F}$ be a Fitting class, and $\mathrm{Rad}_\mathfrak{F}(G) = \{G_\mathfrak{F}\}$. Then $\mathrm{Rad}_\mathfrak{F}$ is a conjugate Fitting functor. If $\mathfrak{F} = \mathfrak{E}_\pi$, then the corresponding Fitting functor $\mathrm{Rad}_\mathfrak{F}$ is denoted by $\mathrm{Rad}_\pi$.

(b) Let $f$ be a π-soluble functor that assigns to each group $G \in \mathfrak{S}^\pi$ the set of all its Hall π-subgroups. By Chunikhin's theorem [17], the functor $f$ is conjugate. This functor is referred to as the *Hall π-functor* and is denoted by $\mathrm{Hall}_\pi$.

(c) Let $\mathfrak{F}$ be a Fitting class, $\pi = \pi(\mathfrak{F})$, and $\mathfrak{X} = \mathfrak{F}\mathfrak{S}^\pi$. If $\mathrm{Inj}_\mathfrak{F}(G) = \{X : X \text{ is an } \mathfrak{F}\text{-injector of } G\}$, then by Theorem II.2.5.3 of [18] it follows that $\mathrm{Inj}_\mathfrak{F}$ is a conjugate Fitting $\mathfrak{X}$-functor.

**Theorem 2.4.** *Let $f$ and $g$ be hereditary Fitting $\mathfrak{X}$-functors. Then the following hold:*

*(1) $f \circ g$ is a Fitting $\mathfrak{X}$-functor;*

*(2) if $f$ and $g$ are conjugate Fitting $\mathfrak{X}$-functors, then $f \circ g$ is a conjugate Fitting $\mathfrak{X}$-functor.*

Proof. 1. We show that condition (i) of Definition 2.1 is satisfied.

Let $G \in \mathfrak{X}$ and let $\alpha : G \to \alpha(G)$ be an isomorphism of $G$. If $f$ and $g$ are Fitting $\mathfrak{X}$-functors and a subgroup $H$ lies in $\alpha((f \circ g)(G))$, then $H = \alpha(X)$ for some subgroup $X \in (f \circ g)(G)$. Then $X \in f(Y)$ for some subgroup $Y \in g(G)$. Hence $H \in \alpha(f(Y))$ for some $Y \in g(G)$. But since $f$ is a Fitting $\mathfrak{X}$-functor, $H \in f(\alpha(Y))$. Let $Y_1 = \alpha(Y)$. Since $Y \in g(G)$, it follows that $Y_1 \in \alpha(g(G))$. Therefore $Y_1 \in g(\alpha(G))$. Thus $H \in f(Y_1)$ for some subgroup $Y_1 \in g(\alpha(G))$. Hence $H \in (f \circ g)(\alpha(G))$ and so

$$\alpha((f \circ g)(G)) \subseteq (f \circ g)(\alpha(G)).$$

To establish the reverse inclusion, let $R \in (f \circ g)(\alpha(G))$. Then $R \in f(X)$ for some subgroup $X \in g(\alpha(G))$. But $g$ is a Fitting $\mathfrak{X}$-functor, and so $X \in \alpha(g(G))$. Hence $X = \alpha(Y)$ for some subgroup $Y \in g(G)$. Therefore $R \in f(\alpha(Y)) = \alpha(f(Y))$ for some subgroup $Y \in g(G)$. This means that $R \in \alpha((f \circ g)(G))$, and thus condition (i) of the definition of a Fitting $\mathfrak{X}$-functor holds.

Now check that condition (ii) of Definition 2.1 is satisfied.

Let $G \in \mathfrak{X}$ and let $N$ be a normal subgroup of $G$. If $X \in (f \circ g)(G)$, then there is a subgroup $Y \in g(G)$ such that $X \in f(Y)$. Since $g$ is a Fitting $\mathfrak{X}$-functor, $Y \cap N \in g(N)$. Furthermore, $Y \cap N \trianglelefteq Y$ and so $X \cap N \in f(Y \cap N)$. Hence $X \cap N \in (f \circ g)(N)$ and

$$(f \circ g)(G) \cap N \subseteq (f \circ g)(N).$$

Suppose that $R \in (f \circ g)(N)$. Then $R \in f(S)$ for some subgroup $S \in g(N)$. As $g$ is a Fitting $\mathfrak{X}$-functor, there is a subgroup $X \in g(G)$ such that $S = X \cap N$. Since $S \trianglelefteq X$, there is a subgroup $Y \in f(X)$ such that

$$R = Y \cap S = Y \cap X \cap N = Y \cap N.$$

Therefore $R \in (f \circ g)(G) \cap N$, and so we prove that $(f \circ g)(G) \cap N = (f \circ g)(N)$.

To prove assertion 2, let $f$ and $g$ be conjugate Fitting $\mathfrak{X}$-functors and let $G_1$ and $G_2$ be subgroups lying in $(f \circ g)(G)$. There are $H_1, H_2 \in g(G)$ such that $G_1 \in f(H_1)$ and $G_2 \in f(H_2)$. Since the Fitting $\mathfrak{X}$-functor $g$ is conjugate, $H_1^g = H_2$ for some $g \in G$. By condition (i) of the definition of a Fitting $\mathfrak{X}$-functor, $G_1^g \in (f(H_1))^g = f(H_1^g) = f(H_2)$. Thus $G_1^g \in f(H_2)$ and $G_2 \in f(H_2)$. But $f$ is a conjugate Fitting $\mathfrak{X}$-functor, and so $(G_1^g)^h = G_2$ for some $h \in H_2$. This implies that the Fitting $\mathfrak{X}$-functor $f \circ g$ is conjugate, and the proof is complete.

## 3. The Class $L_\pi(f)$ and Its Properties

We now explore how the constructive potential of Fitting $\mathfrak{X}$-functors can be used for generating families of Fitting classes. The following definition of a class of groups by a Fitting $\mathfrak{X}$-functor was inspired by the construction known in the theory of classes (and first introduced by Lockett in [11]), of the soluble Fitting class $L_\pi(\mathfrak{F})$ of all groups $G$ whose $\mathfrak{F}$-injectors contain Hall $\pi$-subgroups of $G$.

Following [7], we introduce

DEFINITION 3.1. Let $\mathfrak{X}$ be a nonempty Fitting class, let $f$ be a Fitting $\mathfrak{X}$-functor and let $\pi$ be a set of primes. Define the class of groups $L_\pi(f)$ as follows: $G \in L_\pi(f)$ if and only if $G \in \mathfrak{X}$ and the index $|G : X|$ is a $\pi'$-number for all $X \in f(G)$.

If $\pi = \varnothing$, then put $L_\pi(f) = \mathfrak{X}$. In the case where $\pi = \{p\}$, denote $L_\pi(f)$ by $L_p(f)$. If $\pi = \mathbb{P}$, then $L_{\mathbb{P}}(f) = \bigcap_{p\in\mathbb{P}} L_p(f)$ is denoted by $L(f)$.

**Theorem 3.2.** *Let $\mathfrak{X}$ be a nonempty Fitting class and suppose that $f$ is a Fitting $\mathfrak{X}$-functor. Then for any set of primes $\pi$ the class $L_\pi(f)$ is a Fitting class.*

PROOF. Let $G \in L_\pi(f)$ and suppose that $N$ is a normal subgroup of $G$. If $Y \in f(N)$, then by condition (ii) of Definition 2.1 we have $Y = X \cap N$, where $X$ is a subgroup of $G$ lying in $f(G)$. Then

$$|N : Y| = \frac{|G : X|}{|G : XN|}.$$

Since $G \in L_\pi(f)$ and $X \in f(G)$, the index $|G : X|$ is a $\pi'$-number. Therefore $|N : Y|$ is also a $\pi'$-number, and so $N \in L_\pi(f)$.

Let $N_1$ and $N_2$ be normal subgroups of $G$ such that $G = N_1N_2$ and $N_1$ and $N_2$ belong to $L_\pi(f)$.

If $X \in f(G)$, then by the definition of a Fitting $\mathfrak{X}$-functor we have

$$X \cap N_1 = Y_1, \tag{3.2.1}$$

$$X \cap N_2 = Y_2 \tag{3.2.2}$$

for some $Y_1 \in f(N_1)$ and $Y_2 \in f(N_2)$.

Calculating the index of $X$ in $G$ yields

$$\rho = |G : X| = \frac{|N_1| \cdot |N_2|}{|X| \cdot |N_1 \cap N_2|} = |N_1 : Y_1| \cdot |N_2 : Y_2| \cdot \frac{|Y_1| \cdot |Y_2|}{|X| \cdot |N_1 \cap N_2|}. \tag{3.2.3}$$

Together with (3.2.1)–(3.2.3) this implies

$$\begin{aligned}\rho &= |N_1 : Y_1| \cdot |N_2 : Y_2| \cdot \frac{|Y_1Y_2| \cdot |Y_1 \cap Y_2|}{|X| \cdot |N_1 \cap N_2|}\\ &= |N_1 : Y_1| \cdot |N_2 : Y_2| \cdot \frac{|(X \cap N_1)(X \cap N_2)| \cdot |X \cap N_1 \cap N_2|}{|X| \cdot |N_1 \cap N_2|}\\ &= \frac{|N_1 : Y_1| \cdot |N_2 : Y_2|}{|X : ((X \cap N_1)(X \cap N_2))| \cdot |(N_1 \cap N_2) : (X \cap N_1 \cap N_2)|}.\end{aligned}$$

By hypothesis, the indices $|N_1 : Y_1|$ and $|N_2 : Y_2|$ are $\pi'$-numbers, and so $|G : X|$ is a $\pi'$-number as well. Thus $G \in L_\pi(f)$ and this completes the proof.

Specifying the $\mathfrak{X}$-functor $f$ and applying this theorem, we can distinguish many families of Fitting classes which were known only in the soluble case (for example, see [10, Chapters IX and X]).

EXAMPLES 3.3. Let $\mathfrak{X}$ and $\mathfrak{F}$ be nonempty Fitting classes, let $\pi$ be a set of primes, and let $f$ be a Fitting $\mathfrak{X}$-functor.

1. Suppose that $f$ is a $\pi$-soluble Fitting functor and $f = \mathrm{Rad}_{\mathfrak{F}} \circ \mathrm{Hall}_\pi$. Then $L_\pi(f) = K_\pi(\mathfrak{F})$ is the class of all groups $G$ whose Hall $\pi$-subgroup is an $\mathfrak{F}$-subgroup.

Indeed, if $G \in L_\pi(\mathrm{Rad}_{\mathfrak{F}} \circ \mathrm{Hall}_\pi)$ and $G_\pi$ is a Hall $\pi$-subgroup of $G$, then $(G_\pi)_{\mathfrak{F}} \in (\mathrm{Rad}_{\mathfrak{F}} \circ \mathrm{Hall}_\pi)(G)$. Hence $|G : (G_\pi)_{\mathfrak{F}}|$ is a $\pi'$-number and $(G_\pi)_{\mathfrak{F}}$ is a Hall $\pi$-subgroup of $G$. Thus $(G_\pi)_{\mathfrak{F}} = G_\pi$ and $G \in K_\pi(\mathfrak{F})$.

Let $G \in K_\pi(\mathfrak{F})$. If $X \in (\mathrm{Rad}_{\mathfrak{F}} \circ \mathrm{Hall}_\pi)(G)$, then $X$ is the $\mathfrak{F}$-radical of a Hall $\pi$-subgroup of $G$, i.e. $X = (G_\pi)_{\mathfrak{F}}$. But $G_\pi \in \mathfrak{F}$ and so $(G_\pi)_{\mathfrak{F}} = G_\pi$. Then the index $|G : X| = |G : G_\pi|$ is a $\pi'$-number for all $X \in f(G)$ and $G \in L_\pi(f)$.

Notice that we used the Fitting class $K_\pi(\mathfrak{F})$ in the universe $\mathfrak{S}^\pi$ to describe the radicals of Hall $\pi$-subgroups in [15].

2. Let $\mathfrak{X} = \mathfrak{S}$ and $f = \mathrm{Inj}_{\mathfrak{F}}$. Then $L_\pi(f) = L_\pi(\mathfrak{F})$ is the class of all groups $G$ whose $\mathfrak{F}$-injector has $\pi'$-index. This construction was first introduced by Lockett in [11] and found numerous applications in describing the structure of injectors and characterizing the soluble Fitting classes in [10, Chapter IX].

3. Let $\mathfrak{X} = \mathfrak{S}$ and $f = \mathrm{Rad}_\pi \circ \mathrm{Inj}_{\mathfrak{F}}$. Then $L_\pi(f) = L'_\pi(\mathfrak{F})$ is the class of all groups $G$ whose Hall $\pi$-subgroup is a normal subgroup in an $\mathfrak{F}$-injector of $G$.

Indeed, if $G \in L_\pi(\mathrm{Rad}_\pi \circ \mathrm{Inj}_{\mathfrak{F}})$ and $V$ is an $\mathfrak{F}$-injector of $G$, then $O_\pi(V) \in (\mathrm{Rad}_\pi \circ \mathrm{Inj}_{\mathfrak{F}})(G)$. Therefore $|G : O_\pi(V)|$ is a $\pi'$-number. But then $O_\pi(V)$ is a Hall $\pi$-subgroup of $G$ and so $G \in L'_\pi(\mathfrak{F})$.

Let $G \in L'_\pi(\mathfrak{F})$ and $X \in (\mathrm{Rad}_\pi \circ \mathrm{Inj}_{\mathfrak{F}})(G)$. Then $X$ is the $\mathfrak{E}_\pi$-radical of some $\mathfrak{F}$-injector $V$ of $G$. Hence $X = O_\pi(V)$. By the definition of the class $L'_\pi(\mathfrak{F})$, we have $G_\pi \trianglelefteq V$. This implies that $G_\pi \le O_\pi(V)$ and so $O_\pi(V) = G_\pi$. Thus $|G : X|$ is a $\pi'$-number for all $X \in (\mathrm{Rad}_\pi \circ \mathrm{Inj}_{\mathfrak{F}})(G)$ and $G \in L_\pi(\mathrm{Rad}_\pi \circ \mathrm{Inj}_{\mathfrak{F}})$.

4. Let $\mathfrak{X} = \mathfrak{S}$ and $f = \mathrm{Rad}_{\mathfrak{F}}$. Then $L_\pi(f) = R_\pi(\mathfrak{F})$ is the class of groups $G$ whose Hall $\pi$-subgroup is contained in the $\mathfrak{F}$-radical of $G$. We used this class to describe factorizations of local Fitting classes with nonlocal factors [14].

## 4. Radicals Defined by Fitting $\mathfrak{X}$-Functors

Let $\pi$ be a nonempty set of primes. A Fitting class $\mathfrak{F}$ is *$\pi$-saturated* if $\mathfrak{F}\mathfrak{E}_{\pi'} = \mathfrak{F}$, where $\mathfrak{E}_{\pi'}$ is the class of all $\pi'$-groups.

**Lemma 4.1.** *Let $\pi$ be a nonempty set of primes and let $f$ be a $\pi$-soluble Fitting functor. Then $L_\pi(f)$ is $\pi$-saturated.*

Proof. It is clear that $L_\pi(f) \subseteq L_\pi(f)\mathfrak{E}_{\pi'}$. Suppose that $G \in L_\pi(f)\mathfrak{E}_{\pi'}$. Then $|G : G_{L_\pi(f)}|$ is a $\pi'$-number. Furthermore, if $X \in f(G)$, then by condition (ii) of Definition 2.1 $G_{L_\pi(f)} \cap X \in f(G_{L_\pi(f)})$. But $G_{L_\pi(f)}$ is an $L_\pi(f)$-group; therefore, $|G_{L_\pi(f)} : (G_{L_\pi(f)} \cap X)|$ is a $\pi'$-number. Hence the index

$$|G : X| = \frac{|G : G_{L_\pi(f)}| \cdot |G_{L_\pi(f)} : (G_{L_\pi(f)} \cap X)|}{|X : (G_{L_\pi(f)} \cap X)|}$$

is a $\pi'$-number, and we deduce that $L_\pi(f) = L_\pi(f)\mathfrak{E}_{\pi'}$. The proof is complete.

The following theorem describes some general laws for constructing radicals of $\pi$-soluble groups in terms of $\pi$-soluble Fitting functors and classes $L_\pi(f)$, and this is the main result of the paper.

**Theorem 4.2.** *Let $\pi$ be a nonempty set of primes and suppose that $f$ is a $\pi$-soluble Fitting functor such that for every group $G$ the equality $|A_\pi| = |B_\pi|$ holds for all groups $A_\pi, B_\pi \in (\mathrm{Hall}_\pi \circ f)(G)$. Then for every $\pi$-soluble group $G$ and every subgroup $X$ lying in $f(G)$ such that $X_\pi \le G_\pi$ and $C = \mathrm{Core}_{G_\pi}(X_\pi)$ the following hold:*

*(1) $G_\pi \cap G_{L_\pi(f)} = C$;*

*(2) if $K/\langle C^G \rangle = O_{\pi'}(G/\langle C^G \rangle)$, then $K = G_{L_\pi(f)}$.*

Proof. To prove assertion 1, let $\mathfrak{L} = L_\pi(f)$. Since $X_\pi$ is a Hall $\pi$-subgroup of $X$, it follows that $X_\pi \cap G_{\mathfrak{L}}$ is a Hall $\pi$-subgroup of $X \cap G_{\mathfrak{L}}$. By the definition of a Fitting $\mathfrak{X}$-functor, we have $X \cap G_{\mathfrak{L}} \in f(G_{\mathfrak{L}})$. But $G_{\mathfrak{L}}$ is an $\mathfrak{L}$-group, therefore, the index $|G_{\mathfrak{L}} : (X \cap G_{\mathfrak{L}})|$ is a $\pi'$-number. This implies that the index

$$|G_{\mathfrak{L}} : (X_\pi \cap G_{\mathfrak{L}})| = |G_{\mathfrak{L}} : (X \cap G_{\mathfrak{L}})| \cdot |(X \cap G_{\mathfrak{L}}) : (X_\pi \cap G_{\mathfrak{L}})|$$

is a $\pi'$-number too. Thus $X_\pi \cap G_{\mathfrak{L}} \in \mathrm{Hall}_\pi(G_{\mathfrak{L}})$. But $G_\pi \cap G_{\mathfrak{L}}$ is a Hall $\pi$-subgroup of $G_{\mathfrak{L}}$, and so $X_\pi \cap G_{\mathfrak{L}} = G_\pi \cap G_{\mathfrak{L}}$. Hence $G_\pi \cap G_{\mathfrak{L}} \leq X_\pi$ and $G_\pi \cap G_{\mathfrak{L}} \trianglelefteq G_\pi$. Thus $G_\pi \cap G_{\mathfrak{L}} \leq \mathrm{Core}_{G_\pi}(X_\pi)$.

We now prove the reverse inclusion. Let $F/G_{\mathfrak{L}} = F_\pi(G/G_{\mathfrak{L}})$ be the $\pi$-nilpotent radical of $G/G_{\mathfrak{L}}$. Since $F_{\mathfrak{L}} = F \cap G_{\mathfrak{L}}$ and $G_{\mathfrak{L}} \leq F$, we have $G_{\mathfrak{L}} = F_{\mathfrak{L}}$. Then $F/F_{\mathfrak{L}} \in \mathfrak{E}_{\pi'}\mathfrak{N}_\pi$. Therefore, Lemma 4.1 implies that

$$F \in \mathfrak{L}(\mathfrak{E}_{\pi'}\mathfrak{N}_\pi) = (\mathfrak{L}\mathfrak{E}_{\pi'})\mathfrak{N}_\pi = \mathfrak{L}\mathfrak{N}_\pi.$$

Thus $F/G_{\mathfrak{L}} \in \mathfrak{N}_\pi$, and $F/G_{\mathfrak{L}}$ is a $\pi$-group. Hence $F \leq G_\pi G_{\mathfrak{L}}$.

Since $(XG_{\mathfrak{L}} \cap F)/G_{\mathfrak{L}} \trianglelefteq\trianglelefteq F/G_{\mathfrak{L}}$, it follows that $XG_{\mathfrak{L}} \cap F \trianglelefteq\trianglelefteq G$. Therefore, $X \cap XG_{\mathfrak{L}} \cap F = X \cap F \in f(XG_{\mathfrak{L}} \cap F)$.

On the other hand, the index $|(XG_{\mathfrak{L}} \cap F) : (X \cap F)| = |(X \cap F)G_{\mathfrak{L}} : (X \cap F)| = |G_{\mathfrak{L}} : (X \cap G_{\mathfrak{L}})|$ is a $\pi'$-number.

Let $Y$ be a subgroup lying in $f(XG_{\mathfrak{L}} \cap F)$. Rewrite its index as follows:

$$|(XG_{\mathfrak{L}} \cap F) : Y| = \frac{|XG_{\mathfrak{L}} \cap F|}{|Y_\pi|} \cdot \frac{|Y_\pi|}{|Y|},$$

where $Y_\pi \in (\mathrm{Hall}_\pi \circ f)(XG_{\mathfrak{L}} \cap F)$. Together with the hypothesis of the theorem, this implies that $(X \cap F)_\pi \in (\mathrm{Hall}_\pi \circ f)(XG_{\mathfrak{L}} \cap F)$ satisfies the equality $|Y_\pi| = |(X \cap F)_\pi|$. Then the index $|(XG_{\mathfrak{L}} \cap F) : Y|$ can be written as

$$\frac{|XG_{\mathfrak{L}} \cap F|}{|(X \cap F)_\pi|} \cdot \frac{|Y_\pi|}{|Y|} = \frac{|(XG_{\mathfrak{L}} \cap F) : (X \cap F)| \cdot |(X \cap F) : (X \cap F)_\pi|}{|Y : Y_\pi|}$$

yielding a $\pi'$-number for all $Y \in f(XG_{\mathfrak{L}} \cap F)$. Hence $XG_{\mathfrak{L}} \cap F \in \mathfrak{L}$ and $XG_{\mathfrak{L}} \cap F \leq G_{\mathfrak{L}}$. On the other hand, $F$ and $CG_{\mathfrak{L}}$ are normal subgroups of $G_\pi G_{\mathfrak{L}}$. Therefore $[CG_{\mathfrak{L}}, F] \leq CG_{\mathfrak{L}} \cap F \leq XG_{\mathfrak{L}} \cap F \leq G_{\mathfrak{L}}$. This yields $CG_{\mathfrak{L}} \leq C_G(F/G_{\mathfrak{L}})$. Since $C_G(F/G_{\mathfrak{L}}) \leq F$, it follows that $CG_{\mathfrak{L}} \leq F \cap CG_{\mathfrak{L}} \leq G_{\mathfrak{L}}$, and so $C \leq G_\pi \cap G_{\mathfrak{L}}$ and $C = G_\pi \cap G_{\mathfrak{L}}$.

To prove assertion 2, let $K/\langle C^G \rangle = O_{\pi'}(G/\langle C^G \rangle)$.

By assertion 1, the subgroup $C$ is a Hall $\pi$-subgroup of $G_{\mathfrak{L}}$. But since $C \leq \langle C^G \rangle \leq G_{\mathfrak{L}}$, the index

$$|\langle C^G \rangle : C| = \frac{|G_{\mathfrak{L}} : C|}{|G_{\mathfrak{L}} : \langle C^G \rangle|}$$

is a $\pi'$-number. Therefore the index $|K : C| = |K : \langle C^G \rangle| \cdot |\langle C^G \rangle : C|$ is also a $\pi'$-number and so $C \in \mathrm{Hall}_\pi(K)$. As $C \leq X \cap K \in f(K)$, we have $K \in \mathfrak{L}$ and $K \leq G_{\mathfrak{L}}$.

On the other hand, the index

$$|G_{\mathfrak{L}} : \langle C^G \rangle| = \frac{|G_{\mathfrak{L}} : C|}{|\langle C^G \rangle : C|}$$

is a $\pi'$-number. Hence $G_{\mathfrak{L}}/\langle C^G \rangle \leq O_{\pi'}(G/\langle C^G \rangle) = K/\langle C^G \rangle$ and so $G_{\mathfrak{L}} \leq K$. The proof is complete.

**Corollary 4.3.** *Suppose that a $\pi$-soluble Fitting functor $f$, a group $G$, and Hall $\pi$-subgroups $X_\pi$ and $G_\pi$ satisfy the hypothesis of Theorem 4.2. Then the following are equivalent:*

(1) *$X_\pi$ is a Hall $\pi$-subgroup of some normal subgroup of $G$;*

(2) *$X_\pi$ is a normal subgroup of $G_\pi$;*

(3) *$X_\pi \leq G_{L_\pi(f)}$;*

(4) *$X_\pi$ is a Hall $\pi$-subgroup of $G_{L_\pi(f)}$.*

Proof. 1 ⇒ 2: Suppose that $X_\pi \in \mathrm{Hall}_\pi(K)$ and $K \trianglelefteq G$. Then $X_\pi = G_\pi \cap K$ and $X_\pi \trianglelefteq G_\pi$.

2 ⇒ 3: If $X_\pi \trianglelefteq G_\pi$, then $\mathrm{Core}_{G_\pi}(X_\pi) = X_\pi$. Thus by Theorem 4.2 we have $G_\pi \cap G_{L_\pi(f)} = X_\pi$. Hence $X_\pi \leq G_{L_\pi(f)}$.

3 ⇒ 4: Let $X_\pi \leq G_{L_\pi(f)}$. By hypothesis, $X_\pi \leq G_\pi$ and so $X_\pi \leq G_\pi \cap G_{L_\pi(f)}$. Also, $\mathrm{Core}_{G_\pi}(X_\pi) \leq X_\pi$, and by Theorem 4.2 $G_\pi \cap G_{L_\pi(f)} \leq X_\pi$. Thus $G_\pi \cap G_{L_\pi(f)} = X_\pi$.

4 ⇒ 1: Let $X_\pi$ be a Hall $\pi$-subgroup of $G_{L_\pi(f)}$. Since $G_{L_\pi(f)} \trianglelefteq G$, assertion 1 follows.

Specifying the $\pi$-soluble Fitting functor, this theorem enables us to describe the radicals of Hall $\pi$-subgroups of $\pi$-soluble groups, which is demonstrated by

**Corollary 4.4.** *Suppose that $\mathfrak{F}$ is a Fitting class, $\pi$ is a nonempty set of primes, $H$ is a Hall $\pi$-subgroup of a $\pi$-soluble group $G$ and suppose further that $\mathfrak{K} = (G \in \mathfrak{S}^\pi : \mathrm{Hall}_\pi(G) \subseteq \mathfrak{F})$ is a Fitting class. Then $H \cap G_\mathfrak{K} = H_\mathfrak{F}$ and $G_\mathfrak{K}/\langle H_\mathfrak{F}^G\rangle = O_{\pi'}(G/\langle H_\mathfrak{F}^G\rangle)$.*

Proof. Let $f = \mathrm{Rad}_\mathfrak{F} \circ \mathrm{Hall}_\pi$ be a $\pi$-soluble Fitting functor, let $X \in f(G)$, and let $X_\pi$ be a Hall $\pi$-subgroup of $X$ such that $X_\pi \le H$. Since $L_\pi(f) = \mathfrak{K}$, Theorem 4.2 yields $H \cap G_\mathfrak{K} = H \cap G_{L_\pi(f)} = \mathrm{Core}_H(X_\pi)$. But $X$ is a subgroup lying in $f(G)$; therefore, $X = H_\mathfrak{F}$. Then, as $X$ is a $\pi$-group, we have $X_\pi = H_\mathfrak{F}$. Thus $\mathrm{Core}_H(X_\pi) = H_\mathfrak{F}$. Hence $H \cap G_\mathfrak{K} = H_\mathfrak{F}$.

Applying assertion 2 of Theorem 4.2, we now have

$$G_\mathfrak{K}/\langle H_\mathfrak{F}^G\rangle = G_{L_\pi(f)}/\langle H_\mathfrak{F}^G\rangle = O_{\pi'}(G/\langle H_\mathfrak{F}^G\rangle).$$

Note that the structure of a radical of a Hall $\pi$-subgroup in the soluble case was first studied by Brison in [12].

If $\pi$ is a set of primes, $\mathfrak{F}$ is a Fitting class, and $\sigma = \pi(\mathfrak{F})$, then along the lines of Lockett [11] and Gallego [19], we can define the following classes of groups:

$$\mathfrak{L} = (G \in \mathfrak{F}\mathfrak{S}^\sigma \cap \mathfrak{S}^\pi : V \ge G_\pi), \quad \mathfrak{L}' = (G \in \mathfrak{F}\mathfrak{S}^\sigma \cap \mathfrak{S}^\pi : G_\pi \trianglelefteq V)$$

(here $V$ denotes an $\mathfrak{F}$-injector of $G$). Then with Chunikhin's theorem in [17] and Theorem II.2.5.3 in [18] it is easy to check that $\mathfrak{L}$ and $\mathfrak{L}'$ are Fitting classes.

**Corollary 4.5.** *Let $V$ be an $\mathfrak{F}$-injector of a group $G \in \mathfrak{F}\mathfrak{S}^\sigma \cap \mathfrak{S}^\pi$ and let $H$ and $R$ be Hall $\pi$-subgroups of $G$ and $V$, respectively. Then the following hold:*

*(1) if $C = \mathrm{Core}_H(R)$, then $C = H \cap G_\mathfrak{L}$ and $G_\mathfrak{L}/\langle C^G\rangle = O_{\pi'}(G/\langle C^G\rangle)$;*

*(2) if $C_1 = \mathrm{Core}_H(O_\pi(V))$, then $C_1 = H \cap G_{\mathfrak{L}'}$ and $G_{\mathfrak{L}'}/\langle C_1^G\rangle = O_{\pi'}(G/\langle C_1^G\rangle)$.*

Proof. Assertion 1 is an immediate consequence of Theorem 4.2 since $\mathfrak{L} = L_\pi(\mathrm{Inj}_\mathfrak{F})$.

We prove assertion 2. Let $\mathfrak{X} = \mathfrak{F}\mathfrak{S}^\sigma \cap \mathfrak{S}^\pi$. Suppose that a Fitting $\mathfrak{X}$-functor $f$ is equal to $\mathrm{Rad}_\pi \circ \mathrm{Inj}_\mathfrak{F}$, $X \in f(G)$, and $X_\pi$ is a Hall $\pi$-subgroup of $X$ such that $X_\pi \le H$. Then the equality $L'_\pi(\mathfrak{F}) = L_\pi(f)$ and Theorem 4.2 yield $H \cap G_{\mathfrak{L}'} = \mathrm{Core}_H(X_\pi)$. But $X$ is a subgroup lying in $f(G)$, and hence $X = O_\pi(V)$. Therefore $X_\pi = O_\pi(V)$ and $H \cap G_{\mathfrak{L}'} = \mathrm{Core}_H(O_\pi(V))$. Also, by assertion 2 of Theorem 4.2, $G_{\mathfrak{L}'}/\langle C_1^G\rangle = O_{\pi'}(G/\langle C_1^G\rangle)$.

## 5. $\pi$-Normally Embedded Fitting $\mathfrak{X}$-Functors

In this section, we apply the results of Section 4 to distinguish one more family of $\mathfrak{X}$-functors and study some properties of their products.

Definition 5.1. Let $\pi$ be a set of primes.

(a) A subgroup $X$ of $G$ is said to be *$\pi$-normally embedded* if a Hall $\pi$-subgroup of $X$ is a $\pi$-subgroup of some normal subgroup of $G$.

(b) A Fitting $\mathfrak{X}$-functor $f$ is said to be *$\pi$-normally embedded* if every subgroup $X \in f(G)$ is a $\pi$-normally embedded subgroup of $G$.

Note that for $\pi = \{p\}$ the functor $f$ is called *p-normally embedded*. In the case where $\mathfrak{X} = \mathfrak{S}$ these functors were studied in [7, 8].

Following [7], we introduce

Definition 5.2. We say that a Fitting $\mathfrak{X}$-functor $f$ *satisfies the Frattini argument* if for every group $G \in \mathfrak{X}$, every subgroup $N \trianglelefteq G$, and every $X \in f(G)$, we have $G = N \cdot N_G(X \cap N)$.

**Lemma 5.3.** *Let $f$ be a conjugate Fitting $\mathfrak{X}$-functor. Then $f$ satisfies the Frattini argument.*

Proof. Let $N \trianglelefteq G$ and $X \in f(G)$. Then by condition (ii) of Definition 2.1, we have $X \cap N \in f(N)$. Let $g \in G$. By the definition of an $\mathfrak{X}$-functor, $X^g \in f(G)$. Hence $X^g \cap N = (X \cap N)^g \in f(N)$. Since $f$ is a conjugate Fitting $\mathfrak{X}$-functor, there is an element $n \in N$ such that $(X \cap N)^{gn} = X \cap N$. Therefore $gn \in N_G(X \cap N)$ and $g \in N_G(X \cap N) \cdot N$. We establish that $G = N_G(X \cap N) \cdot N$, and this completes the proof.

**Theorem 5.4.** *Let $\pi$ be a set of primes and let $f$ and $g$ be conjugate $\pi$-soluble Fitting functors. If $Y \in g(G)$, $X \in f(Y)$, $X$ is a $\pi$-normally embedded subgroup of $Y$, and $Y$ is a $\pi$-normally embedded subgroup of $G$, then $X$ is a $\pi$-normally embedded subgroup of $G$. In particular, if the functors $f$ and $g$ are $\pi$-normally embedded, then their product $f \circ g$ is a $\pi$-normally embedded functor.*

PROOF. Let $\mathfrak{L} = L_\pi(g)$ and $R = G_{\mathfrak{L}}$. Then $Y \cap R \in g(R)$. Since $g$ is a conjugate $\pi$-soluble Fitting functor, the Frattini argument yields $G = RN_G(Y \cap R)$. Hence

$$\frac{|G|}{|N_G(Y \cap R)|} = \frac{|R : (Y \cap R)|}{|(R \cap N_G(Y \cap R)) : (Y \cap R)|}. \tag{5.4.1}$$

Since $R \in L_\pi(g)$, the index $|R : (Y \cap R)|$ is a $\pi'$-number. By (5.4.1), the index $|G : N_G(Y \cap R)|$ is also a $\pi'$-number. Then there is a Hall $\pi$-subgroup $H$ of $G$ such that $H \leq N_G(Y \cap R)$. Hence $H \cap Y \cap R \in \mathrm{Hall}_\pi(Y \cap R) \subseteq \mathrm{Hall}_\pi(R)$. But $H \cap R \in \mathrm{Hall}_\pi(R)$. Therefore $H \cap Y \cap R = H \cap R$, and so $H \cap R \leq Y$. Since $Y$ is a $\pi$-normally embedded subgroup of $G$, Corollary 4.3 implies that $\mathrm{Hall}_\pi(Y) \subseteq \mathrm{Hall}_\pi(R)$. Thus $H \cap R \in \mathrm{Hall}_\pi(Y)$.

Suppose that $X_\pi$ is a Hall $\pi$-subgroup of $X$. Then there is an element $y \in Y$ such that $X_\pi \leq (H \cap R)^y$. Since $X$ is a $\pi$-normally embedded subgroup of $Y$, by Corollary 4.3 we have

$$X_\pi = (H \cap R)^y \cap Y_{L_\pi(f)} = (H \cap (R \cap Y)_{L_\pi(f)})^y.$$

Since $(R \cap Y)_{L_\pi(f)}$ is a characteristic subgroup of $R \cap Y$ and $R \cap Y \trianglelefteq (R \cap Y)H$, it follows that $H$ normalizes $(R \cap Y)_{L_\pi(f)}$. Hence, $X_\pi \trianglelefteq H^y$. By assertion 2 of Theorem 2.4, $f \circ g$ is a conjugate Fitting $\mathfrak{X}$-functor, and so Corollary 4.3 yields $X_\pi \in \mathrm{Hall}_\pi(G_{L_\pi(f \circ g)})$. Thus $X$ is a $\pi$-normally embedded subgroup of $G$, and the proof is complete.

## References


1. *Kamornikov S. F. and Sel′kin V. M.*, Subgroup Functors and Classes of Finite Groups [in Russian], Belarusskaya Nauka, Minsk (2003).
2. *Sudbrock W.*, “Sylowfunktionen in endlichen Gruppen,” Rend. Sem. Math. Univ. Padova, **36**, No. 1, 158–184 (1966).
3. *Barnes D. W. and Kegel O. H.,* “Gaschütz functors on finite soluble groups,” Math. Z., Bd 94, Heft 2, 134–142 (1966).
4. *Skiba A. N.*, Algebra of Formations [in Russian], Belarusskaya Nauka, Minsk (1997).
5. *Anderson W.,* Fitting Sets in Finite Soluble Groups, Thes. Doct. Philosophy, Michigan State University (1973).
6. *Schnackenberg F. R.*, “Injectors of finite groups,” J. Algebra, **30**, 548–558 (1974).
7. *Beidleman J. C., Brewster R., and Hauck P.,* “Fittingfunktoren in endlichen auflösbaren Gruppen. I,” Math. Z., Bd 182, 359–384 (1983).
8. *Beidleman J. C., Brewster R., and Hauck P.*, “Fitting functors in finite solvable groups. II,” Math. Proc. Cambridge Philos. Soc., **101**, 37–55 (1987).
9. *Beidleman J. C. and Hauck P.*, “Closure properties for Fitting functors,” Mh. Math., **108**, 1–22 (1989).
10. *Doerk K. and Hawkes T.*, Finite Soluble Groups, Walter de Gruyter, Berlin and New York (1992).
11. *Lockett P.,* “On the theory of Fitting classes of finite soluble groups,” Math. Z., Bd 131, 103–115 (1973).
12. *Brison O. J.*, “Hall operators for Fitting classes,” Arch. Math. (Basel), **33**, 1–9 (1979).
13. *Cusack E.*, “Normal Fitting classes and Hall subgroups,” Bull. Austral. Math. Soc., **21**, 229–236 (1980).
14. *Shpakov V. V. and Vorob′ev N. T.*, “Local factorizations of nonlocal Fitting classes,” Discrete Math. Appl., **18**, No. 4, 439–446 (2008).
15. *Vorob′ev N. T., Vit′ko E. A., and Ivanova N. V.*, “On properties of radicals of Hall subgroups of $\pi$-soluble groups,” Vestnik Vitebsk Univ., **48**, No. 2, 125–129 (2008).
16. *Shemetkov L. A.*, Formations of Finite Groups [in Russian], Nauka, Moscow (1978).
17. *Chunikhin S. A.*, Subgroups of Finite Groups [in Russian], Nauka i Tekhnika, Minsk (1964).
18. *Guo Wenbin,* The Theory of Classes of Groups, Kluwer Acad. Publ., Beijing, New York, Dordrecht, Boston, and London (2000) (Math. Appl.; V. 505).
19. *Gallego M.*, “A note on Hall operators for Fitting classes,” Bull. London Math. Soc., **17**, 248–252 (1985).



E. A. Vit′ko; N. T. Vorob′ev
Masherov Vitebsk State University, Vitebsk, Belarus
*E-mail address*: alenkavit@tut.by; nicholas@vsu.by