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MAXIMAL SUBCLASSES OF LOCAL FITTING CLASSES

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Abstract: A Fitting class $\mathfrak{F}$ is said to be π-*maximal* if $\mathfrak{F}$ is an inclusion maximal subclass of the Fitting class $\mathfrak{S}_\pi$ of all finite soluble π-groups. We prove that $\mathfrak{F}$ is a π-maximal Fitting class exactly when there is a prime $p \in \pi$ such that the index of the $\mathfrak{F}$-radical $G_\mathfrak{F}$ in G is equal to 1 or p for every π-subgroup of G. Hence, there exist maximal subclasses in a local Fitting class. This gives a negative answer to Skiba's conjecture that there are no maximal Fitting subclasses in a local Fitting class (see [1, Question 13.50]).

Keywords: Fitting class, maximal Fitting subclass, local Fitting class, $\mathfrak{F}$-radical, Lockett class, Lausch group, Fitting pair

Introduction

One of the difficult problems in the theory of Fitting classes of finite soluble groups is that of finding a criterion for maximality of a Fitting subclass $\mathfrak{F}$ in a Fitting class $\mathfrak{X}$ which was posed in 1974 by Bryce and Cossey (see [2, p. 170] and also [3, X.4, p. 735]). Recall that a Fitting class $\mathfrak{F}$ is said to be an inclusion *maximal* Fitting subclass of a Fitting class $\mathfrak{X}$ (denoted by $\mathfrak{F} <\cdot\, \mathfrak{X}$) if $\mathfrak{F} \subset \mathfrak{X}$; and from $\mathfrak{F} \subseteq \mathfrak{M} \subseteq \mathfrak{X}$, where $\mathfrak{M}$ is a Fitting class, it follows that $\mathfrak{M} \in \{\mathfrak{F}, \mathfrak{X}\}$. We notice that the problem stated above was solved by Bryce and Cossey [1] only when $\mathfrak{X}$ coincides with the trivial normal Fitting class $\mathfrak{S}$ of all finite soluble groups (see [3, X.4, Theorem 4.26]).

The main result of this paper is a description of a necessary and sufficient condition for the maximality of a Fitting subclass $\mathfrak{F}$ in the Fitting class $\mathfrak{S}_\pi$ of all finite soluble π-groups which is not normal for any nonempty proper subset π of the set of all primes. If $\mathfrak{F} <\cdot\, \mathfrak{S}_\pi$ then we will refer to $\mathfrak{F}$ simply as a π-*maximal Fitting class.*

We establish that $\mathfrak{F}$ is a π-maximal Fitting class exactly when there is a prime $p \in \pi$ such that the index of the $\mathfrak{F}$-radical $G_\mathfrak{F}$ in G is equal to 1 or p for all groups $G \in \mathfrak{S}_\pi$. As the Fitting class $\mathfrak{S}_\pi$ is local, the theorem we prove implies in particular that there exist maximal Fitting subclasses in a nonnormal local Fitting class, corroborating a negative answer to Skiba's question in the Kourovka Notebook [1, Question 13.50]. Remark that the first examples of maximal normal Fitting classes in the normal local class $\mathfrak{S}$ were announced in [4]. We also confirm the existence of nontrivial π-maximal Fitting subclasses by constructing concrete examples.

All groups in this paper are finite and soluble unless specified otherwise. Our definitions and notation follow [3].

1. Preliminaries

A *Fitting class* is a normally hereditary class $\mathfrak{F}$ of groups closed under the products of normal $\mathfrak{F}$-subgroups.

If $\mathfrak{F}$ is a nonempty Fitting class then a subgroup $G_\mathfrak{F}$ of G is called the $\mathfrak{F}$-*radical* of G if $G_\mathfrak{F}$ is the largest among normal subgroups of G belonging to $\mathfrak{F}$. The *product* $\mathfrak{F}\mathfrak{H}$ of Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ is the class of all groups G such that $G/G_\mathfrak{F}$ belongs to $\mathfrak{H}$. It is well known that $\mathfrak{F}\mathfrak{H}$ is a Fitting class and the multiplication of Fitting classes is associative.

For proving the main result, we will use the operations "*" and "$_\star$" introduced by Lockett [5]. For a nonempty Fitting class $\mathfrak{F}$, let $\mathfrak{F}^\star$ denote the least Fitting class containing $\mathfrak{F}$ such that $(G \times H)_{\mathfrak{F}^\star} =$

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$G_{\mathfrak{F}^\star} \times H_{\mathfrak{F}^\star}$ for all G and H. The class $\mathfrak{F}_\star$ is defined as the intersection of all Fitting classes $\mathfrak{X}$ such that $\mathfrak{X}^\star = \mathfrak{F}^\star$. If $\mathfrak{F} = \mathfrak{F}^\star$ then $\mathfrak{F}$ is called a *Lockett class*.

The *Lockett section* of a nonempty Fitting class $\mathfrak{F}$ is the set of all Fitting classes $\mathfrak{X}$ such that $\mathfrak{X}^\star = \mathfrak{F}^\star$. The Lockett section of a Fitting class $\mathfrak{F}$ is denoted by $\mathrm{Locksec}(\mathfrak{F})$ [5]. The *Lockett subsection* of a Fitting class $\mathfrak{F}$ is the set $\{\mathfrak{H} : \mathfrak{H} \in \mathrm{Locksec}(\mathfrak{F}), \mathfrak{H} \subseteq \mathfrak{F}\}$ denoted by $\mathrm{Locksub}(\mathfrak{F})$.

Recall that a class of groups $\mathfrak{F}$ is called a *homomorph* if every factor group of each group in $\mathfrak{F}$ belongs to $\mathfrak{F}$. A homomorph $\mathfrak{H}$ is said to be *saturated* if $G/\Phi(G) \in \mathfrak{H}$ implies $G \in \mathfrak{H}$.

We will use the following properties of Lockett's operations, which are gathered in the following lemma.

Lemma 1.1 [3, X.1.13, X.1.8(b), X.1.15]. *If $\mathfrak{F}$ and $\mathfrak{H}$ are Fitting classes, * and $_\star$ are Lockett's operations, then the following hold:*

(1) *if $\{\mathfrak{F}_i \mid i \in I\}$ is a set of nonempty Fitting classes then $(\bigcap_{i\in I} \mathfrak{F}_i)^\star = \bigcap_{i\in I} \mathfrak{F}_i^\star$;*

(2) [6, Lemma 3] *if $\mathfrak{H}$ is a saturated homomorph then $(\mathfrak{F}\mathfrak{H})^\star = \mathfrak{F}^\star\mathfrak{H}$;*

(3) *if $\mathfrak{F} \subseteq \mathfrak{H}$ then $\mathfrak{F}^\star \subseteq \mathfrak{H}^\star$;*

(4) $(\mathfrak{F}_\star)_\star = (\mathfrak{F}^\star)_\star = \mathfrak{F}_\star \subseteq \mathfrak{F} \subseteq \mathfrak{F}^\star = (\mathfrak{F}_\star)^\star$.

A nonidentity Fitting class $\mathfrak{F}$ is said to be *normal in a Fitting class* $\mathfrak{X}$ (denoted by $\mathfrak{F} \trianglelefteq \mathfrak{X}$) or $\mathfrak{X}$-*normal*, if $\mathfrak{F} \subseteq \mathfrak{X}$ and the $\mathfrak{F}$-radical $G_\mathfrak{F}$ is an $\mathfrak{F}$-maximal subgroup for all $G \in \mathfrak{X}$. Denote by $\mathbb{P}$ the set of all primes. The following lemma presents a well-known characterization of $\mathfrak{X}$-normal Fitting classes.

Lemma 1.2 [4, X.3.3, X.3.7]. *The following hold:*

(1) *if $\mathfrak{F} \neq (1)$ is a Fitting class and $\mathfrak{X}$ is a Lockett class then $\mathfrak{F} \lhd \mathfrak{X}$ if and only if $\mathfrak{F}^\star \lhd \mathfrak{X}$;*

(2) *if $\varnothing \neq \pi \subseteq \mathbb{P}$ then $(1) \neq \mathfrak{F} \lhd \mathfrak{S}_\pi$ only if $\mathfrak{F}^\star = \mathfrak{S}_\pi$; in particular, $(1) \neq \mathfrak{F} \lhd \mathfrak{S}$ if and only if $\mathfrak{F}^\star = \mathfrak{S}$.*

A map $h : \mathbb{P} \to \{\text{Fitting classes}\}$ is called a *Hartley function* or *H-function* [7]. A Fitting class $\mathfrak{F}$ is called *local* if there exists a Hartley function f such that $\mathfrak{F} = \mathfrak{S}_\pi \cap \left(\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'}\right)$, where $\pi = \mathrm{Supp}(f) = \{p \in \mathbb{P} \mid f(p) \neq \varnothing\}$. The following lemma is established in [8].

Lemma 1.3. *Each local Fitting class is a Lockett class.*

As known, if $\mathfrak{F}$ is a nonempty Fitting class and N is a normal subgroup of G then $N_\mathfrak{F} = N \cap G_\mathfrak{F}$.

It is straightforward to verify the following

Lemma 1.4. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be Fitting classes. If $\mathfrak{X}$ is a radical homomorph and $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ then $\mathfrak{F}_1\mathfrak{X} \subseteq \mathfrak{F}_2\mathfrak{X}$.*

2. π-Normal and π-Maximal Classes

Recall that for $\varnothing \neq \pi \subseteq \mathbb{P}$ a Fitting class $\mathfrak{F} \neq (1)$ is said to be *normal in the class* $\mathfrak{S}_\pi$ or briefly π-*normal* (denoted by $\mathfrak{F} \lhd \mathfrak{S}_\pi$), if $\mathfrak{F} \subseteq \mathfrak{S}_\pi$ and for every π-group G its $\mathfrak{F}$-radical is an $\mathfrak{F}$-maximal subgroup of G.

For a Fitting class $\mathfrak{F}$, a subgroup V of G is called an $\mathfrak{F}$-*injector* of G if $V \cap N$ is a normal $\mathfrak{F}$-maximal subgroup of N for every subnormal subgroup N of G. The following lemma, interesting in its own right, reveals a relation between π-maximal and π-normal Fitting classes.

Lemma 2.1. *Every π-maximal Fitting class is π-normal.*

Proof. Let $\mathfrak{F} \lessdot \mathfrak{S}_\pi$ and let $\mathfrak{N}_\pi$ be the class of all nilpotent π-groups. Since $\mathfrak{F} \subseteq \mathfrak{F}\mathfrak{N}_\pi$, it follows that $\mathfrak{F} \subseteq \mathfrak{S}_\pi \cap \mathfrak{F}\mathfrak{N}_\pi \subseteq \mathfrak{S}_\pi$. However, $\mathfrak{F}$ is maximal in $\mathfrak{S}_\pi$. Thus, we have the two possibilities: either $\mathfrak{F} = \mathfrak{S}_\pi \cap \mathfrak{F}\mathfrak{N}_\pi$ or $\mathfrak{S}_\pi \cap \mathfrak{F}\mathfrak{N}_\pi = \mathfrak{S}_\pi$. Consider each of them.

Case 1. $\mathfrak{F} = \mathfrak{S}_\pi \cap \mathfrak{F}\mathfrak{N}_\pi$. In this case (1) of Lemma 1.1 implies that $\mathfrak{F}^\star = (\mathfrak{S}_\pi \cap \mathfrak{F}\mathfrak{N}_\pi)^\star = \mathfrak{S}_\pi^\star \cap (\mathfrak{F}\mathfrak{N}_\pi)^\star$. However, $\mathfrak{S}_\pi$ is a local Fitting class, since it is defined by the H-function f such that

$$f(p) = \begin{cases} \mathfrak{S}_\pi, & \text{if } p \in \pi, \\ \varnothing, & \text{if } p \in \pi'. \end{cases}$$

Therefore, by Lemma 1.3, $\mathfrak{S}_\pi$ is a Lockett class. Moreover, it is easily seen that the Fitting class $\mathfrak{N}_\pi$ is a saturated homomorph. So, (2) of Lemma 1.1 implies that $(\mathfrak{F}\mathfrak{N}_\pi)^\star = \mathfrak{F}^\star\mathfrak{N}_\pi$. Thus, $\mathfrak{F}^\star = \mathfrak{S}_\pi \cap \mathfrak{F}^\star\mathfrak{N}_\pi$. Since $\mathfrak{F} \subseteq \mathfrak{S}_\pi$, (3) of Lemma 1.1 yields $\mathfrak{F}^\star \subseteq \mathfrak{S}_\pi^\star = \mathfrak{S}_\pi$. Observe also that $\mathfrak{F}^\star\mathfrak{N}_\pi \subseteq \mathfrak{S}_\pi\mathfrak{N}_\pi \subseteq \mathfrak{S}_\pi$. Hence, $\mathfrak{F}^\star = \mathfrak{S}_\pi \cap \mathfrak{F}^\star\mathfrak{N}_\pi = \mathfrak{F}^\star\mathfrak{N}_\pi$. As the multiplication of Fitting classes is associative, from Lemma 1.1 it follows that

$$\mathfrak{F}^\star = \mathfrak{F}^\star\mathfrak{N}_\pi = (\mathfrak{F}^\star\mathfrak{N}_\pi)\mathfrak{N}_\pi = \mathfrak{F}^\star\mathfrak{N}_\pi^2 = \big(\mathfrak{F}^\star\mathfrak{N}_\pi^2\big)\mathfrak{N}_\pi = \mathfrak{F}^\star\mathfrak{N}_\pi^3 = \dots = \mathfrak{F}^\star\mathfrak{N}_\pi^n.$$

Thus, $\mathfrak{F}^\star = \mathfrak{F}^\star\mathfrak{N}_\pi^n$ for every natural n.

In view of the fact that $\bigcup_{n=1}^\infty \mathfrak{N}_\pi^n = \mathfrak{S}_\pi$ and $\mathfrak{N}_\pi^n$ is a saturated homomorph, by (3) of Lemma 1.1 we derive the inclusions

$$\mathfrak{S}_\pi \subseteq \bigcup_{n=1}^\infty \mathfrak{F}^\star\mathfrak{N}_\pi^n = \bigcup_{n=1}^\infty \mathfrak{F}^\star = \mathfrak{F}^\star \subseteq \mathfrak{S}_\pi.$$

So $\mathfrak{F}^\star = \mathfrak{S}_\pi$, and (2) of Lemma 1.2 implies that $\mathfrak{F} \lhd \mathfrak{S}_\pi$. Thus the theorem is true in the first case.

Case 2. $\mathfrak{S}_\pi \cap \mathfrak{F}\mathfrak{N}_\pi = \mathfrak{S}_\pi$. In this case (1) of Lemma 1.1 implies that $\mathfrak{S}_\pi^\star \cap (\mathfrak{F}\mathfrak{N}_\pi)^\star = \mathfrak{S}_\pi^\star$. Since $\mathfrak{S}_\pi$ is a Lockett class, it follows that $\mathfrak{S}_\pi^\star = \mathfrak{S}_\pi$. Moreover, by (2) of Lemma 1.1 we infer $(\mathfrak{F}\mathfrak{N}_\pi)^\star = \mathfrak{F}^\star\mathfrak{N}_\pi$. Hence, we derive the equality $\mathfrak{S}_\pi \cap \mathfrak{F}^\star\mathfrak{N}_\pi = \mathfrak{S}_\pi$. This yields $\mathfrak{S}_\pi \subseteq \mathfrak{F}^\star\mathfrak{N}_\pi$. Since $\mathfrak{F} \subseteq \mathfrak{S}_\pi$, from (3) of Lemma 1.1 it follows that $\mathfrak{F}^\star \subseteq \mathfrak{S}_\pi^\star = \mathfrak{S}_\pi$. As $\mathfrak{N}_\pi$ is a radical homomorph, Lemma 1.4 implies that $\mathfrak{F}^\star\mathfrak{N}_\pi \subseteq \mathfrak{S}_\pi\mathfrak{N}_\pi$. Obviously, $\mathfrak{N}_\pi \subseteq \mathfrak{S}_\pi$, and so $\mathfrak{S}_\pi\mathfrak{N}_\pi \subseteq \mathfrak{S}_\pi$. Hence, $\mathfrak{F}^\star\mathfrak{N}_\pi \subseteq \mathfrak{S}_\pi$. Thus, $\mathfrak{F}^\star\mathfrak{N}_\pi = \mathfrak{S}_\pi$. The last equality is equivalent to the assertion that $G/G_{\mathfrak{F}^\star} \in \mathfrak{N}_\pi$ for all $G \in \mathfrak{S}_\pi$.

Let V be an $\mathfrak{F}^\star$-injector of G. Since every subgroup of a nilpotent group is subnormal, $V/G_{\mathfrak{F}^\star}$ is a subnormal subgroup in $G/G_{\mathfrak{F}^\star}$. Hence V is subnormal in G, which implies that $V = G_{\mathfrak{F}^\star}$. Thus $G_{\mathfrak{F}^\star}$ is $\mathfrak{F}^\star$-maximal in G for every π-group G. Therefore $\mathfrak{F}^\star$ is normal in $\mathfrak{S}_\pi$. So by (1) of Lemma 1.2, the Fitting class $\mathfrak{F}$ is normal in $\mathfrak{S}_\pi$.

The lemma is proved.

3. The π-Lausch Group

For proving a criterion of π-maximality, we will use the notion of the Lausch group [9] (also see [3, X.4.2]) and the properties of the group. Recall a procedure of constructing a group of this sort for the Fitting class $\mathfrak{F} = \mathfrak{S}_\pi$.

The notion of the Lausch group is related to that of restricted direct product (see [2, X.4, p. 721]). Recall that for a set of groups $\{G_\mu\}_{\mu\in M}$, their restricted direct product $D_M = \times_{\mu\in M} G_\mu$ consists of all functions $f : M \to \bigcup_{\mu\in M} G_\mu$ of finite support such that $f(\mu) \in G_\mu$ for all $\mu \in M$ with the group operation defined as follows: $(fg)(\mu) = f(\mu)g(\mu)$. Let $N \subseteq M$ and $D_N = \times_{\nu\in N} G_\nu$. By the natural embedding $\varepsilon_N : D_N \to D_M$ we mean the map that sends an element $f_0 \in D_N$ to the following element f of D_M:

$$f(\lambda) = \begin{cases} f_0(\lambda), & \text{if } \lambda \in N, \\ 1, & \text{if } \lambda \notin N. \end{cases}$$

Given groups G and H, an isomorphism $\alpha : G \to H$ is said to be a *subnormal* (*normal*) *embedding* of G into H if $G\alpha \lhd\lhd H$ ($G\alpha \lhd H$). It is clear that ε_N is a normal embedding of D_N into D_M. In a particular case, when $N = \{\nu\}$ is a singleton, we define $G = G_\nu$ and let ε_G stand for the natural embedding of G ($= D_{\{\nu\}}$) in D_M. Following [3], we denote the set of all subnormal embeddings of G into H by $\mathrm{Snemb}(G \to H)$ and the set of normal embeddings, by $\mathrm{Nemb}(G \to H)$.

Let $\mathfrak{E}$ be the class of all finite groups. Fix some set $\mathscr{E}$ containing exactly one representative of each class of isomorphic groups. So, for every group $G \in \mathfrak{E}$, there is a unique group $G_0 \in \mathscr{E}$ such that $G \cong G_0$. Granted a Fitting class $\mathfrak{X} \subseteq \mathfrak{E}$, define $\mathrm{Set}(\mathfrak{X}) = \mathfrak{X} \cap \mathscr{E}$.

Let $\mathfrak{F}$ be a Fitting class of finite groups and $\Delta(\mathfrak{F}) = \times\{G : G \in \mathrm{Set}(\mathfrak{F})\}$. Consider the subgroup

$$\Gamma(\mathfrak{F}) = \langle (g^{-1}\varepsilon_G)(g_\alpha\varepsilon_N) : G, H \in \mathrm{Set}(\mathfrak{F}),\ g \in G,\ \alpha \in \mathrm{Nemb}(G \to H)\rangle,$$

of $\Delta(\mathfrak{F})$. As established in [3, X.4.3(a)], the subgroup $\Gamma(\mathfrak{F})$ is normal in $\Delta(\mathfrak{F})$ and the commutant $\Delta(\mathfrak{F})'$ is contained in $\Gamma(\mathfrak{F})$.

Definition 3.1 [3, X.4.2]. The factor group $\Lambda(\mathfrak{F}) = \Delta(\mathfrak{F})/\Gamma(\mathfrak{F})$ is called the Lausch group of a Fitting class $\mathfrak{F}$.

We will use the construction of the Lausch group in the case when $\mathfrak{F} = \mathfrak{S}_\pi$. The resultant group, denoted by Λ_π, we call the π-*Lausch* group. Thus, $\Lambda_\pi = \Lambda(\mathfrak{S}_\pi)$.

Since $\Delta(\mathfrak{F})$ is a restricted direct product of finite groups, $\Delta(\mathfrak{F})$ is a torsion group. Hence, in light of X.4.3(a) from [3] (see above), the Lausch group of a Fitting class $\mathfrak{F}$ is a torsion abelian group.

For stating the available properties of the Lausch group of a Fitting class $\mathfrak{F}$ we will need further, we also use a notion of Fitting pair for $\mathfrak{F}$. Recall that a pair (A, d) is called a *Fitting pair* for $\mathfrak{F}$ (or $\mathfrak{F}$-*Fitting pair*), if A is an abelian group (infinite in general) and $d : \mathrm{Set}(\mathfrak{F}) \to \{\mathrm{Hom}(G, A) : G \in \mathrm{Set}(\mathfrak{F})\}$ is a map such that for every $G \in \mathrm{Set}(\mathfrak{F})$, the image d_G is a homomorphism of G into A satisfying the following two conditions:

FP1. $d_G = \alpha \circ d_H$ for all $G, H \in \mathrm{Set}(\mathfrak{F})$ and normal embeddings $\alpha : G \to H$.

FP2. $A = \{gd_G : G \in \mathrm{Set}(\mathfrak{F}), g \in G\}$.

An $\mathfrak{F}$-Fitting pair (Λ, δ) is said to be *universal*, if for every $\mathfrak{F}$-Fitting pair (A, d) there exists a homomorphism $\varphi : \Lambda \to A$ such that $d_G = \delta_G \circ \varphi$ for all $G \in \mathrm{Set}(\mathfrak{F})$.

Lemma 3.2 [3, X.4.5(i)]. *Let $\mathfrak{X}$ be a D_0-closed subclass of a Fitting class $\mathfrak{F}$ and $\Xi(\mathfrak{X}) = \{g\delta_G : g \in G \in \mathrm{Set}(\mathfrak{X})\}$. Then $\Xi(\mathfrak{X})$ is a subgroup of the Lausch group $\Lambda(\mathfrak{F})$.*

Lemma 3.3 [3, X.4.14]. *Let Λ_π be the π-Lausch group. Suppose that, for $\mathfrak{X} \in \mathrm{Locksub}(\mathfrak{S}_\pi)$, $\Xi(\mathfrak{X}) = \{x\delta_X : x \in X \in \mathrm{Set}(\mathfrak{X})\}$ is a subset of Λ_π. Then $\Xi : \mathfrak{X} \to \Xi(\mathfrak{X})$ is a lattice isomorphism between the lattice of π-normal Fitting classes and the subgroup lattice of the π-Lausch group Λ_π.*

Lemma 3.4 [3, X.4.15]. *Let $\mathfrak{F}$ be some Fitting class of π-groups, let S be a subgroup of the Lausch group Λ_π, and let $\mathfrak{G}$ be the uniquely determined Fitting class in the Lockett subsection* Locksub($\mathfrak{F}$) *with $\Xi(\mathfrak{G}) = S$. If $G \in \mathrm{Set}(\mathfrak{F})$ then $G_\mathfrak{G} = \{g \in G : g\delta_G \in S\}$.*

4. A Criterion of π-Maximality

Theorem 4.1. *Let π be a nonempty set of primes and let $\mathfrak{F} \subset \mathfrak{S}_\pi$ be a Fitting class. The following are equivalent:*

(1) *$\mathfrak{F}$ is π-maximal;*

(2) *there is a prime $p \in \pi$ such that $|G : G_\mathfrak{F}| \in \{1, p\}$ for all $G \in \mathfrak{S}_\pi$.*

Proof. (1) $\Longrightarrow$ (2) Let $\Xi(\mathfrak{F}) = \{x\delta_X : x \in X \in \mathrm{Set}(\mathfrak{F})\}$. Then, by Lemma 3.2, $\Xi(\mathfrak{F})$ is a subgroup of Λ_π. Remark that the statement X.4.3(a) from [3] yields that Λ_π is an abelian group.

Since $\mathfrak{F} \lessdot \mathfrak{S}_\pi$, it follows from Lemma 2.1 that $\mathfrak{F} \lhd \mathfrak{S}_\pi$. Hence, by (2) of Lemma 1.2 we have $\mathfrak{F}^\star = \mathfrak{S}_\pi = \mathfrak{S}_\pi^\star$, and so $\mathfrak{F} \in \mathrm{Locksec}(\mathfrak{S}_\pi)$. If a Fitting class $\mathfrak{X}$ is contained in $\mathrm{Locksec}(\mathfrak{S}_\pi)$ then $\mathfrak{X}^\star = \mathfrak{S}_\pi$. Thus, in view of $\mathfrak{X} \subseteq \mathfrak{X}^\star$, we infer $\mathfrak{X} \subseteq \mathfrak{S}_\pi$ and $\mathfrak{X} \in \mathrm{Locksub}(\mathfrak{S}_\pi)$. Therefore, $\mathrm{Locksec}(\mathfrak{S}_\pi) = \mathrm{Locksub}(\mathfrak{S}_\pi)$. Thus, by Lemma 3.3, there exists a lattice isomorphism between π-normal classes and subgroups of Λ_π. By Lemma 3.3, it follows from π-maximality of $\mathfrak{F}$ that $\Xi(\mathfrak{F})$ is a maximal subgroup of Λ_π. But since Λ_π is a torsion group, we conclude that $\Lambda_\pi/\Xi(\mathfrak{F}) \cong Z_p$ for some $p \in \pi$. Then Lemma 3.4 implies that for all $G \in \mathfrak{S}_\pi$ the factor group $G/G_\mathfrak{F}$ is isomorphic to some subgroup of $\Lambda_\pi/\Xi(\mathfrak{F})$.

Remark that $\Lambda_\pi/\Xi(\mathfrak{F})$ is a group of prime order p. Therefore, either $G/G_\mathfrak{F} = \Lambda/\Xi(\mathfrak{F}) \cong Z_p$ or $G = G_\mathfrak{F}$. This yields that $|G/G_\mathfrak{F}| \in \{1, p\}$ for every group $G \in \mathfrak{S}_\pi$, proving (2).

(2) $\Longrightarrow$ (1) Suppose that a Fitting class $\mathfrak{F}$ is not maximal in $\mathfrak{S}_\pi$. Then there is a Fitting class $\mathfrak{M}$ such that $\mathfrak{F} \subset \mathfrak{M} \subset \mathfrak{S}_\pi$.

Let G and H be groups from the classes $\mathfrak{M}\backslash\mathfrak{F}$ and $\mathfrak{S}_\pi\backslash\mathfrak{M}$ respectively. By hypothesis, for every π-group its index by its $\mathfrak{F}$-radical is equal to a prime $p \in \pi$, and so $|G/G_\mathfrak{F}| = |H/H_\mathfrak{F}| = p$. However, $\mathfrak{F} \subset \mathfrak{M}$. Thus, $H_\mathfrak{F} \leq H_\mathfrak{M}$.

It follows from $|H/H_{\mathfrak{F}}| = p$ that $H_{\mathfrak{F}}$ is a maximal normal subgroup of H. Thus, in view of $H_{\mathfrak{F}} \le H_{\mathfrak{M}}$, we infer that either $H_{\mathfrak{M}} = H$ or $H_{\mathfrak{F}} = H_{\mathfrak{M}}$. If $H_{\mathfrak{M}} = H$ then $H \in \mathfrak{M}$; a contradiction to choice of H.

Hence,

$$H_{\mathfrak{F}} = H_{\mathfrak{M}}. \tag{4.1.1}$$

Put $T = G \times H$. If we assume $T \in \mathfrak{M}$ then it follows from $H \lhd T$ that $H \in \mathfrak{M}$, which is impossible by the choice of H. Thus, $T_{\mathfrak{M}} < T$.

Since $\mathfrak{F} \subset \mathfrak{M}$, we have $(T_{\mathfrak{M}})_{\mathfrak{F}} = T_{\mathfrak{M}} \cap T_{\mathfrak{F}} = T_{\mathfrak{F}}$.

Remark that $T_{\mathfrak{M}} = G \times H_{\mathfrak{M}}$. Let $T_{\mathfrak{M}} \in \mathfrak{F}$. Then $G \lhd T_{\mathfrak{M}}$ yields $G \in \mathfrak{F}$. The last contradicts the choice of G.

So $T_{\mathfrak{M}} \notin \mathfrak{F}$ and

$$(T_{\mathfrak{M}})_{\mathfrak{F}} = T_{\mathfrak{F}} < T_{\mathfrak{M}}. \tag{4.1.2}$$

On the other hand, by (4.1.1) $H_{\mathfrak{F}} = H_{\mathfrak{M}}$. Hence, $T_{\mathfrak{M}} = G \times H_{\mathfrak{M}} \in \mathfrak{F}$ and $G \lhd T_{\mathfrak{M}}$.

Thus, $T_{\mathfrak{M}} \le T_{\mathfrak{F}}$. The last contradicts (4.1.2). Therefore, $\mathfrak{F} \lessdot \mathfrak{S}_\pi$.

The theorem is proved.

In the case $\pi = \mathbb{P}$ we obtain

Corollary 4.2 [3, Theorem X.4.26]. *Let $\mathfrak{F}$ be a Fitting class. The following are equivalent:*

(i) *$\mathfrak{F}$ is maximal in the class of all finite soluble groups $\mathfrak{S}$;*

(ii) *there is a prime p such that $|G : G_{\mathfrak{F}}| \in \{1, p\}$ for all $G \in \mathfrak{S}$.*

5. Examples

5.1. The idea of the technique for constructing the first example originates from the well-known results of Blessenohl and Gaschütz [10] which relate to creating soluble normal Fitting classes and consist in constructing a special Fitting pair for $\mathfrak{F}$.

Let $\varnothing \ne \pi \subseteq \mathbb{P}$ and let $p \in \pi$ be a fixed prime. Let A be a cyclic group of order $p-1$, and let G be a group from a Fitting class $\mathfrak{F} \subseteq \mathfrak{S}_\pi$. Denote by $\mathscr{L}$ some chief series of G. Let $\mathscr{M} = \{M_1, M_2, \dots, M_r\}$ be the set of all chief p-factors of G and $\mathscr{M} \ne \varnothing$. Then each $g \in G$ induces the automorphism α_g on M_i $(i = 1, 2, \dots, r)$. Let $n = \dim(M_i)$ and let m be a map form the automorphism group $A(M_i)$ of M_i into the group $GL(M_i)$ of nonsingular matrices of order n with entries a_{ij} belonging to the Galois field F_p, i. e. $\alpha_g m = \|a_{ij}\|$. Now, let h be a map of $GL(M_i)$ into the multiplicative group $F_p^\star$ of F_p such that

$$\|a_{ij}\| h = \det \|a_{ij}\|.$$

It is easy that h is a homomorphism, and so the product $d = mh$ is a homomorphism of $A(M_i)$ into $A = F_p^\star$. Thus,

$$\alpha_g mh = \alpha_g d = \det \|a_{ij}\|.$$

Put $\alpha_g d = d_i^p(g)$ and $d_G^p(g) = \prod_{i=1}^r d_i^p(g)$ for all $g \in G$.

If $\mathscr{M} = \varnothing$ then $d_G^p(g) = 1$. It is not hard to notice that $d_G^p : G \to A$ is a homomorphism.

Now, put $p = 3$. It is straightforward to check that (A, d), where $A = F_3^\star$ and $d : \mathfrak{F} \to \bigcup\{\mathrm{Hom}(G, A)\}$, satisfies Condition FP1 of the definition of Fitting pair for $\mathfrak{F}$. Let $\mathfrak{M} = \big(G \in \mathfrak{F} : d_G^3(G) = 1\big)$. Then, applying Theorem IX.2.11 of [3], we derived that $\mathfrak{M}$ is a Fitting class and $G_{\mathfrak{M}} = \mathrm{Ker}(d_G)$. Now the maximality of $\mathfrak{M}$ in $\mathfrak{F}$ follows from Theorem 4.1, since $|A| = 2$; and by the assertion IX.4.2(b) from [3] we have $|G/G_{\mathfrak{M}}| \in \{1, 2\}$ for all $G \in \mathfrak{F}$.

5.2. Let $3 \in \pi \subseteq \mathbb{P}$ and $\mathfrak{F} = \mathfrak{S}_\pi$. Define the class $\mathfrak{M} \subseteq \mathfrak{F}$ of groups as follows: $G \in \mathfrak{M}$ if and only if G acts on its elements of order 3^n as a group of even permutations (n is an arbitrary natural number). In view of Camina's result [11], we infer that $\mathfrak{M}$ is a Fitting class of index 2 in $\mathfrak{F}$, i. e. $|G/G_{\mathfrak{M}}| \in \{1, 2\}$ for all G from $\mathfrak{F}$, and $\mathfrak{M} \lessdot \mathfrak{F}$ by Theorem 4.1.

We conclude by observing that the arbitrary choice of a natural n yields that there are countably many maximal Fitting subclasses in the local Fitting class $\mathfrak{S}_\pi$.

6. Applications

6.1. Existence of nonnormal π-maximal subclasses. Hertzfeld [12] and Skiba (see [13, Example 19.1]) showed that a nonidentity local formation has no subformations that are maximal under inclusion. The question on the existence of maximal Fitting subclasses of a minimal normal Fitting class was posed by Lausch in [14, Question 9.18] and answered in the negative by Vorob′ev [8]. In this connection, the following question stated by Skiba arises naturally.

Problem [1, Question 13.50]. *Let $\mathfrak{F}$ be a local Fitting class. Is it true that the inclusion ordered set of the Fitting classes that lie in $\mathfrak{F}$ and differ from $\mathfrak{F}$ has no maximal elements?*

A negative answer to the above problem is provided by

Theorem 6.1.1. *There are nonnormal π-maximal Fitting classes in the local Fitting class $\mathfrak{S}_\pi$.*

Proof. Let $\varnothing \subset \pi \subset \mathbb{P}$ and let $\mathfrak{F}$ be a Fitting class such that for every π-group G the index $|G : G_{\mathfrak{F}}|$ is in $\{1, p\}$ for some prime $p \in \pi$. Then by Theorem 4.1 we have $\mathfrak{F} <\cdot\, \mathfrak{S}_\pi$ (these classes $\mathfrak{F}$ exist in view of Examples 5.1 and 5.2). Hence, by Lemma 2.1 we have $\mathfrak{F} \lhd \mathfrak{S}_\pi$. But then (2) of Lemma 1.2 yields $\mathfrak{F}^\star = \mathfrak{S}_\pi \neq \mathfrak{S}$, and $\mathfrak{F}$ is a nonnormal Fitting class.

The theorem is proved.

6.2. On a Fitting class with the Lockett condition. A Fitting class $\mathfrak{F}$ is said to be a *Fitting class with the Lockett condition in a Fitting class $\mathfrak{X}$* or *$\mathscr{L}_{\mathfrak{X}}$-class* if $\mathfrak{F} \subseteq \mathfrak{X}$ and $\mathfrak{F}_\star = \mathfrak{F} \cap \mathfrak{X}_\star$.

If $\mathfrak{X} = \mathfrak{S}$ then $\mathfrak{F}$ is simply called an *$\mathscr{L}$-class.*

By Bryce and Cossey's result [15], a problem of constructing each soluble Lockett class $\mathfrak{F}$ as an intersection of a Lockett class and some normal Fitting class is equivalent to checking that $\mathfrak{F}$ satisfies the Lockett condition $\mathfrak{F}_\star = \mathfrak{F} \cap \mathfrak{S}_\star$, where $\mathfrak{S}_\star$ is the inclusion minimal element of the Lockett section of the class $\mathfrak{S}$, i.e. $\mathfrak{S}_\star$ is a minimal normal Fitting class. As showed in [15], the Lockett condition holds for soluble local hereditary Fitting classes. The following general result was established in [6]: Every soluble local Fitting class satisfies Lockett's condition.

By now, one of the most difficult problems in the theory of Fitting classes is that of describing the groups in the Fitting class generated by the symmetric group on three symbols (for example, see [3, XI.3.4, p. 801]). Recall that for $\mathfrak{F} = \operatorname{Fit} S_3$, where S_3 is the symmetric group on three symbols,

$$\operatorname{Fit} S_3 = \cap\{\mathfrak{X} \mid \mathfrak{X} \text{ is the Fitting class and } S_3 \in \mathfrak{X}\}.$$

Distinguishing a maximal subclass in $\mathfrak{F} = \operatorname{Fit} S_3$, we discover new information about this class.

Recall that for Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$, $\mathfrak{F} \vee \mathfrak{H}$ denotes the Fitting class that is generated by the join of $\mathfrak{F}$ and $\mathfrak{H}$.

Theorem 6.2.1. *The Fitting class $\mathfrak{F}$ is an $\mathscr{L}$-class.*

Proof. We claim that $\mathfrak{F}$ meets the Lockett condition. Since $\mathfrak{F} = \mathfrak{F}_\star \vee \mathfrak{F}$, (4) of Lemma 1.1 implies that

$$\mathfrak{F}_\star \vee (\mathfrak{F} \cap (\mathfrak{F}_\star)^\star) = \mathfrak{F}_\star \vee (\mathfrak{F} \cap \mathfrak{F}^\star) = \mathfrak{F}_\star \vee \mathfrak{F}.$$

Applying Corollary 2.6 (see [16, p. 42]), for all $G \in \mathfrak{F} \backslash \mathfrak{F}_\star$ we infer

$$\mathfrak{F}_\star \vee \mathfrak{F} = \{G : (G \times S_3)_{\mathfrak{F}_\star} \text{ is subdirect in } G \times S_3\}.$$

Thus, $(G \times S_3)_{\mathfrak{F}^\star}$ is subdirect in $G \times S_3$ for all $G \in \mathfrak{F} \backslash \mathfrak{F}_\star$. Therefore, in view of Example 2.7.2 (see [16, p. 42]), we have $|G/G_{\mathfrak{F}}| = 2$. Following the proof of (2) $\Longrightarrow$ (1) in Theorem 4.1, we infer that $\mathfrak{F}_\star <\cdot\, \mathfrak{F}$. By (3) and (4) of Lemma 1.1 we see from $\mathfrak{F} \subseteq \mathfrak{S}$ that $\mathfrak{F}_\star \subseteq \mathfrak{S}_\star$. Hence, considering that $\mathfrak{F}_\star \subset \mathfrak{F}$, we obtain $\mathfrak{F}_\star \subseteq \mathfrak{F} \cap \mathfrak{S}_\star \subseteq \mathfrak{F}$. Since $\mathfrak{F}_\star <\cdot\, \mathfrak{F}$, one of the following holds: $\mathfrak{F} \cap \mathfrak{S}_\star = \mathfrak{F}$ and $\mathfrak{F}_\star = \mathfrak{F} \cap \mathfrak{S}_\star$. If the first equality holds then $\mathfrak{S}_\star \supseteq \mathfrak{F}$. This contradicts the fact that $S_3 \notin \mathfrak{S}_\star$ and $\mathfrak{F} \nsubseteq \mathfrak{S}_\star$ due to Camina's example [11].

Thus, $\mathfrak{F}_\star = \mathfrak{F} \cap \mathfrak{S}_\star$, and $\mathfrak{F}$ is an $\mathscr{L}$-class.

The proof of the theorem is complete.

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