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Local factorisations of nonlocal Fitting classes

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Abstract — We consider general rules of constructing of local products of the Fitting classes with the use of nonlocal factors determined by the minimal elements of the Lockett section. We give a simple method to construct local Fitting classes which are factorised by nonlocal Fitting classes of some classes of π-groups and π'-groups. Applying the class $\mathfrak{B}$, we simplify the procedure of construction of a nonlocal factor in the local product. It is shown that in the construction of the local product the nonlocal factor for the appropriate set of primes π is the Fitting class of the form $\mathfrak{B}_*\mathfrak{S}_\pi$, where $\mathfrak{B}_*$ is the minimal element of the Lockett section of the class $\mathfrak{B}$. In this paper, we consider only finite solvable groups.

1. INTRODUCTION

The product of Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ is the class of all groups G whose factorgroups with respect to $\mathfrak{F}$-radical are $\mathfrak{H}$-subgroups [1]. It is well known that a product of two Fitting classes is also a Fitting class and the operation of multiplication of Fitting classes is associative (see, for example, IX.1.12 in [1]). Among the products of Fitting classes, the best known by their applications are local products, that is, such products which are local Fitting classes. In [2], it is proved that a product of any two local Fitting classes is local.

Recall that any mapping f of the set P of all prime numbers into the set of Fitting classes is called a Hartley function or an H-function [3]. A Fitting class $\mathfrak{F}$ is called local if there exists an H-function f such that

$$\mathfrak{F} = \mathfrak{S}_{\pi(\mathfrak{F})} \cap \left(\bigcap_{p\in\pi(\mathfrak{F})} f(p)\mathfrak{N}_p\mathfrak{S}_{p'} \right).$$

Examples of local Fitting classes and local formations which are factorised by non-local factors were first constructed by N. T. Vorobyev and A. N. Skiba (see [4]), and V. A. Vedernikov (see [5]), N. T. Vorobyev (see [6]), respectively, and thereby they gave a positive solution of problems 11.25 a) and 9.58 in [7].

In this paper, we consider general rules of constructing local products of Fitting classes with the use of nonlocal factors which are determined by the minimal elements of Lockett sections [8] (see X.1.12 (b) and X.1.16 in [1]). First of all, we suggest a simple method of constructing local Fitting classes factorised by nonlocal Fitting classes of some classes of

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π-groups and π'-groups (Theorem 1, case 1). In this case, as distinct from [4], we do not use the complicated construction of the Fitting class $\mathfrak{B}$ which is defined in [9]. Moreover, applying the class $\mathfrak{B}$, we simplify, in comparison with [4], the procedure of constructing a nonlocal factor in a local product. It is shown that in construction of a local product, differing from the product in the example in [4], the nonlocal factor for an appropriate set of prime numbers π is the Fitting class of the form $\mathfrak{B}_*\mathfrak{S}_\pi$, where $\mathfrak{B}_*$ is the minimal element of the Lockett section of the class $\mathfrak{B}$. Note also that in addition to [4] we show that each factor of a local product is defined semilocally. In this paper, we consider only finite solvable groups.

Note that the absent definitions and notation can be found in [1, 10] if necessary.

2. AUXILIARY RESULTS

A class of groups $\mathfrak{F}$ is called a Fitting class [1] if $\mathfrak{F}$ is closed with respect to taking normal subgroups and products of normal $\mathfrak{F}$-subgroups. If $\mathfrak{F}$ is a nonempty Fitting class, then the subgroup $G_\mathfrak{F}$ of the group G is called an $\mathfrak{F}$-radical of the group G if G is the maximal normal subgroups of G belonging to $\mathfrak{F}$.

In lemmas below, we give some known properties of $\mathfrak{F}$-radicals and products of Fitting classes which will be used later.

A class of groups $\mathfrak{F}$ is called a homomorph [10] if each factor group of any groups of $\mathfrak{F}$ also belong to $\mathfrak{F}$.

Lemma 1 (Lemma 2.1 in [11]). *Let $\mathfrak{F}$ be a nonempty Fitting class. Then the following assertions are true:*

(1) $\mathfrak{F} \subseteq \mathfrak{F}\mathfrak{H}$ *for any Fitting class* $\mathfrak{H} \neq \varnothing$*;*

(2) *if a Fitting class $\mathfrak{H}$ is a homomorph, then* $\mathfrak{H} \subseteq \mathfrak{F}\mathfrak{H}$

(3) *if $\{\mathfrak{H}_i \mid i \in I\}$ is a nonempty set of Fitting classes, then*

$$\bigcap_{i\in I} \mathfrak{F}\mathfrak{H}_i = \mathfrak{F}\left(\bigcap_{i\in I} \mathfrak{H}_i\right).$$

Lemma 2 ([2]). *A product of any two local Fitting classes is local.*

A homomorph $\mathfrak{F}$ is saturated [10] if the condition $G/\Phi(G) \in \mathfrak{F}$ implies the inclusion $G \in \mathfrak{F}$. A class of groups $\mathfrak{F}$ is called a radical homomorph [10] if it is simultaneously a homomorph and a Fitting class.

Recall that Lockett [8] for any Fitting class $\mathfrak{F}$ defines the class $\mathfrak{F}^*$ as the minimal Fitting class containing $\mathfrak{F}$ and such that for all groups G, H the equality $(G\times H)_{\mathfrak{F}^*} = G_{\mathfrak{F}^*} \times H_{\mathfrak{F}^*}$ is true and the class $\mathfrak{F}_*$ as the intersection of all Fitting classes $\mathfrak{X}$ such that $\mathfrak{X}^* = \mathfrak{F}^*$. A Fitting class $\mathfrak{F}$ is called the Lockett class if $\mathfrak{F} = \mathfrak{F}^*$.

We give the known properties of the Lockett operators * and $_*$ in the following lemma.

Lemma 3 (X.1.2, IX.1.8 in [1]; [8], [11]). *For any nonempty Fitting class $\mathfrak{F}$ the following assertions are true:*

(1) *the action of the operators * and $_*$ on the class $\mathfrak{F}$ are characterised by the relations*

$$\mathfrak{F}_* = (\mathfrak{F}_*)_* = (\mathfrak{F}^*)_* \subseteq \mathfrak{F} \subseteq \mathfrak{F}^* = (\mathfrak{F}_*)^* = (\mathfrak{F}^*)^*;$$

(2) *for any nonempty Fitting class $\mathfrak{H}$ the equality*

$$(\mathfrak{F} \cap \mathfrak{H})^* = \mathfrak{F}^* \cap \mathfrak{H}^*$$

is true;

(3) *if $\mathfrak{F}$ is a homomorph, then $\mathfrak{F}^* = \mathfrak{F}$;*

(4) *if $\mathfrak{H}$ is a radical saturated homomorph, then $(\mathfrak{F}\mathfrak{H})^* = \mathfrak{F}^*\mathfrak{H}$.*

Lemma 4 ([11]). *Each local Fitting class is a Lockett class.*

A Fitting class $\mathfrak{F}$ is called normal [1] if the $\mathfrak{F}$-radical of a group G is the $\mathfrak{F}$-maximal subgroup of the group G for any group G. Note that the intersection of any set of non-unit normal Fitting classes is non-unit normal Fitting class. The minimal non-unit Fitting class is denoted by $\mathfrak{S}_*$.

Lemma 5 (X.5.32 in [1]). *If p and q are different prime numbers, then $\mathfrak{N}_p\mathfrak{N}_q \not\subseteq \mathfrak{S}_*$.*

Recall that some classes have standard notations: $\mathfrak{S}$ is the class of all solvable groups; $\mathfrak{S}_\pi$ is the class of all solvable π-groups; $\mathfrak{N}$ is the class of all nilpotent groups; $\mathfrak{N}_\pi$ is the class of all nilpotent π-groups.

Lemma 6 ([8]). *If $\mathfrak{F}$ is nonempty Fitting class, then the following assertions are equivalent:*

(1) *$\mathfrak{F}$ is a normal Fitting class;*

(2) $\mathfrak{F}\mathfrak{N} = \mathfrak{S}$*;*

(3) $\mathfrak{F}^* = \mathfrak{S}$.

A Fitting class $\mathfrak{F}$ satisfies the Lockett hypothesis or is an $\mathfrak{L}$-class if $\mathfrak{F}_* = \mathfrak{F}^* \cap \mathfrak{S}_*$ (see [11]).

Lemma 7 ([11]). *Each local Fitting class is an $\mathfrak{L}$-class.*

A Fitting class $\mathfrak{F}$ is called π-saturated [12] if $\mathfrak{F}\mathfrak{S}_{\pi'} = \mathfrak{F}$, where $\varnothing \neq \pi \subseteq P$ and $\mathfrak{S}_{\pi'}$ is the class of all solvable π'-group.

3. CLASS $R_\pi(\mathfrak{F})$ AND ITS PROPERTIES

Let

$$\pi = \mathrm{Supp}(f) = \{p \in P \colon f(p) \neq \varnothing\}$$

be the support of an H-function f. Then, following [12], we set

$$SLR(f) = \bigcap_{p \in \pi} f(p)\mathfrak{S}_{p'}$$

and say that the Fitting class $\mathfrak{F}$ is defined semilocally [13] if $\mathfrak{F} = SLR(f)$ for some H-function f. If $\pi = \varnothing$, the we set $SLR(f) = \varnothing$.

The following assertion gives a criterion that a Fitting class is defined semilocally.

Lemma 8. *A Fitting class $\mathfrak{F}$ is defined semilocally if and only if $\mathfrak{F}$ is π-saturated for some π, $\varnothing \subset \pi \subseteq P$.*

Proof. Let $\varnothing \subset \pi \subseteq P$ and $\mathfrak{F}$ be a π-saturated Fitting class. Then $\mathfrak{F}\mathfrak{S}_{\pi'} = \mathfrak{F}$. We construct the H-function f in the following way:

$$f(p) = \begin{cases} \mathfrak{F}, & p \in \pi, \\ \varnothing, & p \in \pi'. \end{cases}$$

Let us show that f defines semilocally the Fitting class $\mathfrak{F}$. Let

$$SLR(f) = \bigcap_{p \in \pi} f(p)\mathfrak{S}_{p'},$$

where $\pi = \mathrm{Supp}(f)$. By the definition $f(p) = \mathfrak{F}$ for all $p \in \pi$, therefore the equality

$$SLR(f) = \bigcap_{p \in \pi} \mathfrak{F}\mathfrak{S}_{p'}$$

is true. Thus, according to assertion 3 of Lemma 1

$$SLR(f) = \mathfrak{F}\left(\bigcap_{p \in \pi} \mathfrak{S}_{p'}\right) = \mathfrak{F}\mathfrak{S}_{\pi'} = \mathfrak{F},$$

and $\mathfrak{F}$ is a Fitting class defined semilocally.

Now let $\mathfrak{F}$ be a Fitting class defined semilocally. Then there exists an H-function f with support π such that

$$\mathfrak{F} = \bigcap_{p \in \pi} f(p)\mathfrak{S}_{p'}.$$

Let us show that $\mathfrak{F}\mathfrak{S}_{\pi'} = \mathfrak{F}$. Since

$$\mathfrak{F}\mathfrak{S}_{\pi'} = \left(\bigcap_{p \in \pi} f(p)\mathfrak{S}_{p'}\right)\mathfrak{S}_{\pi'},$$

by virtue of assertion 3 of Lemma 1

$$\mathfrak{F}\mathfrak{S}_{\pi'} = \bigcap_{p\in\pi} (f(p))\mathfrak{S}_{p'}\mathfrak{S}_{\pi'}.$$

But $p \in \pi$, and therefore $\pi' \subseteq p'$. Hence, $\mathfrak{S}_{\pi'} \subseteq \mathfrak{S}_{p'}$ for all prime numbers $p \in \pi$. Moreover, $\mathfrak{S}_{p'}\mathfrak{S}_{\pi'} = \mathfrak{S}_{p'}$. Therefore,

$$\mathfrak{F}\mathfrak{S}_{\pi'} = \bigcap_{p\in\pi} f(p)(\mathfrak{S}_{p'}\mathfrak{S}_{\pi'}) = \bigcap_{p\in\pi} f(p)\mathfrak{S}_{p'} = \mathfrak{F}.$$

The lemma is proved.

Definition 1. Let $\pi \subseteq P$. We define the class of groups $R_\pi(\mathfrak{F})$, setting

$$G \in R_\pi(\mathfrak{F}) \iff G_\pi \subseteq G_{\mathfrak{F}}.$$

If $\mathfrak{F} = \varnothing$, then we set $R_\pi(\mathfrak{F}) = \varnothing$. In the case where $\pi = \varnothing$ and $\pi = P$, we set $R_\varnothing(\mathfrak{F}) = \mathfrak{S}$ and $R_P(\mathfrak{F}) = \mathfrak{F}$ respectively.

Note that if $\mathfrak{F}$ is a normal Fitting class, then $R_\pi(\mathfrak{F}) = L_\pi(\mathfrak{F})$, where $L_\pi(\mathfrak{F})$ is the class of all groups G whose $\mathfrak{F}$-injections contain the Hall π-group G (see IX.1.14 in [1]).

Lemma 9. *If $\mathfrak{F}$ is a Fitting class and $\pi \subseteq P$, then $R_\pi(\mathfrak{F}) = \mathfrak{F}\mathfrak{S}_{\pi'}$ is a π-saturated (semilocal) Fitting class. In particular, if $\mathfrak{F}$ is a local Fitting class, then $R_\pi(\mathfrak{F})$ is a local Fitting class.*

Proof. Let $G \in R_\pi(\mathfrak{F})$. Then $G_\pi \subseteq G_{\mathfrak{F}}$, and therefore $G/G_{\mathfrak{F}}$ is a π'-group. Since $G_{\mathfrak{F}} \in \mathfrak{F}$ and $G/G_{\mathfrak{F}} \in \mathfrak{S}_{\pi'}$, the inclusion $G \in \mathfrak{F}\mathfrak{S}_{\pi'}$ is true. Therefore, $R_\pi(\mathfrak{F}) \subseteq \mathfrak{F}\mathfrak{S}_{\pi'}$. Let $H \in \mathfrak{F}\mathfrak{S}_{\pi'}$. Then $H/H_{\mathfrak{F}} \in \mathfrak{S}_{\pi'}$ and therefore $H_\pi \subseteq H_{\mathfrak{F}}$. Hence, $H \in R_\pi(\mathfrak{F})$ and $\mathfrak{F}\mathfrak{S}_{\pi'} \subseteq R_\pi(\mathfrak{F})$. Therefore $R_\pi(\mathfrak{F}) = \mathfrak{F}\mathfrak{S}_{\pi'}$ is a Fitting class. Since $R_\pi(\mathfrak{F})\mathfrak{S}_{\pi'} = (\mathfrak{F}\mathfrak{S}_{\pi'})\mathfrak{S}_{\pi'} = \mathfrak{F}(\mathfrak{S}_{\pi'}\mathfrak{S}_{\pi'}) = \mathfrak{F}\mathfrak{S}_{\pi'} = R_\pi(\mathfrak{F})$, we see that $R_\pi(\mathfrak{F})$ is a π-saturated Fitting class. Let $\mathfrak{F}$ be a local Fitting class. Since $\mathfrak{S}_{\pi'}$ is a local Fitting class, according to Lemma 2 $R_\pi(\mathfrak{F}) = \mathfrak{F}\mathfrak{S}_{\pi'}$ is a local Fitting class. The lemma is proved.

Note that in the general case a Fitting class $R_\pi(\mathfrak{F})$ is not local. This fact is confirmed by the following example.

Example 1. Let $\mathfrak{F} = \mathfrak{S}_*$ be the minimal normal Fitting class and $\varnothing \neq \pi \subset P$. Let us show that that in this case the class $R_\pi(\mathfrak{F})$ is not local. Suppose that $R_\pi(\mathfrak{F})$ is a local Fitting class. Since the class $\mathfrak{F}$ is normal, it follows from Lemma 6 that $\mathfrak{F}\mathfrak{N} = \mathfrak{S}$. Since $\mathfrak{F} \subseteq R_\pi(\mathfrak{F})$, we see that $\mathfrak{S} = \mathfrak{F}\mathfrak{N} \subseteq R_\pi(\mathfrak{F})\mathfrak{N} \subseteq \mathfrak{S}$. Therefore, $R_\pi(\mathfrak{F})\mathfrak{N} = \mathfrak{S}$ and by Lemma 6 $R_\pi(\mathfrak{F})$ is a normal Fitting class. By Lemma 9 $R_\pi(\mathfrak{F}) = \mathfrak{F}\mathfrak{S}_{\pi'}$ and by assertion 4 of Lemma 3 $(R_\pi(\mathfrak{F}))^* = (\mathfrak{F}\mathfrak{S}_{\pi'})^* = \mathfrak{F}^*\mathfrak{S}_{\pi'}$. But by Lemma 6 and assertion 1 of Lemma 3 $\mathfrak{F}^* = \mathfrak{S}$ and $(R_\pi(\mathfrak{F}))^* = \mathfrak{S}$. Now, since $R_\pi(\mathfrak{F})$ is local, by Lemma 4 $R_\pi(\mathfrak{F})$ is a Lockett class, and therefore, $R_\pi(\mathfrak{F}) = \mathfrak{S}$.

Now let p and q be prime numbers such that $p \mid (q-1)$, $G = D^n_{q^n}$ be a monolithic group with normal abelian Sylow q-subgroup of exponent q^n and cyclic Sylow q'-subgroup of order p. Let $\pi = \pi(G)$. Then $G_\pi = G$ and according to T. K. Berger's result (see property 3 in [14]) $G \notin \mathfrak{S}_*$. Therefore, $G_{\mathfrak{S}_*} \subset G$. Thus, $G \notin R_\pi(\mathfrak{F})$ and $R_\pi(\mathfrak{F}) \neq \mathfrak{S}$. The obtained contradiction proves that the Fitting class $R_\pi(\mathfrak{F})$ is not local.

4. LOCAL FACTORISATIONS

Lemma 10. *If $\pi \subset P$ and $|\pi| \geq 2$, then the product $(\mathfrak{S}_\pi)_*\mathfrak{S}_{\pi'}$ is not a Lockett class.*

Proof. Suppose that $(\mathfrak{S}_\pi)_*\mathfrak{S}_{\pi'}$ is a Locket class. Then with the use of assertions 1 and 4 of Lemma 3 we obtain $(\mathfrak{S}_\pi)_*\mathfrak{S}_{\pi'} = ((\mathfrak{S}_\pi)_*\mathfrak{S}_{\pi'})^* = ((\mathfrak{S}_\pi)_*)^*\mathfrak{S}_{\pi'} = (\mathfrak{S}_\pi)^*\mathfrak{S}_{\pi'} = \mathfrak{S}_\pi\mathfrak{S}_{\pi'}$. The Fitting class $\mathfrak{S}_\pi = LR(f)$ for the H-function f is such that

$$f(p) = \begin{cases} \mathfrak{S}_\pi, & p \in \pi, \\ \varnothing, & p \in \pi'. \end{cases}$$

Therefore, by Lemma 7 the class $\mathfrak{S}_\pi$ is an $\mathfrak{L}$-class. But then $(\mathfrak{S}_\pi)_*\mathfrak{S}_{\pi'} = (\mathfrak{S}_\pi \cap \mathfrak{S}_*)\mathfrak{S}_{\pi'} = \mathfrak{S}_\pi\mathfrak{S}_{\pi'} \cap \mathfrak{S}_*\mathfrak{S}_{\pi'}$. Therefore, $\mathfrak{S}_\pi\mathfrak{S}_{\pi'} = \mathfrak{S}_\pi\mathfrak{S}_{\pi'} \cap \mathfrak{S}_*\mathfrak{S}_{\pi'}$ and $\mathfrak{S}_\pi\mathfrak{S}_{\pi'} \subseteq \mathfrak{S}_*\mathfrak{S}_{\pi'}$. Hence we obtain $\mathfrak{S}_\pi \subseteq \mathfrak{S}_*\mathfrak{S}_\pi \cap \mathfrak{S}_*\mathfrak{S}_{\pi'} = \mathfrak{S}_*(\mathfrak{S}_\pi \cap \mathfrak{S}_{\pi'})$. Thus, $\mathfrak{S}_\pi \subseteq \mathfrak{S}_*$. However, by Lemma 5, this is impossible for $|\pi| \geq 2$. The obtained contradiction shows that the Fitting class $(\mathfrak{S}_\pi)_*\mathfrak{S}_{\pi'}$ is not a Locket class. The lemma is proved.

Lemma 11. *If $\mathfrak{F}$ is a Lockett class, then the class $R_\pi(\mathfrak{F})$ is a Lockett class.*

Proof. By Lemma 9, $R_\pi(\mathfrak{F}) = \mathfrak{F}\mathfrak{S}_{\pi'}$. Hence, $(R_\pi(\mathfrak{F}))^* = (\mathfrak{F}\mathfrak{S}_{\pi'})^*$. Then by assertion 4 of Lemma 3 $(\mathfrak{F}\mathfrak{S}_{\pi'})^* = \mathfrak{F}^*\mathfrak{S}_{\pi'}$. But $\mathfrak{F}$ is a Lockett class, and therefore, $\mathfrak{F}^* = \mathfrak{F}$. Hence, $\mathfrak{F}^*\mathfrak{S}_{\pi'} = \mathfrak{F}\mathfrak{S}_{\pi'}$. Thus, we obtain $(R_\pi(\mathfrak{F}))^* = R_\pi(\mathfrak{F})$ and $R_\pi(\mathfrak{F})$ is a Lockett class. The lemma is proved.

Note that by virtue of a result of [14] the trivial local Fitting class $\mathfrak{S}$ can be represented in the form of local product $\mathfrak{S} = \mathfrak{F}\mathfrak{S}_*$ for any non-unit normal Fitting class $\mathfrak{F} \neq \mathfrak{S}$. In this case, in view of Lemmas 4 and 6 it easy to see that each of the factors $\mathfrak{F}$ and $\mathfrak{S}_*$ is not local and is not a formation.

In order to construct local products (differing from $\mathfrak{S}$) of nonlocal non-normal Fitting classes which are not formations, in particular, we will use Fitting class $\mathfrak{B}$ introduced in [9].

Recall that the construction of the class $\mathfrak{B}$ reduces to finding some group X contained simultaneously in the classes $\mathfrak{B}$ and $\mathfrak{S}_*$, but not belonging to the class $\mathfrak{B}_*$. For this purpose we use the description of representations of extra-special p-groups over an arbitrary field of characteristic not equal to p (see [1], pp. 166–168). Now let $p = 3$ and R be an extra-special group of order 27 and exponent 3. Then by assertion (ii) of Theorem 9.16 in [1] R has exact absolutely irreducible module W of dimension 3 over the field $GF(7)$ (existence of such module is proved in [9]). Let $Y = [W]R$. We denote by A the group of automorphisms of the group R. Let $B = C_A(Z(R))$, Q be the subgroup of quaternions of the group B and $X = Z(Q)Y$. Then we denote by $\mathfrak{M}$ the class $(G \mid O^{2'}(G/O_{\{2,3\}}(G)) \in S_n D_o(X))$, where $D_o(X)$ is the class of all finite direct products of isomorphic copies of the group X. We introduce the notation $\mathfrak{B} = \mathfrak{M} \cap \mathfrak{S}_7\mathfrak{S}_3\mathfrak{S}_2$. By Theorem 4.5 in [9], the class $\mathfrak{B}$ is a

Lockett class and is not a Fisher class. Moreover, $\mathfrak{B}$ possesses the property given in the following assertion.

Lemma 12 (Theorem 4.4 in [9]). *If $\mathfrak{S}_*$ is the minimal normal Fitting class, then $X \in (\mathfrak{B} \cap \mathfrak{S}_*) \setminus \mathfrak{B}_*$.*

Lemma 13. *If $\pi(X) \subseteq \sigma \subset P$, then the Fitting class $R_\sigma(\mathfrak{B}_*)$ is not a Lockett class.*

Proof. We suppose that $(R_\sigma(\mathfrak{B}_*))^* = R_\sigma(\mathfrak{B}_*)$. Since by assertion 1 of Lemma 3 $(\mathfrak{B}_*)^* = \mathfrak{B}^*$, making use of Lemma 9 and assertion 4 of Lemma 3, we obtain the equality $(R_\sigma(\mathfrak{B}_*))^* = (\mathfrak{B}_*\mathfrak{S}_{\sigma'})^* = (\mathfrak{B}_*)^*\mathfrak{S}_{\sigma'} = \mathfrak{B}^*\mathfrak{S}_{\sigma'} = R_\sigma(\mathfrak{B}^*)$. By Lemma 12, $\mathfrak{B}$ is a Lockett class. Therefore, $R_\sigma(\mathfrak{B}_*) = R_\sigma(\mathfrak{B})$. By virtue of Lemma 12, $X \in \mathfrak{B}$, therefore, $X = X_\sigma = X_{\mathfrak{B}}$. Thus, $X \in R_\sigma(\mathfrak{B}) = R_\sigma(\mathfrak{B}_*)$. But then $X = X_\sigma \subseteq X_{\mathfrak{B}_*} \subseteq X$. Hence it follows that $X = X_{\mathfrak{B}_*}$ and $X \in \mathfrak{B}_*$. We obtain a contradiction. Therefore, $R_\sigma(\mathfrak{B}_*)$ is not a Lockett class. The lemma is proved.

Theorem 1. *Let $\varnothing \neq \sigma \subset P$ and $|\sigma'| \geq 2$. If $\mathfrak{F}$ and $\mathfrak{H} = (\mathfrak{S}_{\sigma'})_*\mathfrak{S}_\sigma$ are Fitting classes, then the product $\mathfrak{F}\mathfrak{H}$ is local and each of the factors $\mathfrak{F}$ and $\mathfrak{H}$ is nonlocal, the product is defined semilocally and is not a formation for each following values of $\mathfrak{F}$:*

(1) *$\mathfrak{F} = (\mathfrak{S}_\sigma)_*\mathfrak{S}_{\sigma'}$ for $|\sigma| \geq 2$;*

(2) *$\mathfrak{F} = R_\sigma(\mathfrak{B}_*)$ for $\sigma \supseteq \pi(X)$.*

Proof. Let $\mathfrak{F} = (\mathfrak{S}_\sigma)_*\mathfrak{S}_{\sigma'}$, where the sets of prime numbers σ and σ' are such that $|\sigma| \geq 2$ and $|\sigma'| \geq 2$. Since $\mathfrak{F}\mathfrak{S}_{\sigma'} = \mathfrak{F}$ and $\mathfrak{H}\mathfrak{S}_\sigma = \mathfrak{H}$, by Lemma 8 the classes $\mathfrak{F}$ and $\mathfrak{H}$ are defined semilocally. Taking into account Lemma 10, we conclude that the classes $\mathfrak{F}$ and $\mathfrak{H}$ are not Lockett classes. Therefore, $\mathfrak{F}$ and $\mathfrak{H}$ by virtue of assertion 3 of Lemma 3 are not formations.

Taking into account assertion 1 of Lemma 3, we see that $\mathfrak{F}\mathfrak{H} = (\mathfrak{S}_\sigma)_*\mathfrak{S}_{\sigma'}(\mathfrak{S}_{\sigma'})_*\mathfrak{S}_\sigma = (\mathfrak{S}_\sigma)_*\mathfrak{S}_{\sigma'}\mathfrak{S}_\sigma$. Let f be an H-function such that

$$f(p) = \begin{cases} (\mathfrak{S}_\sigma)_*\mathfrak{S}_{\sigma'} & \text{if } p \in \sigma', \\ (\mathfrak{S}_\sigma)_*\mathfrak{S}_{\sigma'}\mathfrak{S}_\sigma & \text{if } p \in \sigma. \end{cases}$$

Let us show that $\mathfrak{F}\mathfrak{H} = LR(f)$. Indeed,

$$LR(f) = \left(\bigcap_{p\in\sigma'} (\mathfrak{S}_\sigma)_*\mathfrak{S}_{\sigma'}\mathfrak{N}_p\mathfrak{S}_{p'}\right) \cap \left(\bigcap_{p\in\sigma} (\mathfrak{S}_\sigma)_*\mathfrak{S}_{\sigma'}\mathfrak{S}_\sigma\mathfrak{N}_p\mathfrak{S}_{p'}\right).$$

Since $\mathfrak{S}_{\sigma'}\mathfrak{N}_p = \mathfrak{S}_{\sigma'}$ for all $p \in \sigma'$ and $\mathfrak{S}_\sigma\mathfrak{N}_p = \mathfrak{S}_\sigma$ for all $p \in \sigma$, we see that

$$LR(f) = \left(\bigcap_{p\in\sigma'} (\mathfrak{S}_\sigma)_*\mathfrak{S}_{\sigma'}\mathfrak{S}_{p'}\right) \cap \left(\bigcap_{p\in\sigma} (\mathfrak{S}_\sigma)_*\mathfrak{S}_{\sigma'}\mathfrak{S}_\sigma\mathfrak{S}_{p'}\right).$$

Using assertion 3 of lemma 1, we obtain

$$LR(f) = (\mathfrak{S}_\sigma)_* \mathfrak{S}'_\sigma \left(\bigcap_{p \in \sigma'} \mathfrak{S}_{p'} \right) \cap \left((\mathfrak{S}_\sigma)_* \mathfrak{S}_{\sigma'} \mathfrak{S}_\sigma \left(\bigcap_{p \in \sigma} \mathfrak{S}_{p'} \right) \right).$$

Taking into account that

$$\bigcap_{p \in \sigma'} \mathfrak{S}_{p'} = \mathfrak{S}_\sigma, \qquad \bigcap_{p \in \sigma} \mathfrak{S}_{p'} = \mathfrak{S}_{\sigma'},$$

we obtain

$$LR(f) = (\mathfrak{S}_\sigma)_* \mathfrak{S}_{\sigma'} \mathfrak{S}_\sigma \cap (\mathfrak{S}_\sigma)_* \mathfrak{S}_{\sigma'} \mathfrak{S}_\sigma \mathfrak{S}_{\sigma'} = (\mathfrak{S}_\sigma)_* \mathfrak{S}_{\sigma'} \mathfrak{S}_\sigma = \mathfrak{F}\mathfrak{H}.$$

Thus, in case 1 $\mathfrak{F}\mathfrak{H}$ is a local product of Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$.

In the case where $\mathfrak{F} = R_\sigma(\mathfrak{B}_*)$, the theorem can be proved similarly with the use of Lemmas 8, 9, 11, and 12.

The theorem is proved.

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