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Problems related to the Lockett Conjecture on Fitting classes of finite groups ☆

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ABSTRACT

The existence of a solvable non-normal Fitting class $\mathcal{F}$ which is not a Lockett class but for which the Lockett Conjecture still holds are studied. We also prove that there exists an ω-local Fitting class $\mathcal{F}$ which does not satisfy the Lockett conjecture but the Lockett conjecture still holds under a given condition. As a consequence of our result, a generalized version of the Lausch's problem in the well-known Kourovka Notebook is answered.

1 INTRODUCTION

All groups in this paper are finite. The reader is assumed to familiar with the theory of formations and Fitting classes.

Let $\mathcal{F}$ be a Fitting class. If $\mathcal{F}$ is non-empty, then it is clear that every group G has a unique $\mathcal{F}$-maximal normal subgroup $G_{\mathcal{F}}$ which is called the $\mathcal{F}$-radical of G.

A Fitting class $\mathcal{F}$ is called a radical homomorph if $\mathcal{F}$ is closed under homomorphic images. A radical homomorph $\mathcal{H}$ is called saturated if $G \in \mathcal{H}$ whenever $G/\Phi(G) \in \mathcal{H}$. If a saturated radical homomorph $\mathcal{H}$ forms a formation, then we

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call $\mathcal{H}$ a saturated Fitting formation. A non-empty Fitting class $\mathcal{F}$ is said to be normal if the $\mathcal{F}$-radical $G_{\mathcal{F}}$ is a $\mathcal{F}$-maximal subgroup of G for every group G (see [7] and [8]).

It is well known that many problems related to Fitting classes can be studied by using the operators "*" and "$_*$" defined by Lockett [9]. Actually every non-empty Fitting class $\mathcal{F}$ can be compared with the Fitting classes $\mathcal{F}^*$ and $\mathcal{F}_*$, where $\mathcal{F}^*$ is the smallest Fitting class containing $\mathcal{F}$ such that the $\mathcal{F}^*$-radical of the direct product $G \times H$ of any two groups G and H is equal to the direct product of the $\mathcal{F}^*$-radical of G and the $\mathcal{F}^*$-radical of H. Finally $\mathcal{F}_*$ is the intersection of all Fitting classes $\mathcal{X}$ such that $\mathcal{X}^* = \mathcal{F}^*$ (cf. [5, Ch. X], [9]). In view of this fact, we call a Fitting class $\mathcal{F}$ a *Lockett class* if $\mathcal{F} = \mathcal{F}^*$.

It was proved by Lockett [9] that in the class of all solvable groups, the following inclusions hold for any Fitting class $\mathcal{F}$:

$$\mathcal{F}_* \subseteq \mathcal{F} \subseteq \mathcal{F}^* \quad \text{and} \quad \mathcal{F}_* \subseteq \mathcal{F}^* \cap \mathcal{X} \subseteq \mathcal{F}^*,$$

where $\mathcal{X}$ is some normal Fitting class. In connection with the inclusions, Lockett proposed the following conjecture [9].

Lockett Conjecture [9, p. 135]. In the class $\mathcal{S}$ of all solvable groups, Lockett conjectured that for a Fitting class $\mathcal{F}$, there exists a normal Fitting class $\mathcal{X}$ such that $\mathcal{F} = \mathcal{F}^* \cap \mathcal{X}$.

For a Fitting class $\mathcal{F}$, if there exists a normal Fitting class $\mathcal{X}$ such that $\mathcal{F} = \mathcal{F}^* \cap \mathcal{X}$, then the Fitting class $\mathcal{F}$ is said to satisfy the Lockett conjecture.

We denote by ω (or π) a non-empty set of primes; by $\mathcal{S}$ the class of all solvable groups; by $\mathcal{S}_\pi$ the class of all solvable π-groups; by $\mathcal{E}$ the class of all groups; by $\mathcal{E}_\pi$ the class of all π-groups and by (1) the class of all identity groups.

Let f be a function which maps: $\mathcal{P} \to \{\text{Fitting classes}\}$. The support of f is defined by $\mathrm{Supp}(f) = \{p \in \mathcal{P} \mid f(p) \neq \phi\}$. Let $\sigma = \mathrm{Supp}(f)$ and $LR(f) = \mathcal{E}_\sigma \cap (\bigcap_{p \in \sigma} f(p)\mathcal{N}_p\mathcal{E}_{p'})$. A Fitting class $\mathcal{F}$ is said to be local if there exists a function f such that $\mathcal{F} = LR(f)$. In this case, f is called the local function of the local Fitting class $\mathcal{F}$ (cf. [10]). Moreover, if $f(p) \subseteq \mathcal{F}$, for all primes p, then f is called an inner (or integrated) local function of $\mathcal{F}$.

It is clear that every solvable normal Fitting class $\mathcal{F}$ satisfies the Lockett conjecture since by [5, X; 3.7], every non-identity Fitting class $\mathcal{F}$ is normal if and only if $\mathcal{F}^* = \mathcal{S}$. It is known that the Lockett conjecture holds for all solvable local hereditary Fitting classes and that every solvable Fitting class $\mathcal{F}$ satisfies the Lockett conjecture if and only if $\mathcal{F}_* = \mathcal{F}^* \cap \mathcal{S}_*$ (see [4]). In 1978, Berger [2] used another method to prove that the Lockett conjecture holds for the same family of Fitting classes. In 1979, Beidlemen and Hauck [1] proved that the Lockett conjecture holds for the Fitting classes $\mathcal{X}\mathcal{N}$ and $\mathcal{X}\mathcal{S}_\pi\mathcal{S}_{\pi'}$, where $\mathcal{X}$ is an arbitrarily non-empty Fitting class. As a further development, Vorob'ev [13] further verified that the Lockett conjecture holds for any solvable local Fitting class. Later on, Gallego [6] extended the Lockett conjecture to the class $\mathcal{E}$ of all finite groups and he proved that if $\mathcal{F}$ is a local Fitting class with $\mathcal{F} \subseteq \mathcal{E}$, then $\mathcal{F}_* = \mathcal{F}^* \cap \mathcal{E}_*$.

A counterexample in [3] shows that there exists a solvable non-local Fitting class (which is a Lockett class) for which the Lockett conjecture does not hold. Therefore, one naturally asks whether we can describe the Fitting classes that satisfies the Leckett conjecture?

In this connection, the following two problems naturally arise.

Problem 1. Does there exist a soluble non-normal Fitting class which is not an Lockett class but for which the Lockett conjecture holds?

Problem 2. For every partly local Fitting class $\mathcal{F}$ (namely, the ω-local Fitting class defined by Shemetkov and Skiba in [10]), does the generalized version of the Lockett conjecture hold (that is, $\mathcal{F}_* = \mathcal{F}^* \cap \mathcal{E}_*$)?

The main purpose of this paper is to give answers to the above two problems (see Corollary 2.6, Theorem B and Example 3.5). As one of the corollaries of our results, a generalized version of Lausch's problem in the well-known Kourovka Notebook [11, Problem 8.30] is answered.

All unexplained notations and terminologies are standard. The reader is referred to Doerk and Hawkes [5] if necessary.

2. THE ANSWER TO PROBLEM 1

We first cite some properties of the operators "$_*$" and "*".

Lemma 2.1 ([9] and [5, Ch.X]). *Let $\mathcal{F}$ and $\mathcal{H}$ be two Fitting classes. Then:*

(a) *If $\mathcal{F} \subseteq \mathcal{H}$, then $\mathcal{F}^* \subseteq \mathcal{H}^*$ and $\mathcal{F}_* \subseteq \mathcal{H}_*$;*
(b) $(\mathcal{F}_*)_* = \mathcal{F}_* = (\mathcal{F}^*)_* \subseteq \mathcal{F} \subseteq \mathcal{F}^* = (\mathcal{F}_*)^* = (\mathcal{F}^*)^*$;
(c) *$\mathcal{F}^* \subseteq \mathcal{F}_*\mathcal{A}$, where $\mathcal{A}$ is the class of all Abelian groups;*
(d) *If $\{\mathcal{F}_i \mid i \in I\}$ is a set of Fitting classes, then $(\bigcap_{i\in I} \mathcal{F}_i)^* = \bigcap_{i\in I} \mathcal{F}_i^*$.*
(e) *If $\mathcal{H}$ is a saturated radical homomorph, then $(\mathcal{F}\mathcal{H})^* = \mathcal{F}^*\mathcal{H}$.*
(f) *If $\mathcal{F}$ is a homomorph (in particular, a formation), then $\mathcal{F}$ is a Lockett class.*

If G is a group and $\mathcal{X}$ is a class of groups, then we write $\sigma(G) = \{p \mid p \in \mathcal{P}$ and $p \| G|\}$, $\sigma(\mathcal{X}) = \bigcup\{\sigma(G) \mid G \in \mathcal{X}\}$ and $\mathrm{Char}(\mathcal{X}) = \{p \mid p \in \mathcal{P}$ and $Z_p \in \mathcal{X}\}$, where $\mathcal{P}$ is the set of all primes. $\mathrm{Char}(\mathcal{X})$ is called the characteristic of $\mathcal{X}$ (see [5]).

Lemma 2.2 ([5, X]). (i) $\mathrm{Char}(\mathcal{F}^*) = \mathrm{Char}(\mathcal{F})$, *for every Fitting class $\mathcal{F}$;*
(ii) *If $\mathcal{F}$ is a solvable Fitting class, then:* (a) $\sigma(\mathcal{F}) = \mathrm{Char}(\mathcal{F})$*;* (b) $p \in \mathrm{Char}(\mathcal{F})$ *if and only if $\mathcal{N}_p \subseteq \mathcal{F}$.*

Lemma 2.3. *Let $\mathcal{F}$, $\mathcal{X}$ and $\mathcal{Y}$ be Fitting classes. Then $\mathcal{F}(\mathcal{X} \cap \mathcal{Y}) = \mathcal{F}\mathcal{X} \cap \mathcal{F}\mathcal{Y}$.*

Proof. The proof is obvious and we omit the details. □

Following [10], an ωd-group is a group whose order is divisible by at least one element in ω. Let $\mathcal{E}_{\omega d}$ be the class of groups whose every composition factor is an ωd-group. Let $\emptyset \neq \omega \subseteq \mathcal{P}$. Then, we call a function $f : \omega \cup \{\omega'\} \rightarrow$ {Fitting class} an ω-local function. Write $LR_\omega(f) = \{G \mid G^{\omega d} \in f(\omega')$ and $F^p(G) \in f(p)$, for all $p \in \omega \cap \pi(G)\}$, where $G^{\omega d}$ is the $\mathcal{E}_{\omega d}$-residual of G, $F^p(G)$ is the $\mathcal{N}_p\mathcal{E}_{p'}$-residual of G. Now, we call the Fitting class $\mathcal{F}$ an ω-local Fitting class if there exists an ω-local function f such that $\mathcal{F} = LR_\omega(f)$. Note that if $\omega = \mathcal{P}$, then an ω-local Fitting class is just a local Fitting class. Clearly, the classes $\mathcal{E}_\pi, \mathcal{S}_\pi, \mathcal{N}_\pi, (1)$ are local Fitting classes.

For a non-empty formation $\mathcal{F}$, we denote by $G^{\mathcal{F}}$ the $\mathcal{F}$-residual of G. Let $\mathcal{X}, \mathcal{Y}$ be two sets of groups. We denote by Fit($\mathcal{X}$) the Fitting class generated by $\mathcal{X}$ and write $\mathcal{X} \vee \mathcal{Y} = \mathrm{Fit}(\mathcal{X} \cup \mathcal{Y})$.

Lemma 2.4 [10, Theorem 9]. *Let $\mathcal{F}$ be a Fitting class. Then the following statements are equivalent.*

(a) *$\mathcal{F}(F^p)\mathcal{N}_p \subseteq \mathcal{F}$, for all $p \in \omega$, where*

$$\mathcal{F}(F^p) = \begin{cases} \mathrm{Fit}\big(F^p(G)\big) \mid G \in \mathcal{F}, & \text{if } p \in \sigma(\mathcal{F}); \\ \emptyset, & \text{if } p \in \sigma'(\mathcal{F}). \end{cases}$$

(b) *$\mathcal{F} = LR_\omega(f)$, where $f(\omega') = \mathcal{F}$ and $f(p) = \mathcal{F}(F^p)\mathcal{N}_p$, for all $p \in \omega$;*

(c) *$\mathcal{F}$ is an ω-local Fitting class.*

Lemma 2.5. *If $\mathcal{F}$ and $\mathcal{H}$ are Fitting classes, then $(\mathcal{F}_* \cap \mathcal{H}_*)_* = (\mathcal{F} \cap \mathcal{H})_*$.*

Proof. By Lemma 2.1 (b), $\mathcal{F}_* \subseteq \mathcal{F}$ and $\mathcal{H}_* \subseteq \mathcal{H}$. Hence, $\mathcal{F}_* \cap \mathcal{H}_* \subseteq \mathcal{F} \cap \mathcal{H}$. Now, by Lemma 2.1(a), $(\mathcal{F}_* \cap \mathcal{H}_*)_* \subseteq (\mathcal{F} \cap \mathcal{H})_*$. On the other hand, by Lemma 2.1, $(\mathcal{F} \cap \mathcal{H})_* \subseteq \mathcal{F}_*$ and $(\mathcal{F} \cap \mathcal{H})_* \subseteq \mathcal{H}_*$. Hence, by Lemma 2.1 again,

$$(\mathcal{F} \cap \mathcal{H})_* = \big((\mathcal{F} \cap \mathcal{H})_*\big)_* \subseteq (\mathcal{F}_* \cap \mathcal{H}_*)_*.$$

Thus, $(\mathcal{F}_* \cap \mathcal{H}_*)_* = (\mathcal{F} \cap \mathcal{H})_*$. □

The following theorem gives a positive answer to Problem 1.

Theorem A. *Let $\mathcal{F}, \mathcal{H}$ be soluble non-normal Fitting class such that $\mathcal{F}$ and $\mathcal{F}^*\mathcal{H}$ satisfy the Lockett conjecture. If $\mathcal{F} \cap \mathcal{H} = (1)$ and $\mathcal{H}$ is a saturated formation, then*

(a) *$\mathcal{F}_*\mathcal{H}$ satisfies the Lockett conjecture;*

(b) *$\mathcal{F}_*\mathcal{H}$ is an non-Lockett class (in particular, $\mathcal{F}_*\mathcal{H}$ is a non-local Fitting class) whenever $\mathcal{N}_p\mathcal{N}_q \subseteq \mathcal{F}^*$, for some distinct primes p, q.*

Proof. (a) Since $\mathcal{F}^*\mathcal{H}$ satisfies the Lockett conjecture, $(\mathcal{F}^*\mathcal{H})_* = (\mathcal{F}^*\mathcal{H})^* \cap \mathcal{S}_*$. However, since $\mathcal{H}$ is a saturated radical homomorph, by Lemma 2.1(e), $(\mathcal{F}^*\mathcal{H})^* = (\mathcal{F}^*)^*\mathcal{H}$. Moreover, by Lemma 2.1(b), $(\mathcal{F}^*)^* = \mathcal{F}^*$. Hence,

(2.1) $\qquad (\mathcal{F}^*\mathcal{H})_* = \mathcal{F}^*\mathcal{H} \cap \mathcal{S}_*.$

Now, by Lemma 2.1(b) and Lemma 2.5,

$$\begin{aligned}(\mathcal{F}^*\mathcal{H})_* &= \big((\mathcal{F}^*\mathcal{H})_*\big)_* = \big((\mathcal{F}^*\mathcal{H}) \cap \mathcal{S}_*\big)_* \\ &= \big((\mathcal{F}^*\mathcal{H})_* \cap (\mathcal{S}_*\mathcal{H})_*\big)_* = (\mathcal{F}^*\mathcal{H} \cap \mathcal{S}_*\mathcal{H})_*.\end{aligned}$$

Since $\mathcal{H}$ is a Fitting formation, $\mathcal{F}^*\mathcal{H} \cap \mathcal{S}_*\mathcal{H} = (\mathcal{F}^* \cap \mathcal{S}_*)\mathcal{H}$ (cf. [13, Lemma 4]). Thus,

$$(2.2)\qquad (\mathcal{F}^*\mathcal{H})_* = \big((\mathcal{F}^* \cap \mathcal{S}_*)\mathcal{H}\big)_*.$$

Since $\mathcal{F}$ satisfies the Lockett conjecture, by (2.1) and (2.2), we obtain $(\mathcal{F}_*\mathcal{H})_* = \mathcal{F}^*\mathcal{H} \cap \mathcal{S}_*$. Now, by Lemma 2.1, $(\mathcal{F}_*\mathcal{H})^* = (\mathcal{F}_*)^*\mathcal{H} = \mathcal{F}^*\mathcal{H}$. Thus, $\mathcal{F}_*\mathcal{H}$ satisfies the Lockett conjecture.

(b) Assume that $\mathcal{F}_*\mathcal{H}$ is an Lockett class, this is, $\mathcal{F}_*\mathcal{H} = (\mathcal{F}_*\mathcal{H})^*$. Then, by Lemma 2.1, $\mathcal{F}_*\mathcal{H} = \mathcal{F}^*\mathcal{H}$. However, since $\mathcal{F}$ satisfies the Lockett conjecture and $\mathcal{H}$ is a formation, $\mathcal{F}^*\mathcal{H} \cap \mathcal{S}_*\mathcal{H} = (\mathcal{F}^* \cap \mathcal{S}_*)\mathcal{H} = \mathcal{F}^*\mathcal{H}$, and consequently $\mathcal{F}^* \subseteq \mathcal{F}^*\mathcal{H} \subseteq \mathcal{S}_*\mathcal{H}$. On the other hand, by Lemma 2.1(c) and (2.1), we know that if $G \in \mathcal{F}^*$, then $G/G_{\mathcal{S}_*} \in \mathcal{A} \cap \mathcal{S}_{\sigma(\mathcal{F}^*)}$. But, by Lemma 2.2, $\mathcal{A} \cap \mathcal{S}_{\sigma(\mathcal{F}^*)} = \mathcal{A} \cap \mathcal{S}_{\sigma(\mathcal{F})} \subseteq \mathcal{N}_{\sigma(\mathcal{F})} \subseteq \mathcal{F}$. Hence, $\mathcal{F}^* \subseteq \mathcal{S}_*\mathcal{F}$ and by Lemma 2.3,

$$\mathcal{F}^* \subseteq \mathcal{S}_*(\mathcal{F} \cap \mathcal{H}) = \mathcal{S}_*.$$

However, by our hypothesis, we have $\mathcal{N}_p\mathcal{N}_q \subseteq \mathcal{F}^*$. This contradicts [5, X; 5.32]. Thus, $\mathcal{F}_*\mathcal{H}$ is a non-Lockett class. Since every local Fitting class is a Lockett class (cf. [13, Lemma 5]), $\mathcal{F}_*\mathcal{H}$ is a non-local Fitting class. Thus, Theorem A is proved. □

Corollary 2.6. *For any set π of primes with $|\pi| \geqslant 2$, $(\mathcal{S}_\pi)_*\mathcal{S}_{\pi'}$ is a solvable non-normal Fitting class which is non-Lockett class but for which the Lockett conjecture holds.*

Proof. Clearly, $(\mathcal{S}_\pi)_*\mathcal{S}_{\pi'}$ is a solvable non-normal Fitting class. Since $\mathcal{S}_\pi$ is a local Fitting class, by [13, Lemma 5], $\mathcal{S}_\pi$ is a Lockett class and so $\mathcal{S}_\pi^* = \mathcal{S}_\pi$. Since the product of two local Fitting classes $\mathcal{S}_\pi$ and $\mathcal{S}_{\pi'}$ is still a local Fitting class (cf. [14, Lemma 4]), $\mathcal{S}_\pi^*\mathcal{S}_{\pi'} = \mathcal{S}_\pi\mathcal{S}_{\pi'}$ is a local Fitting class. By [13, Theorem], $\mathcal{S}_\pi^*\mathcal{S}_{\pi'}$ satisfies the Lockett conjecture. Since $|\pi| \geqslant 2$, there are distinct primes p, q such that $\mathcal{N}_p\mathcal{N}_q \subseteq \mathcal{S}_{\{p,q\}} \subseteq \mathcal{S}_\pi^* = \mathcal{S}_\pi$. This shows that all conditions of Theorem A are satisfied. Therefore, $(\mathcal{S}_\pi)_*\mathcal{S}_{\pi'}$ is non-Lockett class. □

3. THE ANSWER TO PROBLEM 2 AND SOME APPLICATIONS

In this section, we generalize the Lockett conjecture from the class $\mathcal{S}$ to the class $\mathcal{E}$. In the class $\mathcal{E}$, we say a Fitting class $\mathcal{F}$ satisfies the Lockett conjecture if $\mathcal{F}_* = \mathcal{F}^* \cap \mathcal{E}_*$. We now prove the following theorem.

Theorem B. *Let $\mathcal{F}$ be a ω-local Fitting class. If* $\operatorname{Char}(\mathcal{F}) \subseteq \omega$, *then $\mathcal{F}$ satisfies the Lockett conjecture.*

Proof. Since $\mathcal{F}$ is ω-local, by Lemma 2.4, $\mathcal{F}(F^p) \subseteq \mathcal{F}$, for all $p \in \omega$. Consequently, $\mathcal{F}(F^p) \subseteq \mathcal{F}$, for all $p \in \mathrm{Char}(\mathcal{F})$. By [10], the ω-local Fitting class $\mathcal{F}$ can be defined as follows:

$$(3.1) \qquad \mathcal{F} = \left(\bigcap_{p\in\pi_2} \mathcal{E}_{p'}\right) \cap \left(\bigcap_{p\in\pi_1} f(p)\mathcal{N}_p\mathcal{E}_{p'}\right) \cap f(\omega')\mathcal{E}_{\omega d},$$

where f is an ω-local function of $\mathcal{F}$, $\pi_1 = \omega \cap \mathrm{Supp}(f)$ and $\pi_2 = \omega \setminus \pi_1$. It follows that $\mathcal{F} \subseteq f(p)\mathcal{N}_p\mathcal{E}_{p'}$, for every $p \in \pi_1$. Now, by Lemma 2.4(b), $\mathcal{F} \subseteq \mathcal{F}(F^p)\mathcal{N}_p\mathcal{E}_{p'}$, for all $p \in \mathrm{Supp}(f) \cap \omega$. Since $\mathrm{Char}(\mathcal{F}) \subseteq \omega$, for every $p \in \mathrm{Char}(\mathcal{F})$, the following inclusion holds:

$$\mathcal{F}(F^p)\mathcal{N}_p \subseteq \mathcal{F} \subseteq \mathcal{F}(F^p)\mathcal{N}_p\mathcal{E}_{p'}.$$

Thus, by [6, Theorem 4.8(c)], $\mathcal{F}$ is a Lockett class. By a result in [12], $\mathcal{F}$ can be defined by a largest inner ω-local function F with $F(p)\mathcal{N}_p = F(p) \subseteq \mathcal{F}$, for all $p \in \omega$. This shows that for all $p \in \omega$,

$$(3.2) \qquad F(p)\mathcal{N}_p \subseteq \mathcal{F} \subseteq F(p)\mathcal{E}_{p'} = F(p)\mathcal{N}_p\mathcal{E}_{p'}.$$

Now, we claim that $\mathcal{F} \subseteq \mathcal{F}_*\mathcal{E}_{p'}\mathcal{N}_{p'}$. In fact, since $\mathcal{E}_{p'}\mathcal{N}_p$ is a saturated Fitting formation, $(\mathcal{F}_*\mathcal{E}_{p'}\mathcal{N}_p)^* = (\mathcal{F}_*)^*\mathcal{E}_{p'}\mathcal{N}_p$ by Lemma 2.1(e). However, by Lemma 2.1(b), $(\mathcal{F}_*)^* = \mathcal{F}^*$. Hence, by [6, Theorem 4.8(c)], $(\mathcal{F}_*\mathcal{E}_{p'}\mathcal{N}_p)^* = \mathcal{F}\mathcal{E}_{p'}\mathcal{N}_p$. On the other hand, $\mathcal{F}_*\mathcal{E}_{p'}\mathcal{N}_p$ is a local Fitting class (cf. [13, Corollary 1]), hence it is also a Lockett class by [13, Lemma 5]. Thus, $\mathcal{F}_*\mathcal{E}_{p'}\mathcal{N}_p = \mathcal{F}\mathcal{E}_{p'}\mathcal{N}_p$ and so $\mathcal{F} \subseteq \mathcal{F}_*\mathcal{E}_{p'}\mathcal{N}_p$. Hence our claim holds. This implies that for all $G \in \mathcal{F}$, $G/G_{\mathcal{F}_*\mathcal{E}_{p'}} \in \mathcal{N}_p$. Moreover, by (3.2), $G/G_{F(p)} \in \mathcal{E}_{p'}$, for all $G \in \mathcal{F}$. Obviously, if $G \in \mathcal{F}$, then $G_{\mathcal{F}_*\mathcal{E}_{p'}} \in \mathcal{F}$ and so $G_{\mathcal{F}_*\mathcal{E}_{p'}} = G_{\mathcal{F}_*\mathcal{E}_{p'}\cap\mathcal{F}}$. Thus,

$$(G/G_{\mathcal{F}_*\mathcal{E}_{p'}})/(G_{\mathcal{F}_*\mathcal{E}_{p'}\cap\mathcal{F}}G_{F(p)}/G_{\mathcal{F}_*\mathcal{E}_{p'}}) \simeq G/G_{\mathcal{F}_*\mathcal{E}_{p'}\cap\mathcal{F}}G_{F(p)} \in \mathcal{N}_p$$

and

$$(G/G_{F(p)})/(G_{\mathcal{F}_*\mathcal{E}_{p'}\cap\mathcal{F}}G_{F(p)}/G_{F(p)}) \simeq G/G_{\mathcal{F}_*\mathcal{E}_{p'}\cap\mathcal{F}}G_{F(p)} \in \mathcal{E}_{p'}.$$

This induces that $G = G_{\mathcal{F}_*\mathcal{E}_{p'}\cap\mathcal{F}}G_{F(p)}$. Obviously, $G_{\mathcal{F}_*\mathcal{E}_{p'}\cap\mathcal{F}} \in (\mathcal{F}_*\mathcal{E}_{p'} \cap \mathcal{F}) \vee F(p)$ and $G_{F(p)} \in (\mathcal{F}_*\mathcal{E}_{p'} \cap \mathcal{F}) \vee F(p)$. Therefore, $G \in (\mathcal{F}_*\mathcal{E}_{p'} \cap \mathcal{F}) \vee F(p)$ and so $\mathcal{F} \subseteq (\mathcal{F}_*\mathcal{E}_{p'} \cap \mathcal{F}) \vee F(p)$. The reverse inclusion is obvious. Thus, $\mathcal{F} = (\mathcal{F}_*\mathcal{E}_{p'} \cap \mathcal{F}) \vee F(p)$. It follows from [6, Theorem 4.11] that

$$(3.3) \qquad \mathcal{F} \cap \mathcal{E}_* \subseteq \mathcal{F}_*\mathcal{E}_{p'}.$$

We now prove that $\mathcal{F} \cap \mathcal{E}_* = \mathcal{F}_*$. In fact, by Lemma 2.1, $\mathcal{F}_* \subseteq \mathcal{E}_*$ and $\mathcal{F}_* \subseteq \mathcal{F}$, and consequently, $\mathcal{F}_* \subseteq \mathcal{F} \cap \mathcal{E}_*$. Assume that $\mathcal{F} \cap \mathcal{E}_* \nsubseteq \mathcal{F}_*$ and let G be a group in $(\mathcal{F} \cap \mathcal{E}_*) \setminus \mathcal{F}_*$ of minimal order. Then, G has a unique maximal normal subgroup M such that $M = G_{\mathcal{F}_*}$. Since $G \in \mathcal{F}$, by Lemma 2.1(c), $G/M \in \mathcal{A}$ and thereby, G/M is a composition factor of order p. Hence, by [5, IX; 1.7], $p \in \mathrm{Char}(\mathcal{F})$. However,

$G/M \in \mathcal{E}_{p'}$ by (3.3), thus, $G/M \in \mathcal{N}_p \cap \mathcal{E}_{p'} = (1)$ and so $G = M$. This contradiction shows that $\mathcal{F} \cap \mathcal{E}_* \subseteq \mathcal{F}_*$. Thus $\mathcal{F}_* = \mathcal{F} \cap \mathcal{E}_*$. Now, by [6, Theorem 4.8(c)] and (3.2), we have $\mathcal{F} = \mathcal{F}^*$. This induces that $\mathcal{F}_* = \mathcal{F}^* \cap \mathcal{E}_*$ and hence $\mathcal{F}$ satisfies Lockett conjecture. This completes the proof. □

If we let $\omega = \mathcal{P}$, then, by Theorem B, we immediately re-obtain the following result of Gallego [6].

Corollary 3.1 [6]. *Every local Fitting class satisfies Lockett conjecture.*

If $\mathcal{F} = \mathcal{S}$ and $\omega = \mathcal{P}$, then by Theorem B, we re-obtain the following Berger's result which answers the Laue's problem positively (see [7, Problem II]).

Corollary 3.2 ([2] and [5, X; 6.15]). $\mathcal{S}_* = \mathcal{S}^* \cap \mathcal{E}_* = \mathcal{S} \cap \mathcal{E}_*$.

In the Kourovka Notebook [11], Lausch proposed the following problem:

The Lausch's Problem [11, Problem 8.30]. Let $\mathcal{F}, \mathcal{H}$ be a solvable Fitting class such that $\mathcal{F} \cap \mathcal{S}_* = \mathcal{F}_*$ and $\mathcal{H} \cap \mathcal{S}_* = \mathcal{H}_*$. Will the equality $(\mathcal{F} \cap \mathcal{H}) \cap \mathcal{S}_* = (\mathcal{F} \cap \mathcal{H})_*$ hold?

For solvable local Fitting classes $\mathcal{F}, \mathcal{H}$, Vorob'ev in [13] has already given an affirmative answer to the Lausch's problem. Our following corollary also gives an affirmative answer to the Lausch's problem in the class $\mathcal{E}$ of all finite groups (in particular, in $\mathcal{S}$) when $\mathcal{F}, \mathcal{H}$ are ω-local Fitting classes and $\mathrm{Char}(\mathcal{F} \cap \mathcal{H}) \subseteq \omega$ (in particular, when $\mathcal{F}, \mathcal{H}$ are local Fitting classes).

Corollary 3.3. *Let $\mathcal{F}, \mathcal{H}$ be two ω-local Fitting classes satisfying the Lockett conjecture and suppose their characteristics are subsets of ω. Then $\mathcal{F} \cap \mathcal{H}$ satisfies the Lockett conjecture and $(\mathcal{F} \cap \mathcal{H})_* = (\mathcal{F} \cap \mathcal{H}) \cap \mathcal{E}_*$.*

Proof. Obviously, $\mathcal{F} \cap \mathcal{H}$ is a ω-local Fitting class and $\mathrm{Char}(\mathcal{F} \cap \mathcal{H}) \subseteq \omega$. By Theorem B, $\mathcal{F} \cap \mathcal{H}$ satisfies the Lockett conjecture. By using the same arguments as in Theorem B, we also see that $\mathcal{F} \cap \mathcal{H}$ is a Lockett class, that is, $(\mathcal{F} \cap \mathcal{H})^* = \mathcal{F} \cap \mathcal{H}$. Therefore, $(\mathcal{F} \cap \mathcal{H})_* = (\mathcal{F} \cap \mathcal{H}) \cap \mathcal{E}_*$. □

The following example shows that there exists an ω-local Fitting class $\mathcal{F}$ such that $\mathrm{Char}(\mathcal{F}) \subseteq \omega$ but $\mathcal{F}$ is not a local Fitting class.

Example 3.4. Let $\mathcal{F} = \mathcal{H}\mathcal{N}_p$, where $\mathcal{H} = \mathrm{Fit}\, A$ is a Fitting class generated by some finite simple non-Abelian group A. Then, $\mathrm{Char}(\mathcal{H}) = \emptyset \subset \sigma(\mathcal{H})$ (see [5, Exercise IX, §1.4]). We claim that $\mathcal{F}$ is ω-local, for $\omega = \{p\}$. Indeed, let

$$\mathcal{F}(F^p) = \begin{cases} \mathrm{Fit}\big(F^p(G)\big) \mid G \in \mathcal{F}, & \text{if } p \in \sigma(\mathcal{F}); \\ \emptyset, & \text{if } p \in \sigma'(\mathcal{F}). \end{cases}$$

Then $\mathcal{F}(F^p) \subseteq \mathcal{H}$ and hence $\mathcal{F}(F^p)\mathcal{N}_p \subseteq \mathcal{H}\mathcal{N}_p = \mathcal{F}$. This leads to $\mathcal{F}$ is ω-local and so our claim is established. Clearly, $\sigma(\mathcal{F}) = \sigma(\mathcal{H}) \cup \{p\}$ and $\text{Char}(\mathcal{F}) = \{p\}$. Hence, $\text{Char}(\mathcal{F}) \neq \sigma(\mathcal{F})$. However, by [6, 4.9(b)], $\mathcal{F}$ is local if and only if $\text{Char}(\mathcal{F}) = \sigma(\mathcal{F})$ and $\mathcal{F}(F^p)\mathcal{N}_p \subseteq \mathcal{F} \subseteq \mathcal{F}(F^p)\mathcal{N}_p\mathcal{E}_{p'}$. This shows that $\mathcal{F}$ is not a local Fitting class.

In closing, we give an example to show that Theorem B is not true in general without the condition "$\text{Char}(\mathcal{F}) \subseteq \omega$".

Example 3.5. Let R be an extra-special (cf. [5, A; 20.3]) group of order 27 with exponent 3 and $M = PSL(2,3)$. Then, by [5, B; 9.16], M has a faithful irreducible R-module W over the field $GF(7)$. Let $Y = [W]R$. Denote by A the automorphism group of R. Let $B = C_A(Z(R))$ and Q the quaternion subgroups of B and $X = Z(Q)Y$ respectively. Let

$$\mathcal{M} = \left(G \mid O^{2'}\left(G/O_{\{2,3\}}(G)\right) \in S_n D_0(X)\right),$$

where $D_0(X)$ is the class of all finite direct products of the groups which is isomorphic to X. By a result in [3], $\mathcal{K} = \mathcal{M} \cap \mathcal{S}_7\mathcal{S}_3\mathcal{S}_2$ is a Lockett class but for which Lockett conjecture does not hold, that is, $\mathcal{K}^* = \mathcal{K}$ and $\mathcal{K}_* \neq \mathcal{K}^* \cap \mathcal{S}_*$. Let $\mathcal{F} = \{\mathcal{F}_i \mid i \in I\}$ be the family of all solvable Fitting classes satisfying the following conditions:

(a) For every $i \in I$, the product $(\mathcal{K})_*\mathcal{F}_i$ is an ω_i-local Fitting class, for some $\emptyset \neq \omega_i \subseteq \mathcal{P}$;
(b) $\mathcal{F}_i \cap \mathcal{F}_j = (1)$, for all $i, j \in I$ with $i \neq j$;
(c) $\mathcal{F}_i$ is a radical saturated homomorph, for all $i \in I$.

Obviously, $\mathcal{F} \neq \emptyset$ (for example, we can take $\mathcal{F}_i = \mathcal{N}_{p_i}$, for all $p_i \in \mathcal{P}$). Assume that $\mathcal{K}_i = (\mathcal{K})_*\mathcal{F}_i$ satisfies the Lockett conjecture, for all $i \in I$. Then, for every $i \in I$, $((\mathcal{K})_*\mathcal{F}_i)_* = (\mathcal{K}_*\mathcal{F}_i)^* \cap \mathcal{S}_*$. Since $\mathcal{K}$ is a Lockett class, by Lemma 2.1, we see that $(\mathcal{K}_*\mathcal{F}_i)^* = (\mathcal{K}_*)^*\mathcal{F}_i = \mathcal{K}^*\mathcal{F}_i = \mathcal{K}\mathcal{F}_i$ and hence

$$\begin{aligned}\bigcap_{i\in I}(\mathcal{K}_*\mathcal{F}_i)_* &= \bigcap_{i\in I}(\mathcal{K}\mathcal{F}_i \cap \mathcal{S}_*)\\ &= \left(\bigcap_{i\in I}\mathcal{K}\mathcal{F}_i\right) \cap \mathcal{S}_* = \mathcal{K}\left(\bigcap_{i\in I}\mathcal{F}_i\right) \cap \mathcal{S}_* = \mathcal{K} \cap \mathcal{S}_*.\end{aligned}$$

By Lemma 2.1(b), $(\mathcal{K}_*\mathcal{F}_i)_* \subseteq \mathcal{K}_*\mathcal{F}_i$, for all $i \in I$. Hence, $\bigcap_{i\in I}(\mathcal{K}_*\mathcal{F}_i)_* \subseteq \mathcal{K}_*$. On the other hand, since $\mathcal{K}_* \subseteq \mathcal{K}_*\mathcal{F}_i$ for every $i \in I$, by Lemma 2.1 again, $\mathcal{K}_* = (\mathcal{K}_*)_* \subseteq \bigcap_{i\in I}(\mathcal{K}_*\mathcal{F}_i)_*$. Thus $\bigcap_{i\in I}(\mathcal{K}_*\mathcal{F}_i)_* = \mathcal{K}_*$. It follows that $\mathcal{K}_* = \mathcal{K} \cap \mathcal{S}_*$, a contradiction. Hence, there exists $i_0 \in I$ such that $\mathcal{K}_{i_0}$ is an ω_{i_0}-local Fitting class which does not satisfy the Lockett conjecture. By condition (b), it is clear that $\text{Char}(\mathcal{R}_{i_0}) \nsubseteq \omega_{i_0}$.

This example hence illustrates that the answer to Problem 2 is in general negative.

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