LOCALITY OF SOLVABLE SUBGROUP-CLOSED FITTING CLASSES

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Most of the known results in the theory of formations of finite groups are concerned with local formations (cf. monograph [1] and [2]). In this connection, Professor L. A. Shemetkov suggested that I study the dual problem – the problem of defining and investigating analogs of local formations in the theory of Fitting classes, that is, classes of finite groups that are closed under passage to normal subgroups and products of normal subgroups. The first attempts to define local solvable Fitting classes (though the definitions were different) were made by Hartley [3] and D'Arcy [4], who were studying properties of injectors. The present author, using Shemetkov's notion of a local group function [1], obtained the first results concerning the construction and investigation of local Fitting classes [5, 6]. It turned out that many familiar classes of groups are local Fitting classes. In particular, I have shown [5, 6] that the subgroup-closed classes of groups $\mathfrak{N}, \mathfrak{S}, \mathfrak{S}_{\pi}, \mathfrak{S}_{\pi}\mathfrak{S}_{\pi'}$ are local Fitting classes. In this paper a very large set of local Fitting classes will be indicated. To be precise:

<u>THEOREM.</u> Every nonempty solvable subgroup-closed Fitting class is local.

Let $\mathfrak{F}$ and $\mathfrak{H}$ be Fitting classes. Then their product $\mathfrak{F}\mathfrak{H}$ is the class of all groups G, such that, $G/G_{\mathfrak{F}} \in \mathfrak{H}$ where $G_{\mathfrak{F}}$ is the $\mathfrak{F}$-radical of G [2].

It is well known that the product of Fitting classes is associative [7].

If there exists a local group function f [1], such that, f(p) is a Fitting class for all primes p, we will say that $\mathfrak{F}$ is a local Fitting class [6] if $\mathfrak{F}=\mathfrak{G}_{\pi(\mathfrak{F})}\cap(\bigcap_{p\in\pi(\mathfrak{F})}f(p)\mathfrak{N}_p\mathfrak{G}_{p'})$. In that case we call f a radical function of $\mathfrak{F}$. If $\pi = \varnothing$, we put $\mathfrak{G}_{\varnothing}=\mathfrak{E}$. In all other notations and definitions we follow Shemetkov's monograph [1] and book [2]. In all cases we consider only finite groups, and in the theorem – finite solvable groups.

<u>LEMMA 1.</u> If $\mathfrak{F}$ is a Fitting class and $\{\mathfrak{H}_i \mid i\in I\}$ a set of Fitting classes, then

$$\bigcap_{i\in I}\mathfrak{F}\mathfrak{H}_i=\mathfrak{F}(\bigcap_{i\in I}\mathfrak{H}_i).$$

The proof is accomplished by a direct check.

We call a product of Fitting classes $\mathfrak{F}\mathfrak{H}$ local if $\mathfrak{F}\mathfrak{H}$ is a local Fitting class.

<u>LEMMA 2.</u> Let π, ω be sets of primes and $\mathfrak{F}=\mathfrak{G}_{\pi}$, $\mathfrak{H}=\mathfrak{G}_{\omega}$. Then the product of Fitting classes $\mathfrak{F}\mathfrak{H}$ is local, with a radical function f, such that,

$$f(p)=\begin{cases}\mathfrak{F}\mathfrak{H}, & \text{if} \quad p\in\omega,\\ \mathfrak{F}, & \text{if} \quad p\in\pi\setminus\omega.\end{cases}$$

In particular,

$$\mathfrak{F}\mathfrak{H}=\mathfrak{G}_{\pi\cup\omega}\cap\mathfrak{G}_{\pi}\mathfrak{G}_{\omega}\mathfrak{G}_{\omega'}\cap\mathfrak{G}_{\pi}\mathfrak{G}_{\pi\setminus\omega}\mathfrak{G}_{(\pi\setminus\omega)'}.$$

<u>Proof.</u> Let $\mathfrak{M}$ be a Fitting class with the above radical function f and let $\sigma = \pi \cup \omega$. Then $\mathfrak{M}=\mathfrak{G}_{\sigma}\cap(\bigcap_{p\in\sigma}f(p)\mathfrak{N}_p\mathfrak{G}_{p'})$. In view of the definition of f, we get

$$\mathfrak{M}=\mathfrak{G}_{\sigma}\cap(\bigcap_{p\in\omega}\mathfrak{G}_{\pi}\mathfrak{G}_{\omega}\mathfrak{G}_{p'})\cap(\bigcap_{p\in\pi\setminus\omega}\mathfrak{G}_{\pi}\mathfrak{G}_{p'}).$$

But then, by Lemma 1

$$\mathfrak{M}=\mathfrak{G}_{\sigma}\cap\mathfrak{G}_{\pi}\mathfrak{G}_{\omega}\mathfrak{G}_{\omega'}\cap\mathfrak{G}_{\pi}\mathfrak{G}_{\pi\setminus\omega}\mathfrak{G}_{(\pi\setminus\omega)'}=\mathfrak{G}_{\sigma}\cap\mathfrak{G}_{\pi}\mathfrak{G}_{\omega}(\mathfrak{G}_{\omega'}\cap\mathfrak{G}_{(\pi\setminus\omega)'}).$$

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Consequently, again applying Lemma 1, we obtain

$$\mathfrak{M}=\mathfrak{S}_\sigma\cap\mathfrak{S}_\pi\mathfrak{S}_\omega\mathfrak{S}_{\sigma'}=\mathfrak{S}_\pi\mathfrak{S}_\omega=\mathfrak{F}\mathfrak{H}.$$

This proves the lemma.

Following Shemetkov [1], we define a partial order on the set Ω of radical functions, as follows: $f_k \le f_\ell$ if and only if $f_k(p) \subseteq f_\ell(p)$ for all primes $p (f_k, f_\ell \in \Omega)$.

<u>LEMMA 3.</u> The intersection of any nonempty set of local Fitting classes is a local Fitting class.

<u>Proof.</u> Let $\mathfrak{F}=\bigcap_{i\in I}\mathfrak{F}_i$ where $\mathfrak{F}_i$ is a Fitting class with radical function f_i, $i \in I$. Construct the group function $f=\bigcap_{i\in I}f_i$. For any $i \in I$ and any group $G \neq 1$, we have $f_i(G) = \bigcap_{p\in\pi(G)}f_i(p)$; hence

$$f(G)=\bigcap_{i\in I}(\bigcap_p f_i(p))=\bigcap_p(\bigcap_{i\in I}f_i(p))=\bigcap_p f(p)$$

for all primes $p \in \pi(G)$. Consequently, f is a radical function. We claim that f locally determines $\mathfrak{F}$. Let $\pi = \pi(\mathfrak{F})$ and

$$\mathfrak{M}=\mathfrak{S}_\pi\cap(\bigcap_{p\in\pi}f(p)\mathfrak{N}_p\mathfrak{S}_{p'}).$$

Since $f \le f_i$ for any $i \in I$, it clearly follows that $\mathfrak{M}\subseteq\mathfrak{F}$. Let G be a group in $\mathfrak{F}$. Then $G/G_{f_i(p)} \in\mathfrak{N}_p\mathfrak{S}_{p'}$ for every $i \in I$, $p \in \pi$. Consequently, $G/\bigcap_{i\in I}G_{f_i(p)}\in\mathfrak{N}_p\mathfrak{S}_{p'}$. But $\bigcap_{i\in I}G_{f_i(p)} = G_{f(p)}$. Thus, $G \in f(p)\mathfrak{N}_p\mathfrak{S}_{p'}$ for all primes $p \in \pi$. In addition, $G\in\mathfrak{S}_\pi$. Consequently, $G\in\mathfrak{S}_\pi\cap(\bigcap_{p\in\pi}f(p)\mathfrak{N}_p\mathfrak{S}_{p'})=\mathfrak{M}$.

This proves the lemma.

The next lemma gives one possible way to construct local products of Fitting classes and is of independent interest.

<u>LEMMA 4.</u> Every finite product of Fitting classes $\mathfrak{F}=\prod_{i=1}^n\mathfrak{F}_i (n\geq 2)$ where $\mathfrak{F}_i=\mathfrak{S}_{\pi_i}$ for some set of primes π_i, is local.

<u>Proof.</u> We prove the lemma by induction on the number of factors n. If n = 2 the assertion is true by Lemma 2.

Suppose now that n > 2 and that all products of length less than n are local. Let $\mathfrak{H}=\prod_{i=1}^{n-2}\mathfrak{F}_i$. Then, by the associativity of multiplication of Fitting classes (see [5, Lemma 1]) and by Lemma 2,

$$\mathfrak{F}=\mathfrak{H}(\mathfrak{F}_{n-1}\mathfrak{F}_n)=\mathfrak{H}(\mathfrak{S}_\sigma\cap\mathfrak{S}_\pi\mathfrak{S}_\omega\mathfrak{S}_{\omega'}\cap\mathfrak{S}_\pi\mathfrak{S}_{\pi\setminus\omega}\mathfrak{S}_{(\pi\setminus\omega)'}).$$

where $\pi_{n-1} = \pi$, $\pi_n = \omega$, $\sigma = \pi \cup \omega$. Consequently, by Lemma 1, $\mathfrak{F}=\mathfrak{H}\mathfrak{S}_\sigma\cap\mathfrak{X}\mathfrak{S}_\omega\mathfrak{S}_{\omega'}\cap\mathfrak{X}\mathfrak{S}_{\pi\setminus\omega}\mathfrak{S}_{(\pi\setminus\omega)'}$ where $\mathfrak{X}=\mathfrak{H}\mathfrak{S}_\pi$. But $\mathfrak{H}\mathfrak{S}_\pi$ is a product of length n − 1, and therefore it is local by induction, and the products $\mathfrak{X}\mathfrak{S}_\omega\mathfrak{S}_{\omega'}$ and $\mathfrak{X}\mathfrak{S}_{\pi\setminus\omega}\mathfrak{S}_{(\pi\setminus\omega)'}$ are local by Corollary 2 to the theorem of [6]. Thus, $\mathfrak{F}$ is a local product by Lemma 3.

This completes the proof.

Let Fit $\mathfrak{X}$ denote the Fitting class generated by a set of groups $\mathfrak{X}$ [5]. Following [1], we call a radical function f of a class $\mathfrak{F}$ internal if $f(p) \subseteq \mathfrak{F}$ and complete if $f(p) = f(p)\mathfrak{N}_p$ for all primes p.

<u>LEMMA 5.</u> The following assertions are true:

1) Every local Fitting class $\mathfrak{F}$ has a unique minimal complete radical function f, such that, for any prime $p \in \pi(\mathfrak{F})$,

$$f(p)=\mathrm{Fit}\{G\in\mathfrak{F}\mid G^{\mathfrak{N}_p}\cong H^{\mathfrak{N}_p\mathfrak{S}_{p'}}(H\in\mathfrak{F})\}\mathfrak{N}_p;$$

2) if $\mathfrak{F}, \mathfrak{H}$ are Fitting classes with minimal complete radical functions f, h, respectively, then $\mathfrak{F}\subseteq\mathfrak{H}$ if and only if $f \le h$.

Proof. Let Ω be the set of all complete radical functions of a local Fitting class $\mathfrak{F}$. Obviously, $\Omega \neq \varnothing$. Let ψ be an arbitrary element of Ω. Then $\mathfrak{F}=\mathfrak{G}_{\pi(\mathfrak{F})}\cap(\bigcap_{p\in\pi(\mathfrak{F})}\psi(p)\cdot\mathfrak{N}_p\mathfrak{G}_{p'})$. Let $\varphi(p)$ denote the set of groups

$$\{G\in\mathfrak{F} \mid G^{\mathfrak{N}_p}\cong H^{\mathfrak{N}_p\mathfrak{G}_{p'}}\ (H\in\mathfrak{F})\},\quad p\in\pi(\mathfrak{F}).$$

If $G \in \varphi(p)$, then $G^{\mathfrak{N}_p}\cong H^{\mathfrak{N}_p\mathfrak{G}_{p'}}$ for some groups $H\in\mathfrak{F}$. Since $H \in \mathfrak{F}$, it follows that $G^{\mathfrak{N}_p}\cong H^{\mathfrak{N}_p\mathfrak{G}_{p'}}\in\psi(p)$, and therefore $G\in\psi(p)\mathfrak{N}_p=\psi(p)$. Thus, for every prime $p \in \pi(\mathfrak{F})$, we have $\varphi(p) \subseteq \psi(p)$. Consequently, $f(p)=(\mathrm{Fit}\,\varphi(p))\mathfrak{N}_p\subseteq(\mathrm{Fit}\,\psi(p))\mathfrak{N}_p=\psi(p)$ for all p in $\pi(\mathfrak{F})$. Thus, $f \leq \psi$ and

$$\mathfrak{M}=\mathfrak{G}_{\pi(\mathfrak{F})}\cap(\bigcap_{p\in\pi(\mathfrak{F})}f(p)\mathfrak{N}_p\mathfrak{G}_{p'})\subseteq\mathfrak{F}.$$

We now prove that $\mathfrak{F}\subseteq\mathfrak{M}$. Let $X \in \mathfrak{F}$. Since by [5, Lemma 1] $(X^{\mathfrak{G}_{p'}})^{\mathfrak{N}_p}=X^{\mathfrak{N}_p\mathfrak{G}_{p'}}$ $(p\in\pi(\mathfrak{F}))$ it follows that $X^{\mathfrak{G}_{p'}}\in\varphi(p)\subseteq f(p)$. Consequently, $X\in f(p)\mathfrak{N}_p\mathfrak{G}_{p'}$ for all primes $p \in \pi(\mathfrak{F})$. In addition, $X\in\mathfrak{G}_{\pi(\mathfrak{F})}$. Thus, $X\in\mathfrak{G}_{\pi(\mathfrak{F})}\cap(\bigcap_{p\in\pi(\mathfrak{F})}f(p)\mathfrak{N}_p\mathfrak{G}_{p'})=\mathfrak{M}$.

2) If $f \leq h$, then obviously $\mathfrak{F}\subseteq\mathfrak{H}$. Let $\mathfrak{F}\subseteq\mathfrak{H}$. But then, by part 1 of the lemma and the definition of the operation Fit, we immediately obtain $f \leq h$.

This completes the proof of the lemma.

Any set $\{\mathfrak{F}_i \mid i\in I\}$ of local Fitting classes will now be considered partially ordered by the inclusion relation $\subseteq$.

LEMMA 6. The union of any nonempty chain of local Fitting classes is a local Fitting class.

Proof. Let $\{\mathfrak{F}_i \mid i\in I\}$ be a chain of local Fitting classes and let $\mathfrak{F}=\bigcup_{i\in I}\mathfrak{F}_i$. Obviously, $\mathfrak{F}$ is a Fitting class. We must prove that $\mathfrak{F}$ is local. By Lemma 5, for each $i \in I$ the Fitting class $\mathfrak{F}_i$ has a unique minimal complete radical function f_i. Let $\Omega = \{f_i \mid i \in I\}$ be the chain of these functions and define $f = \bigcup_{i\in I}f_i$. We first show that f is a radical function. That $f(p)$ is a Fitting class for all primes p is obvious. We now show that for any group $G \neq 1$ we have $f(G) = \bigcap_p f(p)$, where p runs through all prime divisors of $\pi(G)$. Since $f(G) = \bigcup_{i\in I}f_i(G)$ and for any $i \in I$ and $p \in \pi(G)$ we have $f_i(G) = \bigcap_p f_i(p)$, it follows that $f(G) \subseteq \bigcap_p f(p)$. We now verify the reverse inclusion. Let $|G| = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$ and $X \in \bigcap_p(\bigcup_{i\in I}f_i(p))$. Then $X \in (\bigcup_{i\in \ell}f_i(p_1)) \cap (\bigcup_{i\in I}f_i(p_2)) \cap \ldots \cap (\bigcup_{i\in I}f_i(p_k))$. Consequently, $X \in f_{i_1}(p_1) \cap f_{i_2}(p_2) \cap \ldots \cap f_{i_k}(p_k)$ for some $i_1, i_2, \ldots, i_k \in I$. We may assume without loss of generality that $f_{i_1} \leq f_{i_2} \leq \ldots \leq f_{i_k}$. Thus, $X \in f_{i_k}(p_1) \cap f_{i_k}(p_2) \cap \ldots \cap f_{i_k}(p_k)$. Consequently, $X \in \bigcap_p f_{i_k}(p) \subseteq \bigcup_{i\in I}(\bigcap_p f_i(p))$. Thus, $\bigcap_p f(p) \subseteq f(G)$ and f is a radical function. It remains to verify that f is a radical function of $\mathfrak{F}$. Let $\mathfrak{M}=\mathfrak{G}_{\pi(\mathfrak{F})}\cap(\bigcap_{p\in\pi(\mathfrak{F})}f(p)\mathfrak{N}_p\mathfrak{G}_{p'})$. Since $f_i \leq f$, we have $\mathfrak{F}\subseteq\mathfrak{M}$. Let L be a group in $\mathfrak{M}$. Then $L^{\mathfrak{N}_p\mathfrak{G}_{p'}}\in f(p)$ for all primes $p \in \pi(\mathfrak{F})$. But $f(p) = \bigcup_{i\in I}f_i(p)$. Consequently, there exists $i_\ell \in I$, such that, $L^{\mathfrak{N}_p\mathfrak{G}_{p'}}\in f_{i_\ell}(p)$ for all primes $p \in \pi(\mathfrak{F})$. Thus, $L\in\bigcap_{p\in\pi(\mathfrak{F}_i)}f_{i_e}(p)\mathfrak{N}_p\mathfrak{G}_{p'}$. Since $L\in\mathfrak{G}_{\pi(\mathfrak{F})}$ it follows that $L\in\mathfrak{G}_{\pi}(\mathfrak{F}_{i_m})$ for some $i_m \in I$. The fact that $\mathfrak{F}_{i_m}$ and $\mathfrak{F}_{i_\ell}$ are elements of a chain implies that either $\mathfrak{F}_{i_m}\subseteq\mathfrak{F}_{i_\ell}$ or $\mathfrak{F}_{i_\ell}\subseteq\mathfrak{F}_{i_m}$. But then either $\mathfrak{G}_{\pi(\mathfrak{F}_{i_m})}\subseteq\mathfrak{G}_{\pi(\mathfrak{F}_{i_e})}$ or $\mathfrak{G}_{\pi(\mathfrak{F}_{i_e})}\subseteq\mathfrak{G}_{\pi(\mathfrak{F}_{i_m})}$ and by Lemma 5 either $f_{i_m} \leq f_{i_\ell}$ or $f_{i_\ell} \leq f_{i_m}$. In either case, it is easy to see that $L \in \mathfrak{F}$. Thus, $\mathfrak{M}=\mathfrak{F}$.

This proves the lemma.

In the following proof of the theorem we will need the concept of a solvable primitive saturated formation [8], which we now recall. Let $\mathcal{F}_0$ be the family of formations consisting of the formations $\varnothing$, $\mathfrak{E}$, $\mathfrak{S}$. For any $i > 0$, we define a family of formations $\mathcal{F}_i$ inductively, as follows: $\mathfrak{F}\in\mathcal{F}_i$ if and only if either $\mathfrak{F}\in\mathcal{F}_{i-1}$ or $\mathfrak{F}$ is a formation with local screen f such that $f(p)\in\mathcal{F}_{i-1}$ for all primes p. Let $\mathcal{F}$ be the family of all formations $\mathfrak{F}$, such that, $\mathfrak{F}=\bigcup_j\mathfrak{F}_j$, where $\mathfrak{F}_j\in\bigcup_i\mathcal{F}_i$ and $\mathfrak{F}_j\subseteq\mathfrak{F}_{j+1}$. We call a formation $\mathfrak{Y}$ primitive if $\mathfrak{Y}\in\mathcal{F}$.

Proof of the Theorem. Let $\mathfrak{F}$ be a solvable subgroup-closed Fitting class. Then by [9, Theorem 1.1], $\mathfrak{F}$ is a formation. But by [8, Theorems 1, 4], any subgroup-closed radical formation is primitive saturated. There are now two possibilities with regard to $\mathfrak{F}$.

1. $\mathfrak{F}\subseteq\mathfrak{N}^k$ for some natural number k, i.e., $\mathfrak{F}$ has bounded nilpotent length. In that case, by [10, Lemma 2.3], $\mathfrak{F}=\cap_{i=1}^{\infty}\mathfrak{F}_i$ where each Fitting class $\mathfrak{F}_i$ is a product of Fitting classes of finite length of the form $\mathfrak{S}_{\pi_1}, \mathfrak{S}_{\pi_2}, \ldots, \mathfrak{S}_{\pi_n}$ where $\pi_1, \pi_2, \ldots, \pi_n$ are sets of primes. Consequently, by Lemma 4, $\mathfrak{F}_i$ is a local Fitting class for any $i \in I$, and therefore, by Lemma 3, $\mathfrak{F}$ is local.

2. The nilpotent length of $\mathfrak{F}$ is unbounded. Let $\mathfrak{F}_i=\mathfrak{F}\cap\mathfrak{N}^i$ where $i \geq 1$. It is easy to see that $\mathfrak{F}_i$ is a subgroup-closed Fitting class of bounded nilpotent length for every $i \geq 1$. Consequently, $\mathfrak{F}$ is a local Fitting class for all $i \geq 1$, by case 1. But then $\mathfrak{F}=\cup_{i=1}^{\infty}\mathfrak{F}_i$ is the union of a chain of local Fitting classes, so that $\mathfrak{F}$ is local by Lemma 6.

This completes the proof.

COROLLARY 1. If $\mathfrak{F}, \mathfrak{H}$ are nonempty solvable subgroup-closed Fitting classes, then their product $\mathfrak{F}\mathfrak{H}$ and intersection $\mathfrak{F}\cap\mathfrak{H}$ are local.

An important problem in the theory of solvable Fitting classes is concerned with the description of Fitting classes that satisfy Lockett's condition $\mathfrak{F}_*=\mathfrak{F}^*\cap\mathfrak{S}_*$ where $\mathfrak{F}_*=\cap\{\mathfrak{X}|\mathfrak{X}^*=\mathfrak{F}^*\}$, $\mathfrak{S}_*$ is a minimal normal Fitting class and $\mathfrak{F}^*=\{G\in\mathfrak{S}|(G\times G)_{\mathfrak{F}}=(G_{\mathfrak{F}}\times G_{\mathfrak{F}})\langle g, g^{-1}\rangle|g\in G\}$ (see, e.g., [10]). It was shown in [6] that every solvable local Fitting class $\mathfrak{F}\supseteq\mathfrak{N}$ satisfies Lockett's condition. Arguments analogous to those used in [6] will show that any solvable local class is a Fitting class with Lockett's condition. Therefore, in view of this observation and the theorem, we obtain a concrete realization of the main results of [6]:

COROLLARY 2. Every nonempty solvable subgroup-closed Fitting class is a Fitting class with Lockett's condition.

COROLLARY 3. If $\{\mathfrak{F}_i|i\in I\}$ is a set of nonempty solvable subgroup-closed Fitting classes, then $\mathfrak{F}=\cap_{i\in I}\mathfrak{F}_i$ is a Fitting class with Lockett's condition.

Corollary 2 implies a positive solution to Problem 8.30 of [11] for the case of subgroup-closed Fitting classes.

The first definition of solvable local Fitting classes is due to Hartley [3], who proposed to define them as classes $\cap_p f(p)\mathfrak{S}_{p'}\mathfrak{N}_p$ where f is some radical function. However, even for $\mathfrak{F}=\mathfrak{N}_p\mathfrak{S}_{p'}$ it was proved in [3, Sec. 4.2] that $\mathfrak{F}$ is a subgroup-closed Fitting class but not local in the sense of Hartley's condition. We have now eliminated these difficulties: it is clear that every local Fitting class in the sense of Hartley's definition is local in our sense. In particular, the products $\mathfrak{F}=\mathfrak{N}_p\mathfrak{S}_{p'}$ and $\mathfrak{H}=\mathfrak{S}_{p'}\mathfrak{N}_p$ are local, by Lemma 2.

In conclusion, we note that the dual of our theorem in the formation theory of finite groups is false. For example, the solvable formations $\mathfrak{A}, \mathfrak{A}\mathfrak{N}$ are subgroup-closed, but not local (see [7]). Our theorem in its full generality cannot be inverted: if $\mathfrak{X}$ is a solvable Fitting class that is not subgroup-closed, then it is easy to see that the product $\mathfrak{X}\mathfrak{N}$ is a local Fitting class but it is not subgroup-closed.

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ALMOST LOCALLY REPRESENTABLE VARIETIES OF LIE ALGEBRAS

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In the theory of algebras with identity relations an important role is played by varieties extremal in relation to some property.

It is said that a variety $\mathfrak{V}$ almost possesses some property P if any proper subvariety of it possesses the indicated property, but $\mathfrak{V}$ itself does not belong to the number of varieties defined by the property P.

Of greatest interest are the situations when the set of varieties minimal in relation to the given property proves to be finite. It is known, for example, that any variety of Lie algebras generated by a finite algebra will be Cross [1]. At the same time, over any finite field there exists a unique almost Cross solvable variety of Lie algebras [2]. In the case of a null characteristic of the ground field let us note examples of the smallest variety of Lie algebras in which not any standard identity is satisfied (see [3, 4]), and also four solvable varieties of almost polynomial growth [5].

Let us recall that a variety of Lie algebras $\mathfrak{V}$ on a field Φ is called locally representable if any finitely-generated algebra from $\mathfrak{V}$ is representable, that is, imbeddable in a finite-dimensional algebra over some extension of the ground field Φ. The class of locally representable varieties of Lie algebras is quite wide (see [6]). Such, for example, are all varieties $\mathfrak{N}_c\mathfrak{A}\cap\mathfrak{A}\mathfrak{N}_c$ over an arbitrary field. Here $\mathfrak{A}$ is an abelian variety, $\mathfrak{N}_c$ is the variety of Lie algebras of nilpotent degree no higher than c.

The goal of the note is to show that over a field of null characteristic there exist only two almost locally representable varieties of Lie algebras, and over an infinite field of positive characteristic – one.

Let us introduce the necessary notation. We will write the left-normalized product $[[\ldots[x_1, x_2], \ldots], x_n]$ without brackets $x_1x_2\ldots x_n$. If $x_2 = \ldots = x_n = y$, then we will denote this same product by x_1y^n.

Let us denote by $\mathfrak{V}_1$ the centrally metabelian variety of Lie algebras over the field Φ, that is, the variety defined by the identity

$$(x_1x_2)(x_3x_4)x_5=0. \tag{1}$$

Let us consider yet another variety. Let G be the three-dimensional Heisenberg algebra over a field Φ with basis $\{x, y, z\}$ and multiplication table $xy = z$, $xz = yz = 0$. The representation of G is well known on the ring of polynomials $\Phi[t]$, given in the following manner: $xf(t) = f'(t)$; $yf(t) = tf(t)$; $zf(t) = f(t)$ $(f(t) \in \Phi[t])$. Considering $\Phi[t]$ as a G-module, one can construct the semidirect product $G \leftthreetimes \Phi[t]$, which is the Lie algebra lying in the variety $\mathfrak{A}\mathfrak{N}_2$. Let us denote by $\mathfrak{V}_2$ the variety generated by the algebra $G\leftthreetimes\Phi[t]$.

THEOREM. Let Φ be an infinite field and $\mathfrak{V}$ be an almost locally representable variety of Φ-Lie algebras.

a) If $\operatorname{char}\Phi > 2$, then $\mathfrak{V}=\mathfrak{V}_1$.

b) If $\operatorname{char}\Phi = 2$, then $\mathfrak{V}$ is given by the identity (1) and the family of identities

$$x_1y_1\ldots y_k(x_2x_3)+x_2y_1\ldots y_k(x_3x_1)+x_3y_1\ldots y_k(x_1x_2)=0 \\ (k=1, 2, \ldots), \tag{2}$$

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