A CRITERION FOR THE LOCALITY OF FORMATION PRODUCTS

N. T. Vorob'ev

Questions about the construction of formations occupy a central position in the theory of formations. In addition to methods of constructing formations by means of group functions and screens (see Chap. I of [1], also [2-8]), there exist methods for constructing from two formations, say $\mathfrak{F}$ and $\mathfrak{H}$, a third, which is naturally called a formation product. The simplest case of a formation product was introduced by L. A. Shemetkov [1]. Five important types of formation products are known (see [5-7]): the formation products $\mathfrak{F} *_i \mathfrak{H}$ of the i-th kind, $1 \leqslant i \leqslant 5$. There arises the following problem: Find those formations $\mathfrak{F}$ and $\mathfrak{H}$ for which the formation $\mathfrak{F} *_i \mathfrak{H}$ is local.

In the present note we consider this problem, which was suggested to the author by L. A. Shemetkov, for the formation products of the second kind.

Suppose we are studying a nonempty class of groups $\mathfrak{U}$, closed under the operations S, Q, and $\mathrm{Ext}_{\mathfrak{A}}$, such that each group G in $\mathfrak{U}$ possesses at least one $\mathfrak{F}$-projector and any two $\mathfrak{F}$-protors are conjugate in G ($\mathfrak{F}$ is a local formation in $\mathfrak{U}$). In what follows, by a group we will always mean a group of class $\mathfrak{U}$, and by a class of groups (in particular, a formation) we will mean a subclass (formation) of $\mathfrak{U}$.

Definition 1. Suppose $\mathfrak{F}$ is a local formation and $\mathfrak{H}$ an arbitrary formation. Let $\mathfrak{F} *_2 \mathfrak{H}$ denote the class of all groups in which an $\mathfrak{F}$-projector belongs to $\mathfrak{H}$.

If $\mathfrak{H} = \varnothing$, then $\mathfrak{F} *_2 \mathfrak{H} = \varnothing$.

The class $\mathfrak{F} *_2 \mathfrak{H}$ is obviously a formation, which, in general, is not local. Note that special cases of the formation $\mathfrak{F} *_2 \mathfrak{H}$ have been studied by various authors. In the class of solvable groups, Doerk [8], D'Arcy [9], and Beidleman and Makan [10] studied the formation $\mathfrak{F} *_2 f(p)$ (f a maximal inner local screen of the formation $\mathfrak{F}$ and p a prime), Blessenohl [11] the formation $\mathfrak{S}_\pi *_2 \mathfrak{H}$ and Doerk [12] the formation $\mathfrak{F} *_2 \mathfrak{H}$ in the case where $\mathfrak{H}$ is local. The formation $\mathfrak{F} *_2 f(p)$ in the class of groups with $\pi(\mathfrak{F})$-solvable $\mathfrak{F}$-coradical was studied in [5-7].

We will denote by $\mathfrak{U}_\pi$ the formation of all π-groups of class $\mathfrak{U}$.

A local screen f will be called:

1) complete if $\mathfrak{N}_p f(p) = f(p)$ for each prime p;

2) S-closed if the formation f(p) is S-closed for each prime p.

All other notation and definitions can be found in Shemetkov's monograph [1].

We first state as lemmas, without giving the proofs, three results from [5, 6].

LEMMA 1. Suppose f_1, f_2 are maximal inner local screens of the formations $\mathfrak{F}_1$, $\mathfrak{F}_2$ respectively. Then $\mathfrak{F}_1$ is a subformation of $\mathfrak{F}_2$ if and only if $f_1 \leqslant f_2$.

Recall that if $\mathfrak{F}$ is a formation with maximal inner local screen f, then we have denoted by f* (see [5]) the local screen such that $f^*(p) = \mathfrak{F} *_2 f(p)$ for each prime p.

LEMMA 2. Suppose $\mathfrak{F}$ is a formation with maximal inner local screen f. If each group in $\mathfrak{U}$ has a $\pi(\mathfrak{F})$-solvable $\mathfrak{F}$-coradical, then:

1) an $\mathfrak{F}$-projector F of the group G covers each f*-central chief factor of G;

2) each chief factor of G covered by an $\mathfrak{F}$-projector F of G is f*-central.

LEMMA 3. If $\mathfrak{F}$ is a nonempty formation, π is some set of primes, and $\mathfrak{U}_\pi \mathfrak{F} = \mathfrak{F}$, then the local screen f such that

---

Vitebsk State Pedagogic Institute. Translated from Matematicheskie Zametki, Vol. 34, No. 2, pp. 165-170, August, 1983. Original article submitted July 9, 1982.

$$f(p)=\begin{cases}\mathfrak{F}, & \text{if } p\in\pi,\\ \mathfrak{U}_{\pi'}\mathfrak{F}, & \text{if } p\in\pi'\end{cases}$$

is a maximal inner local screen of the formation $\mathfrak{U}_{\pi'}\mathfrak{F}$.

LEMMA 4. If $\mathfrak{F}$ is a local formation and φ a complete local screen, then the local screen φ^* such that $\varphi^*(p)=\mathfrak{F}*_2\varphi(p)$ for each prime p is complete.

Proof. Obviously, $\varphi^*(p)\subseteq\mathfrak{N}_p\varphi^*(p)$. Suppose $G\in\mathfrak{N}_p\varphi^*(p)$ and F is an $\mathfrak{F}$-projector of G. Then $F/F\cap O_p(G)\simeq FO_p(G)/O_p(G)\in\varphi(p)$. Since $F\cap O_p(G)\subseteq O_p(F)$, it follows that $F/O_p(F)\in\varphi(p)$. But φ is a complete screen. Therefore, $F\in\mathfrak{N}_p\varphi(p)=\varphi(p)$. It follows that $G\in\varphi^*(p)$. The lemma is proved.

Definition 2. Suppose $\mathfrak{F}$ is a local formation and f a local screen of $\mathfrak{F}$. A subgroup H of a group G will be called an f-$\mathcal{L}$-subgroup if H covers only f-central chief factors of G.

If $\mathfrak{F}$ is a local formation with maximal inner local screen f, then examples of f^*-$\mathcal{L}$-subgroups (f-$\mathcal{D}$-subgroups) in groups with $\pi(\mathfrak{F})$-solvable $\mathfrak{F}$-coradical are $\mathfrak{F}$-projectors of groups ($\mathfrak{F}$-normalizers of groups).

LEMMA 5. Suppose $\mathfrak{F}_1\supseteq\mathfrak{F}_2$ are formations with maximal inner local screens f_1, f_2 respectively. Suppose f is a maximal inner local screen of the formation $l\,\mathrm{form}(\mathfrak{F}_1*_2\mathfrak{F}_2)$. Then:

1) $\varphi\leqslant f$, where φ is the local screen such that $\varphi(p)=\mathfrak{F}_1*_2f_2(p)$ for each prime p;

2) if an $\mathfrak{F}_i$-projector of each group G is an f_i^*-$\mathcal{D}$-subgroup of G (f_i^* is the local screen such that $f_i^*(p)=\mathfrak{F}_i*_2f_i(p)$ for all primes p), i = 1, 2, then:

a) f(p) is a subformation of $\varphi_1(p)$, where $\varphi_1(p)=\mathfrak{U}_{p'}\varphi(p)$, φ is the same screen as in 1), and p is a prime such that $\mathfrak{F}_2\subseteq f_1(p)$;

b) $f\leqslant\varphi_1$ and $f=\varphi'$ if $\mathfrak{F}_2\subseteq f_1(p)$ for all primes $p\in\pi(\mathfrak{F}_1)$, and the screens φ and φ_1 are the same as in 1) and a).

Proof. We will prove the first assertion of the lemma. Assume it is false. Then there exists a prime p such that $\varphi(p)$ is not contained in f(p). Choose in the class $\varphi(p)\setminus f(p)$ a group G having the smallest order. In view of Theorem 3.3 of [1], the screen f is complete, hence $O_p(G)=1$. Consider the regular wreath product $\Gamma=P\wr G$, where $|P|$ = p. Then $\Gamma=N\leftthreetimes G$ and $N=O_p(\Gamma)=F_p(\Gamma)$. Suppose F is an $\mathfrak{F}_1$-projector of G. Since $G\in\varphi(p)$, we have $F\in f_2(p)$. But $\mathfrak{F}_1\supseteq\mathfrak{F}_2$. Consequently, by Lemma 1, $F\in f_1(p)$. Applying Lemma 2 of [13], we see that the subgroup $FC_N(F^{f_1(p)})=FN$ is an $\mathfrak{F}_1$-projector of Γ. It follows from Theorem 3.3 of [1] that the screen f_2 is complete and $F/F\cap N\simeq FN/N\in f_2(p)$, hence $FN\in\mathfrak{N}_pf_2(p)=f_2(p)$. Consequently, $\Gamma\in\varphi(p)\subseteq l\,\mathrm{form}(\mathfrak{F}_1*_2\mathfrak{F}_2)$. It follows that $G\simeq\Gamma/N\in f(p)$. Contradiction. Assertion 1) is proved.

Let us prove assertion 2). In view of Lemma 3 and the definition of the formation $l\,\mathrm{form}(\mathfrak{F}_1*_2\mathfrak{F}_2)$, in order to establish a) it suffices to show that $\mathfrak{F}_1*_2\mathfrak{F}_2$ is a subformation of the formation $\mathfrak{U}_{p'}\varphi(p)$, where p is a prime such that $\mathfrak{F}_2\subseteq f_1(p)$. Suppose $G\in\mathfrak{F}_1*_2\mathfrak{F}_2$ and H/K is a pd-chief factor of G. Then $G/C_G(H/K)\in f_1^*(p)$. Consequently, an $\mathfrak{F}_1$-projector F of G covers H/K. Since $\mathfrak{F}_2\subseteq\mathfrak{F}_1$ and $G\in\mathfrak{F}_1*_2\mathfrak{F}_2$, it is easy to see that F is an $\mathfrak{F}_2$-projector of G. Therefore, by hypothesis, $G/C_G(H/K)\in f_2^*(p)$. It follows that

$$G/O_{p'}(G)/F_p(G)/O_{p'}(G)\simeq G/F_p(G)\in f_2^*(p)\cap(\mathfrak{F}_1*_2\mathfrak{F}_2).$$

It is easy to see that $f_2^*(p)\cap(\mathfrak{F}_1*_2\mathfrak{F}_2)=\varphi(p)$. But it follows from Lemma 4 that φ is a complete screen. Consequently, $G/O_{p'}(G)\in\varphi(p)$, hence $G\in\mathfrak{U}_{p'}\varphi(p)$. Thus, $\mathfrak{F}_1*_2\mathfrak{F}_2\subseteq\mathfrak{U}_{p'}\varphi(p)$.

Assertion b) now follows directly from Lemmas 1 and 3 and from 1). The lemma is proved.

THEOREM. Suppose $\mathfrak{F}_1$, $\mathfrak{F}_2$ are local formations. If an $\mathfrak{F}_i$-projector of each group G of class $\mathfrak{U}$ is an f_i^*-$\mathcal{D}$-subgroup of G (i = 1, 2) and $\mathfrak{F}_1=\mathfrak{U}_\pi$, then the formation $\mathfrak{F}_1*_2\mathfrak{F}_2$ is local.

Proof. Obviously, $\mathfrak{F}_1*_2\mathfrak{F}_2\subseteq l\,\mathrm{form}(\mathfrak{F}_1*_2\mathfrak{F}_2)$. We will prove that $l\,\mathrm{form}(\mathfrak{F}_1*_2\mathfrak{F}_2)\subseteq\mathfrak{F}_1*_2\mathfrak{F}_2$. Assume this is not so. Choose in the class $l\,\mathrm{form}(\mathfrak{F}_1*_2\mathfrak{F}_2)\setminus(\mathfrak{F}_1*_2\mathfrak{F}_2)$ a group G having the smallest order. Then G has a unique minimal normal subgroup K, which coincides with $G^{\mathfrak{F}_1*_2\mathfrak{F}_2}$.

Suppose K is a π'-group and F an $\mathfrak{F}_1$-projector of G. Then $F\simeq F/F\cap K\in\mathfrak{F}_2$, hence $G\in\mathfrak{F}_1*_2\mathfrak{F}_2$. Contradiction.

Assume that $O_{\pi'}(G) = 1$. If K is non-Abelian, then obviously, $C_G(K) = 1$. Consequently, $G \in f(K)$, where f is a maximal inner local screen of the formation l form $(\mathfrak{F}_1 * {}_2\mathfrak{F}_2)$. It is easy to see that $\mathfrak{F}_1$ has a maximal inner local screen f_1 such that

$$f_1(p) = \begin{cases} \mathfrak{F}_1, & \text{if } p \in \pi, \\ \varnothing, & \text{if } p \in \pi'. \end{cases}$$

It follows that $\mathfrak{F}_1 \cap \mathfrak{F}_2 \subseteq \mathfrak{F}_1 \subseteq f_1(p)$ for all primes $p \in \pi$. Let ψ be a maximal inner local screen of $\mathfrak{F}_1 \cap \mathfrak{F}_2$ (the formation $\mathfrak{F}_1 \cap \mathfrak{F}_2$ is local by virtue of Lemma 3.7 of [1]). Then, by assertion 2) of Lemma 5, we have $f(K) \subseteq f(p) \subseteq \mathfrak{F}_1 *_2 \mathfrak{F}_2$ for some prime $p \in \pi$. But $\mathfrak{F}_1 * {}_2\psi(p) \subseteq \mathfrak{F}_1 * {}_2\mathfrak{F}_2$. Therefore, $G \in \mathfrak{F}_1 * {}_2\mathfrak{F}_2$. Contradiction.

It remains to consider the following situation: K is an Abelian p-group, $p \in \pi$. Since $\mathfrak{F}_1 \cap \mathfrak{F}_2 \subseteq f_1(p)$, it follows from assertion 2) of Lemma 5 that $G/C_G(K) \in \mathfrak{F}_1 * {}_2\psi(p)$. Consequently, $F/C_F(K) \simeq FC_G(K)/C_G(K) \in \psi(p)$. Look at the chief series

$$F_0 = F \supset F_1 \supset \ldots \supset F_k = F \cap K \supset F_{k+1} \supset \ldots \supset F_t = 1 \qquad (*)$$

of F. In the section $F \supset \ldots \supset F \cap K$ all factors of the series (*) are $(\mathfrak{F}_1 \cap \mathfrak{F}_2)$-central, since $F/F \cap K \simeq FK/K \in \mathfrak{F}_1 \cap \mathfrak{F}_2$. It is easy to see that $C_F(K) \subseteq C_F(F_{i-1}/F_i)$, $i = k, k+1, \ldots, t$. Consequently, $F/C_F(F_{i-1}/F_i) \in \psi(p)$. Thus, $F \in \mathfrak{F}_1 \cap \mathfrak{F}_2 \subseteq \mathfrak{F}_2$, hence $G \in \mathfrak{F}_1 * {}_2\mathfrak{F}_2$. Contradiction. The theorem is proved.

<u>COROLLARY 1.</u> Suppose $\mathfrak{F}_i$ is a local formation (i = 1, 2). If each group in $\mathfrak{U}$ has a $\pi(\mathfrak{F}_i)$-solvable $\mathfrak{F}_i$-coradical and $\mathfrak{F}_1 = \mathfrak{U}_\pi$, then $\mathfrak{F}_1 *_2 \mathfrak{F}_2$ is a local formation.

<u>Proof.</u> By Lemma 2, each $\mathfrak{F}_i$-projector of a group in $\mathfrak{U}$ is an f_i^*-$\mathscr{D}$-subgroup of this group (i = 1, 2). Then $\mathfrak{F}_1 * {}_2\mathfrak{F}_2$ is local by the theorem just proved.

From Corollary 1 we easily obtain the following well-known result of Blessenohl, which has been proved by other methods.

<u>COROLLARY 2 (Blessenohl [11]).</u> Suppose $\mathfrak{U} = \mathfrak{S}$ and $\mathfrak{F}$ is a local formation. If H is a π-Hall subgroup of G and $H/H \cap \Phi(G) \in \mathfrak{F}$, then $H \in \mathfrak{F}$.

<u>Proof.</u> By Theorem 15.2 of [1], the set of all $\mathfrak{S}_\pi$-projectors of G coincides with the set of all π-Hall subgroups of G. Since $H\Phi(G)/\Phi(G) \simeq H/H \cap \Phi(G) \in \mathfrak{F}$, it follows that $G/\Phi(G) \in \mathfrak{S}_\pi * {}_2\mathfrak{F}$. But by Theorem 4.2 of [1] and Corollary 1 the formation $\mathfrak{S}_\pi *_2 \mathfrak{F}$ is saturated. Therefore, $G \in \mathfrak{S}_\pi *_2 \mathfrak{F}$, hence $H \in \mathfrak{F}$.

LITERATURE CITED

1. L. A. Shemetkov, Formations of Finite Groups [in Russian], Nauka, Moscow (1978).
2. L. A. Shemetkov, "Two trends in the development of nonsimple finite groups," Usp. Mat. Nauk, <u>30</u>, No. 2, 179-211 (1975).
3. L. A. Shemetkov, "Graduated formations of groups," Mat. Sb., <u>94</u>, No. 2, 264-268 (1974).
4. L. A. Shemetkov, "Screens of a product of formations," Dokl. Akad. Nauk BSSR, <u>25</u>, No. 8, 677-680 (1981).
5. N. T. Vorob'ev, "Maximal screens of local formations," Algebra Logika, <u>18</u>, No. 2, 137-161 (1979).
6. N. T. Vorob'ev, "Embedding of local screens," Mat. Zametki, <u>30</u>, No. 2, 305-311 (1981).
7. N. T. Vorob'ev, "Maximal screens of induced formations," Dokl. Akad. Nauk BSSR, <u>23</u>, No. 2, 115-117 (1979).
8. K. Doerk, "Zur Theorie der Formationen endlicher auflösbarer Gruppen," J. Algebra, <u>13</u>, No. 3, 345-373 (1969).
9. P. D'Arcy, "$\mathfrak{F}$-abnormality and the theory of finite solvable groups," J. Algebra, <u>28</u>, No. 2, 342-361 (1974).
10. J. C. Beidleman and A. K. Makan, "On saturated formations which are special relative to the strong cover-avoidance property," Proc. Am. Math. Soc., <u>47</u>, No. 1, 29-36 (1975).
11. D. Blessenohl, "Über Formationen und Halluntergruppen endlicher auflösbarer Gruppen," Math. Z., <u>142</u>, No. 3, 299-300 (1975).
12. K. Doerk, "Zur Sättigung einer Formationen endlicher auflösbarer Gruppen," Arch. Math., <u>28</u>, No. 5, 561-571 (1977).
13. P. D'Arcy, "On formations of finite groups," Arch. Math., <u>25</u>, No. 1, 3-7 (1974).