EMBEDDING OF LOCAL SCREENS

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The notion of a screen, introduced by Shemetkov in [1, 2], plays an important role in the problems of construction and classification of formations. A screen is a mapping f of the class $\mathfrak{G}$ of all finite groups into a set of classes of groups such that the following conditions are fulfilled for each group G:

1) f(G) is a formation;

2) $f(G) \subseteq f(G^\alpha) \cap f(\mathrm{Ker}\,\alpha)$ for each homomorphism α of the group G;

3) $f(1) = \mathfrak{G}$, where 1 is the trivial group.

Following Shemetkov, we will assume each set Ω of screens to be partially ordered by the relation $\leqslant$, defined in the following manner. If $f_1, f_2 \in \Omega$, then the screen f_1 will be said to be embedded in the screen f_2 (in symbols, $f_1 \leqslant f_2$,) if $f_1(G) \subseteq f_2(G)$ for each group G. There arises the problem to determine the conditions under which a screen f_1 is embedded in a screen f_2. The present note is devoted to the consideration of this problem for local screens. Three criteria for the embedding of local screens are obtained. Let us recall that a screen f is said to be local [1] if the following conditions are fulfilled:

1) f(R) = f(S) for each pair of nontrivial p-groups and each prime p (in this case, the value of f on nontrivial p-groups is denoted by f(p));

2) $f(G) = \bigcap_p f(p)$ for each nontrivial group G, where p runs over all prime divisors of the order of G.

All the groups considered in the present note are finite. We will denote the class of all p-groups, where p is a prime, by $\mathfrak{G}_p$. Let $\mathfrak{X}$ be a certain class of groups. A screen f is called an $\mathfrak{X}$-screen if $f(G) \subseteq \mathfrak{X}$ for each group G. Where necessary, definitions and notation, not given here, can be found in [1, 3-5]. In the sequel, we will use the following lemmas, which are of independent interest.

LEMMA 1. Let H be a subgroup of a group G such that G = HF(G). If $\mathfrak{F}$ is an arbitrary nonempty formation, then $H^{\mathfrak{F}} \subseteq G^{\mathfrak{F}}$.

Proof. Let G be a group of the least order for which the lemma is not valid. If $G \in \mathfrak{F}$, then it follows by Lemma 1.5 of [6] that $H \in \mathfrak{F}$, and the lemma is valid. Let us suppose that G does not belong to $\mathfrak{F}$. Let K be a minimal normal subgroup of G contained in $G^{\mathfrak{F}}$ and $H^{\mathfrak{F}}$ and $G^{\mathfrak{F}}$ be the $\mathfrak{F}$-coradicals of the groups H and G, respectively. Then it follows by Lemma 1.5 of [1] that $H^{\mathfrak{F}}K/K$ and $G^{\mathfrak{F}}/K$ are the $\mathfrak{F}$-coradicals of the groups HK/K and G/K, respectively. By induction, $H^{\mathfrak{F}} \subseteq G^{\mathfrak{F}}$. The lemma is proved.

LEMMA 2. Let all the minimal normal subgroups of a group G be solvable. If G has at most two minimal normal subgroups and $O_p(G) = 1$ for a certain prime p, then G has an exact irreducible representation over a finite field of characteristic p.

Proof. Let us suppose that the group G has no proper normal subgroups. Then the order of G is equal to a prime number q that is different from p. Let X be a Schmid group of order $p^m q$ with a normal elementary Abelian Sylow p-subgroup X_p. Then a Sylow q-subgroup X_q of X is isomorphic to G. But X_q is isomorphically embedded in the group GL(m, p) of all automorphisms of the Sylow p-subgroup X_n of X. Consequently, the representation

$$\varphi: \quad G \to \mathrm{GL}(m, p)$$

is also an exact irreducible representation over a field of p elements.

Let M be the Socle of the group G. If M is a solvable minimal normal subgroup of G, then M contains a normal subgroup K such that the group M/K is cyclic and $K_G = 1$. Let us

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suppose that G has exactly two different solvable minimal normal subgroups R and S. Then $M = R \times S$. If the orders of the subgroups R and S are relatively prime, then

$$R = \langle r \rangle \times K_1, \quad S = \langle s \rangle \times K_2,$$

where K_1 and K_2 are subgroups of prime index in R and S, respectively.

Let $K = K_1 \times K_2$. Then it is obvious that the group M/K is cyclic and $K_G = 1$. Let us suppose that the orders of the groups R and S are not relatively prime. Suppose that

$$R = \mathop{\times}_{i=1}^{k} \langle r_i \rangle, \quad S = \mathop{\times}_{j=1}^{l} \langle s_j \rangle$$

and

$$K = \langle r_1 \rangle \times \langle r_2 \rangle \times \ldots \times \langle r_{k-1} \rangle \times \langle s_1 \rangle \times \langle s_2 \rangle \times \ldots \times \langle s_{l-1} \rangle \times \langle r_k s_l \rangle.$$

Then it is easily seen that the group M/K is cyclic and $K_G = 1$. Thus, M always contains a normal subgroup K such that $M = \langle m \rangle \times K$ and $K_G = 1$, where $n = |\langle m \rangle|$ is either prime or a product of two prime numbers. Since n and p are relatively prime, it follows that each field of characteristic p contains exactly n different n-th roots $\varepsilon, \varepsilon^2, \ldots, \varepsilon^n = 1$ of the unity. Let us consider the mapping

$$\varphi: m^\alpha K \to \varepsilon^\alpha,$$

where $\alpha = 1, 2, \ldots, n$. It is obvious that φ is a one-dimensional representation of the group M and $\operatorname{Ker} \varphi = K$. Let φ^G be the representation of the group G induced by the representation φ, and τ be the irreducible component of the matrix $(\varphi^G(g))$, situated in the upper left corner. It is easily seen that $\operatorname{Ker} \tau \cap M = K_G = 1$. Consequently, $\operatorname{Ker} \tau = 1$, and therefore τ is the desired representation of G. The lemma is proved.

A subgroup H of a group G is called a $\mathcal{DM}$-subgroup [7] if H either covers or avoids each principal factor of the group G, by a check it is easy to establish that the following lemma is valid.

LEMMA 3. If H is a $\mathcal{DM}$-subgroup of a group G, then the order of H is equal to the product of the orders of all the principal factors of a certain principal series of G that are covered by H.

Definition. Let f be a screen. A subgroup H of a group G is called an f-$\mathcal{DM}$-subgroup if H covers each f-central principal factor of G and avoids each f-excentral principal factor of it.

Let us recall that a screen f is called a screen of a formation $\mathfrak{F}$ [1] if $\mathfrak{F} = \langle f \rangle$, where $\langle f \rangle$ is the set of those groups which have f-central series.

LEMMA 4. Let f be a local screen of a formation $\mathfrak{F}$. If a group G has an f-$\mathcal{DM}$-subgroup H and $G^{\mathfrak{F}}$ is nilpotent, then H is an $\mathfrak{F}$-projection of the group G.

Proof. Let G be the group of the least order for which the lemma is not valid. Since $G^{\mathfrak{F}}$ is nilpotent, it follows that G = HF(G). Let

$$G_0 = 1 \subset G_1 \subset G_2 \subset \ldots \subset G_t = G \tag{1}$$

be a principal series of G. Let us consider the series

$$1 \subseteq H \cap G_1 \subseteq H \cap G_2 \subseteq \ldots \subseteq H \cap G_t = H. \tag{2}$$

Suppose that G_{i+1}/G_i is a principal factor of the series (1), where $i = 0, 1, \ldots, t$. If H avoids G_{i+1}/G_i, then

$$H \cap G_{i+1}/H \cap G_i \simeq G_i/G_i.$$

But if H covers G_{i+1}/G_i, then

$$H \cap G_{i+1}/H \cap G_i \simeq G_{i+1}/G_i.$$

Consequently, all the factors of series (2) are f-central in H. Therefore $H \in \mathfrak{F}$.

Let K be a minimal normal subgroup of G that is contained in $G^{\mathfrak{F}}$. By induction, HK/K is an $\mathfrak{F}$-projection of the group G/K. Therefore, by virtue of Proposition VI.7.9 of [5], to prove the lemma it is sufficient to show that H is an $\mathfrak{F}$-projection of the group HK. If

$K \subseteq H$, then this is obvious. Let K be not contained in H. Then K is an f-excentral principal factor of G. Since G = HF(G), it follows that K is an f-excentral principal factor of the group HK. Consequently, $K = (HK)^{\mathfrak{F}}$, and therefore H is an $\mathfrak{F}$-projection of G. The lemma is proved.

It follows from Theorem 5.6 of [3] that if the $\mathfrak{F}$-coradical of the group G is nilpotent, then each $\mathfrak{F}$-projection of the group G coincides with an $\mathfrak{F}$-normalizer of G. Taking this fact into account, we get the following corollary from the above lemma.

COROLLARY. Suppose that f_1 and f_2 are two local screens of the formation $\mathfrak{F}$. If the group G has an f_1-$\mathcal{DM}$-subgroup H_1 and an f_2-$\mathcal{DM}$-subgroup H_2 and, moreover, $G^{\mathfrak{F}}$ is nilpotent, then the subgroups H_1 and H_2 coincide and are $\mathfrak{F}$-normalizers of G.

THEOREM 1. Let f_1 and f_2 be two local screens. Suppose that $\mathfrak{X}$ is a certain class of groups, and f_1^* and f_2^* are also local $\mathfrak{X}$-screens such that $f_i^*(p) = \mathfrak{G}_p f_i(p)$ for each prime p, i = 1, 2. Then the following statements are equivalent:

1) $f_1^* \leqslant f_2^*$;

2) Each f_1-central principal factor of an arbitrary group G belonging to $\mathfrak{X}$ is f_2-central.

Proof. Suppose that $f_1^* \leqslant f_2$. It is obvious that each f_1-central principal factor of arbitrary group G is f_1^*-central. Let H/K be an f_1-central principal factor of G. It is easily seen that $G/C_G(H/K)$ does not have nontrivial normal p-subgroups for any prime $p \in \pi(H/K)$. Consequently, H/K is an f_1-central principal factor of G. Hence $\langle f_1\rangle = \langle f_1^*\rangle$. Analogously, $\langle f_2\rangle = \langle f_2\rangle$. Therefore $\langle f_1\rangle \subseteq \langle f_2\rangle$, and statement 2) is valid.

Suppose that statement 2) holds. We can set $f_1 = f_1^*$ and $f_2 = f_2^*$. Let us suppose that there exists a prime number p such that $f_1(p)$ is not contained in $f_2(p)$. Then the formations $f_1(p)$ and $f_2(p)$ are obviously nonempty. Let G be a group of the least order in $f_1(p)$ that does not belong to $f_2(p)$. Then $G^{f_2(p)}$ is obviously the only minimal normal subgroup of G. It is easily seen that $O_p(G) = 1$. Let us consider $\Gamma = C_p \wr G$ — the regular interlacing of the cyclic group of order p with the group G. Then $\Gamma = N \leftthreetimes G$, where N is an elementary Abelian p-group. It is obvious that $\Gamma \in f_1(p)$ and $N = O_p(\Gamma) = F_p(\Gamma)$. Since each principal factor of the group Γ whose order divides p is, by the condition, f_2-central, it follows that $\Gamma^{f_2(p)} \subseteq F_p(\Gamma)$. It follows from Lemma 1 that $G^{f_2(p)} \subseteq \Gamma^{f_2(p)}$. Therefore $G^{f_2(p)} \subseteq N$. But this is possible only when $G \in f_2(p)$. We have obtained a contradiction. The theorem is proved.

A screen f is said to be inner [1] if $f(G) \subseteq \langle f\rangle$ for each group G.

COROLLARY. Let f_1 and f_2 be maximal (with respect to embedding) inner local screens of the formations $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively. Then the following statements are equivalent:

1) $f_1 \leqslant f_2$;

2) $\mathfrak{F}_1$ is a subformation of $\mathfrak{F}_2$.

Definition. Let f_1 and f_2 be local screens. Define $\mathfrak{H}$ as the class of all those groups which have an f_i-$\mathcal{DM}$-subgroup, i = 1, 2. Let $\mathfrak{H}^*$ denote the set of all those groups G in $\mathfrak{H}$, for which the following condition is fulfilled: If H_i is an f_i-$\mathcal{DM}$-subgroup of G, i = 1, 2, then $H_1 \cap H_2$ is an $(f_1 \cap f_2)$-$\mathcal{DM}$-subgroup of G.

THEOREM 2. Let f_1 and f_2 be local screens and let f_1^* and f_2^* be local $\mathfrak{H}^*$-screens such that $f_i^*(p) = \mathfrak{G}_p f_i(p)$ for each prime p, i = 1, 2. Then the following statements are equivalent:

1) $f_1^* \leqslant f_2^*$;

2) Each f_1-$\mathcal{DM}$-subgroup of an arbitrary group G from $\mathfrak{H}^*$, is contained in an f_2-$\mathcal{DM}$-subgroup of G.

Proof. Let $f_1^* \leqslant f_2^*$, G be a group from $\mathfrak{H}^*$, and H_i be an f_i-$\mathcal{DM}$-subgroup of G for i = 1, 2. By Theorem 1, each f_1-central principal factor of an arbitrary group G from $\mathfrak{H}^*$ is f_2-central. Then the subgroup $H_1 \cap H_2$ covers each f_2-central principal factor of G and therefore, by Lemma 3, $|H_1| \leqslant |H_1 \cap H_2|$. Consequently, $H_1 \subseteq H_2$.

Suppose that each f_1-$\mathcal{DM}$-subgroup H_1 of an arbitrary group G from $\mathfrak{H}^*$ is contained in an f_2-$\mathcal{DM}$-subgroup H_2 of G. Let R/S be an f_1-central principal factor of G. It is obvious that H_2 covers R/S. Consequently, R/S is an f_2-central principal factor of G. It follows by Theorem 1 that $f_1^* \leqslant f_2^*$. The theorem is proved.

A formation $\mathfrak{X}$ is said to be S-closed if it is closed with respect to taking of subgroups.

THEOREM 3. Suppose that $\mathfrak{F}$ and $\mathfrak{X}$ are formations such that $\mathfrak{F} \subseteq \mathfrak{X}$, $\mathfrak{F}$ is local, $\mathfrak{X}$ is S-closed, and each group from $\mathfrak{X}$ has solvable $\mathfrak{F}$-coradical. Let f be a local screen of the formation $\mathfrak{F}$, f_1 be a local $\mathfrak{X}$-screen such that $f_1(p) = \mathfrak{G}_p f(p)$ for each prime p, and f_2 be a local screen such that $f_2(p) = f_1(p) \cap \mathfrak{F}$ for each prime p. If each group G from $\mathfrak{X}$ has an f_i-$\mathcal{DM}$-subgroup (i = 1, 2) and each f_2-$\mathcal{DM}$-subgroup of an arbitrary f_1-$\mathcal{DM}$-subgroup of G is an f_2-$\mathcal{DM}$-subgroup of G, then $f_1 = f_2$.

Proof. By Lemma 1.3 of [1], the local screen f_2 is an inner local screen of the formation $\mathfrak{F}$. It is obvious that $f_2 \leqslant f_1$. We prove that $f_1 \leqslant f_2$. Let us suppose that there exists a prime p such that $f_1(p)$ is not contained in $\mathfrak{F}$. Obviously, $f_1(p) = \varnothing$. We select a group G of the least order in the class $f_1(p) \setminus \mathfrak{F}$. Then G has a unique minimal normal subgroup K, which coincides with $G^{\mathfrak{F}}$. Since the formation $\mathfrak{F}$ is saturated, it follows that $K = C_G(K)$. Obviously, $O_p(G) = 1$. Consequently, by Lemma 2, the group G is the irreducible group of automorphisms of a p-group N. Let $\Gamma = N \rtimes G$ be the extension of the group G by means of N. It is obvious that N is an f_1-central principal factor of the group Γ. Let F* be an f_1-$\mathcal{DM}$-subgroup of Γ. Then it is easily seen that F^*/N is an f_1-$\mathcal{DM}$-subgroup of Γ/N. It follows by Lemma 1.2 of [3] that $(\Gamma/N)^{\mathfrak{F}}$ is nilpotent. Therefore, using Lemma 3, we see that F^*/N is an $\mathfrak{F}$-projection of the group Γ/N. Since $\Gamma/N = (F^*/N)$ F(Γ/N), it follows from Lemma 1.5 of [6] that $\in f_1(p)$. But then it is easily seen that $F^* \in \mathfrak{F}$. Let F be an f_2-$\mathcal{DM}$-subgroup of the group F^*. Obviously, $F = F^*$. Consequently, F^* is an f_2-$\mathcal{DM}$-subgroup of Γ, and therefore

$$G \cong \Gamma/N \in f_2(p) \subseteq \mathfrak{F}.$$

We have obtained a contradiction. The theorem is proved.

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