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Analytic geometry in the space

Study guide

for the self-organized work of the students of the specialty "Applied Informatics"

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This study guide is intended for self-organized work of the first-year students of the Mathematical Faculty taught with a specialization in ‘Applied Informatics’. Theoretical material is outlined; the examples of problems solution are presented. Problems for solving in practical classes and individual assignments are also attached.

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Content

Introduction

This study guide is intended for self-organized work of the students of the Faculty of Mathematics taught with a specialization "Applied Informatics". It combines the lecture notes on the subject "Analytical Geometry" (section "Analytical Geometry in the space"), examples of problems solving, problems for the solution at the practical classes and individual tasks for a self-sustaining solutions. The volume of material calculated on the basis of the theoretical training time available taking into account students' self-organized work. This study guide can't completely replace the lecture and the practical courses on Analytic Geometry. Many results are given without proofs, and some topics are omitted.

Number of options for an individual practice assignment is to be chosen in accordance with the serial number of the student in the teacher's records. Before solving the problem, examine the example of its solution. In case if there are unforeseen difficulties, you should consult the teacher.

Icon ■ in the text signifies the completion of the proof.

Chapter 1. Vector algebra and coordinate systems

§1. *Cartesian coordinate system in the space*

Let three perpendicular axes are chosen in the space and let O be their point of intersection. We call O *the initial point of the coordinate system (or the origin)* and the axes are called *the coordinate axes*. We denote them as Ox, Oy, and Oz. In addition we require, that $90°$ rotation, which matches the positive x-direction with the positive y-direction, should come counterclockwise, if one look at the plane Oxy from the positive z-direction. Then we say that three axes together with the point O form the *Cartesian coordinate system* $Oxyz$ in the space. Planes Oxy, Oxz and Oyz are called *coordinate planes*.

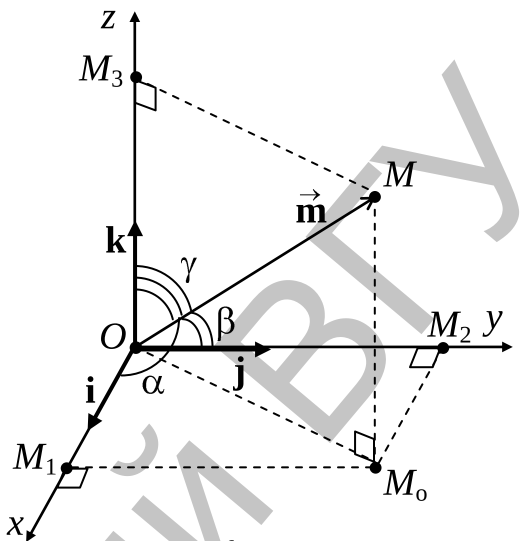


*fig.*1

Let M be an arbitrary point in the space. Let's drop a perpendicular MM_o to the plane Oxy. Then M_o is called the projection of the point M on the plane Oxy. Also we draw a perpendicular MM_3 to the axis Oz. Let the point M_o have coordinates (x, y) on the plane Oxy, and the point M_3 has a coordinate z on the axis Oz. Then we say, that the point M has coordinates (x, y, z) in the space and we write $M(x, y, z)$. The coordinate x is called *the abscissa* of the point M, and the coordinate y is called *the ordinate* of the point M, and the coordinate z is called *the applicate*. The triple of numbers (x, y, z) is called *the Cartesian coordinates of the point* M.

The unit vectors, which co-directed with the positive directions of the coordinate axes are denoted as $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, $\mathbf{k}$ and the are called *the basic orts*.

The vector $\vec{\mathbf{m}}=\overrightarrow{OM}$ is called *the radius- vector* of the point M. Let's consider further $\vec{\mathbf{m}}$ as an arbitrary vector, because the point M was an arbitrary one. According to the rule of parallelogram, $\vec{\mathbf{m}}=\overrightarrow{OM_o}+\overrightarrow{OM_3}$ and $\overrightarrow{OM_o}=\overrightarrow{OM_1}+\overrightarrow{OM_2}$. Hence,

$$\vec{\mathbf{m}}=\overrightarrow{OM_1}+\overrightarrow{OM_2}+\overrightarrow{OM_3}.$$

We see, that $\overrightarrow{OM_1}\,||\,\vec{\mathbf{i}}$, $\overrightarrow{OM_2}\,||\,\vec{\mathbf{j}}$, $\overrightarrow{OM_3}\,||\,\vec{\mathbf{k}}$. According to the first criterion of collinearity, there are numbers x, y, z, such that

$$\overrightarrow{OM_1}=x\vec{\mathbf{i}},\quad \overrightarrow{OM_2}=y\vec{\mathbf{j}},\quad \overrightarrow{OM_3}=z\vec{\mathbf{k}}.$$

Finally, we get

$$\vec{\mathbf{m}}=x\vec{\mathbf{i}}+y\vec{\mathbf{j}}+z\vec{\mathbf{k}}. \tag{1}$$

This equality is called a decomposition of the vector $\vec{\mathbf{m}}$ by the basis $\{\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}}\}$. The triple of numbers (x, y, z) is called the coordinates of the vector $\vec{\mathbf{a}}$. We write as follows: $\vec{\mathbf{m}}(x, y, z)$. We see, that x coincides with a coordinate of the point M_1 on the axis Ox and y coincides with a coordinate of the point M_2 on the axis Oy and z coincides with a coordinate of the point M_3 on the axis Oz. Thus the point M has coordinates just the same as the vector $\vec{\mathbf{m}}(x, y, z)$.

So, everybody must learn the rule: *coordinates of a point coincide with coordinates of its radius vector*.

Let $\alpha = \angle(\vec{\mathbf{i}}, \vec{\mathbf{m}})$, $\beta = \angle(\vec{\mathbf{j}}, \vec{\mathbf{m}})$, $\gamma = \angle(\vec{\mathbf{k}}, \vec{\mathbf{m}})$ be the angles between coordinate axes and the vector $\vec{\mathbf{m}}$. Then the values $\cos\alpha, \cos\beta, \cos\gamma$ are called *the directing cosines of the vector* $\vec{\mathbf{m}}$ направляющими косинусами вектора $\vec{\mathbf{d}}$. They has the property:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

In the same way as on the plane we can prove, that

$$\begin{cases} x = |\vec{\mathbf{m}}|\cos\alpha, \\ y = |\vec{\mathbf{m}}|\cos\beta, \\ z = |\vec{\mathbf{m}}|\cos\gamma. \end{cases}$$

In particular, if the vector $\vec{\mathbf{m}}$ is unit, then it has coordinates

$$\vec{\mathbf{m}}(\cos\alpha, \cos\beta, \cos\gamma).$$

The following theorem is proved in the space analogously as on the plane.

***Theorem* 1** (*the second criterion of collinearity of two vectors*). *Two non-null vectors are collinear if and only if there's coordinates are proportional*:

$$\vec{\mathbf{a}}(x_1, y_1, z_1) \,||\, \vec{\mathbf{b}}(x_2, y_2, z_2) \iff \frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}.$$

Suppose, that two points are given in the space: $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $\vec{\mathbf{c}} = \overrightarrow{AB}$. Then

$$\vec{\mathbf{c}}(x_2 - x_1, y_2 - y_1, z_2 - z_1). \tag{2}$$

So we can formulate the rule: *to find the coordinates of the vector, one must subtract the coordinates of its end from the coordinates of its origin*.

Let's remind now a definition.

Definition. Let AB be a segment and a point C lies on the straight line AB. We say that the point C *divides the segment* AB *in the ratio* $\lambda_1 : \lambda_2$, if the equality

$$\lambda_2 \overrightarrow{AC} = \lambda_1 \overrightarrow{CB}. \tag{3}$$

takes place. The number $\lambda = \lambda_1/\lambda_2$ $(\overrightarrow{AC} = \lambda\overrightarrow{CB})$ is called *the simple ratio of three points* A, B, C and we denote it as (AB, C).

Suppose, that we know the coordinates: $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, and we know $\lambda = (AB, C)$. The problem is to find the unknown coordinates $C(x, y)$. The following formulas can be proved like on the plain:

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda},\ \ y = \frac{y_1 + \lambda y_2}{1 + \lambda},\ \ z = \frac{z_1 + \lambda z_2}{1 + \lambda}. \tag{4}$$

In particular, if C divides the segment AB on two equal parts, then

$$x = \frac{x_1 + x_2}{2},\ \ y = \frac{y_1 + y_2}{2},\ \ z = \frac{z_1 + z_2}{2}. \tag{5}$$

So, the coordinates of the midpoint of the segment AB are the arithmetic means of the coordinates of the points A and B.

§2. *The scalar product in the space and the computation formula for the scalar product*

The definition of the scalar product in the space

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\angle(\vec{\mathbf{a}}, \vec{\mathbf{b}})$$

and its properties are the same, as on the plane. For example, we can uncover the parentheses, as if the vectors are numbers. We are going to remind **the** very important theorem.

Theorem 2. **1.** *A scalar square of a vector is equal to the square of its length* $(\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = |\vec{\mathbf{a}}|^2)$.

2. *Non-null vectors* $\vec{\mathbf{a}}$ *and* $\vec{\mathbf{b}}$ *are perpendicular if and only if its scalar product is equal to zero* $(\vec{\mathbf{a}} \perp \vec{\mathbf{b}} \Leftrightarrow \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0)$.

Let the Cartesian coordinate system be given in the space, and $\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}}$ be the basic orts. We know, that these vectors are unit and they are mutually perpendicular. Consequently

$$\vec{\mathbf{i}} \cdot \vec{\mathbf{i}} = \vec{\mathbf{j}} \cdot \vec{\mathbf{j}} = \vec{\mathbf{k}} \cdot \vec{\mathbf{k}} = 1,\ \ \vec{\mathbf{i}} \cdot \vec{\mathbf{j}} = \vec{\mathbf{i}} \cdot \vec{\mathbf{k}} = \vec{\mathbf{j}} \cdot \vec{\mathbf{k}} = 0, \tag{5}$$

and it is true for their products in the inverse order.

Let $\vec{\mathbf{a}}(a_1, a_2, a_3)$, $\vec{\mathbf{b}}(b_1, b_2, b_3)$. According to the properties of the scalar product

$$\begin{aligned}\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} &= (a_1\vec{\mathbf{i}} + a_2\vec{\mathbf{j}} + a_3\vec{\mathbf{k}}) \cdot (b_1\vec{\mathbf{i}} + b_2\vec{\mathbf{j}} + b_3\vec{\mathbf{k}}) = \\ &= a_1b_1\vec{\mathbf{i}} \cdot \vec{\mathbf{i}} + a_1b_2\,\vec{\mathbf{i}} \cdot \vec{\mathbf{j}} + a_1b_3\,\vec{\mathbf{i}} \cdot \vec{\mathbf{k}} + a_2b_1\vec{\mathbf{j}} \cdot \vec{\mathbf{i}} + a_2b_2\,\vec{\mathbf{j}} \cdot \vec{\mathbf{j}} + a_2b_3\,\vec{\mathbf{j}} \cdot \vec{\mathbf{k}} + \\ &\quad + a_3b_1\mathbf{k} \cdot \vec{\mathbf{i}} + a_3b_2\,\mathbf{k} \cdot \vec{\mathbf{j}} + a_3b_3\vec{\mathbf{k}} \cdot \vec{\mathbf{k}}.\end{aligned}$$

We use (6) and as a final result we get the main formula and its implications:

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_1b_1 + a_2b_2 + a_3b_3. \tag{6}$$

$$\Rightarrow\ \vec{\mathbf{a}}^{\,2} = \vec{\mathbf{a}}\cdot\vec{\mathbf{a}} = a_1^2 + a_2^2 + a_3^2 \qquad (7)$$

$$\Rightarrow\ |\vec{\mathbf{a}}| = \sqrt{\vec{\mathbf{a}}^{\,2}} = \sqrt{a_1^2 + a_2^2 + a_3^2} \qquad (8)$$

$$\Rightarrow\ \cos\angle(\vec{\mathbf{a}},\vec{\mathbf{b}}) = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}\,. \qquad (9)$$

If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, then $\vec{AB}\,(x_2 - x_1, y_2 - y_1, z_2 - z_1) \Rightarrow$

$$|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\,. \qquad (10)$$

§3. *Vector product*

Let's remind, that we have defined the notion of oriented angle between two vectors on the plane. Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be two non-null vectors. We plot two given non- zero vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ aside from an arbitrary point O: $\vec{\mathbf{a}} = \vec{OA}$, $\vec{\mathbf{b}} = \vec{OB}$. If the shortest rotation from the ray OA to the ray OB is performed counterclockwise (fig. 2), we consider that $\alpha > 0$. If this angle is performed clockwise, we consider that $\alpha < 0$. If $\alpha > 0$, then a pair of vectors $(\vec{\mathbf{a}},\vec{\mathbf{b}})$ is called *the right pair*, and if $\alpha < 0$ it is called *the left pair*.

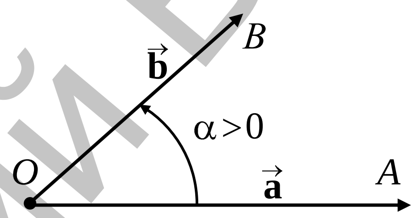


Fig2

These notions have no sense in the space. If we look at the plane OAB from one side, than we will notice, that the shortest rotation from the ray OA to the ray OB is performed in one direction, and if we look at the same plane from the other side, we will notice the shortest rotation is performed in the other direction.

Three vectors in the space are called *coplanar*, if they are parallel to same plane. Suppose that three non-coplanar vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ are given. We plot them from an arbitrary point O:

$$\vec{\mathbf{a}} = \vec{OA},\quad \vec{\mathbf{b}} = \vec{OB},\quad \vec{\mathbf{c}} = \vec{OC}.$$

A triple of the vectors $(\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}})$ is called *the right triple*, if the shortest rotation from the ray OA to the ray OB is performed counterclockwise, if we look at the plane from the point C. If this rotation is performed clockwise, then *a triple of the vectors* $(\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}})$ is called *the left triple*. The right triple $(\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}})$ is drawn on fig. 3.

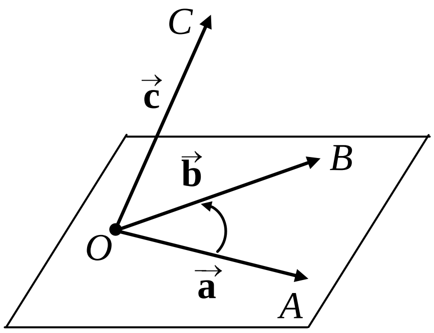


fig.3

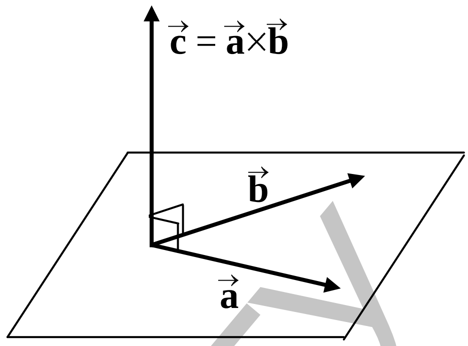


fig.4

Definition. *The vector product of two non-null vectors* $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is the vector $\vec{\mathbf{c}}$, such that

1. $\vec{\mathbf{c}} \perp \vec{\mathbf{a}}$, $\vec{\mathbf{c}} \perp \vec{\mathbf{b}}$;

2. the triple $(\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}})$ is the right one;

3. $|\vec{\mathbf{c}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin\angle(\vec{\mathbf{a}}, \vec{\mathbf{b}})$.

(fig. 3). We write $\vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$ (the following notation is also common used: $\vec{\mathbf{c}} = [\,\vec{\mathbf{a}}, \vec{\mathbf{b}}\,]$).

If one of the vectors (or both vectors) is null, then $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{o}}$.

The vector product can be also called *the cross product*.

***Теорема* 3.** *The modulus of the vector product of two vectors* $\vec{\mathbf{a}}$ *and* $\vec{\mathbf{b}}$ *is numerically equal to the area of the parallelogram, which is drawn on the directed segments* $\overrightarrow{OA}$ *and* $\overrightarrow{OB}$ *presenting these vectors laid aside from the one point* (fig. 5).

Proof. $S = |\overrightarrow{OA}|\,|\overrightarrow{OB}|\sin\angle AOB =$

$= |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\sin\angle(\vec{\mathbf{a}}, \vec{\mathbf{b}}) = |\vec{\mathbf{a}} \times \vec{\mathbf{b}}|$. ■

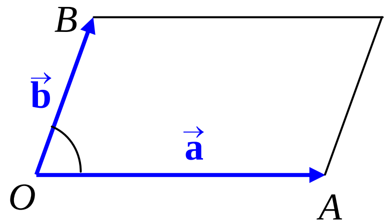


fig.5

Corollary. *Two vectors are collinear if and only if their vector product is equal to the null vector*: $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{o}} \;\Leftrightarrow\; \vec{\mathbf{a}} \,||\, \vec{\mathbf{b}}$. *In particular,* $\vec{\mathbf{a}} \times \vec{\mathbf{a}} = \vec{\mathbf{o}}$ *for any vector* $\vec{\mathbf{a}}$.

Indeed, $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{o}} \;\Leftrightarrow\; S = 0 \;\Leftrightarrow\;$ the sides of the parallelogram are parallel or the length of one side is equal to zero. Remind that the null vector is collinear for any vector.

Properties of the vector product.

1. $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}$,

2. $(\lambda\vec{\mathbf{a}}) \times \vec{\mathbf{b}} = \lambda(\vec{\mathbf{a}} \times \vec{\mathbf{b}})$,

3. $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}}$.

We are going to prove these properties, after we get the computation formula for the vector product in the Cartesian coordinate system.

§4. *Formula for the computation the vector product*

***Теорема* 3.** *Let coordinates of two vectors are given in the Cartesian coordinate system*: $\vec{\mathbf{a}}(a_1, a_2, a_3)$, $\vec{\mathbf{b}}(b_1, b_2, b_3)$. *Then its vector product is calculated by the formula*:

$$\vec{\mathbf{a}}\times\vec{\mathbf{b}}=\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\vec{\mathbf{i}}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\vec{\mathbf{j}}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\vec{\mathbf{k}}=$$
$$=(a_1b_2-a_2b_1)\vec{\mathbf{i}}-(a_1b_3-a_3b_1)\vec{\mathbf{j}}+(a_2b_3-a_3b_2)\vec{\mathbf{k}}. \qquad (11)$$

Proof of the theorem will be given on the lectures.

***Corollary* 1.** $\vec{\mathbf{a}}\times\vec{\mathbf{b}}=-\vec{\mathbf{b}}\times\vec{\mathbf{a}}$.

Really, according to the properties of the determinant, if we change the position of two lines, then the sigh of the determinant changes:

$$\vec{\mathbf{a}}\times\vec{\mathbf{b}}=\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=-\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\end{vmatrix}=-\vec{\mathbf{b}}\times\vec{\mathbf{a}}. \blacksquare$$

***Corollary* 2.** $(\lambda\vec{\mathbf{a}})\times\vec{\mathbf{b}}=\lambda(\vec{\mathbf{a}}\times\vec{\mathbf{b}})$.

Indeed, according to the properties of the determinant, the common factor of the elements of one line is carried over the determinant sign:

$$(\lambda\vec{\mathbf{a}})\times\vec{\mathbf{b}}=\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ \lambda a_1 & \lambda a_2 & \lambda a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\lambda\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\lambda(\vec{\mathbf{a}}\times\vec{\mathbf{b}}). \blacksquare$$

***Corollary* 3.** $\vec{\mathbf{a}}\times(\vec{\mathbf{b}}+\vec{\mathbf{c}})=\vec{\mathbf{a}}\times\vec{\mathbf{b}}+\vec{\mathbf{a}}\times\vec{\mathbf{c}}$.

Actually, according to the properties of the determinant

$$\vec{\mathbf{a}}\times(\vec{\mathbf{b}}+\vec{\mathbf{c}})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}=\vec{\mathbf{a}}\times\vec{\mathbf{b}}+\vec{\mathbf{a}}\times\vec{\mathbf{c}}.\blacksquare$$

***Corollary* 4.** $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{k}\times\mathbf{i}=\mathbf{j}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$.

Prove these equalities independantly, using formula (11).

It is convenient to remember these equalities with the help of the diagram. The product of two orts, taken one after another in the direction of the arrow gives us the third ort. The product of two orts, taken one after another in the opposite direction gives us the third ort with the minus sign.

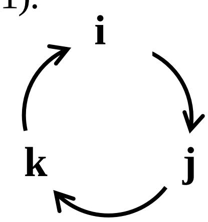


*fig.*6

§5. *The mixed product*

Definition. *The mixed product of tree vectors* $\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}$ is the number $(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}}$. It is denoted as $\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}}$ или $(\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}})$. It also can be called the triple product or the triple scalar product.

***Теорема* 4.** *The modulus of the mixed product of three non-coplanar vec-*

tors $\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}$ *is equal numerically to the volume of the parallelepiped constructed on the directed segments* $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, *which present these vectors laid aside from one point.*

Proof. Let the basis of the parallelepiped is the parallelogram, constructed on the directed segments $\overrightarrow{OA}\ \overrightarrow{OB}$. Let h be the height, dropped from the point C on the basis. Let α be the angle between h and the side OC (fig. 7). Then

$$h = |OC|\cos\alpha\,,\ S_{\text{осн}} = |\vec{\mathbf{a}}\times\vec{\mathbf{b}}|.$$

Let $\beta = \angle(\vec{\mathbf{a}}\times\vec{\mathbf{b}},\, \vec{\mathbf{c}})$.

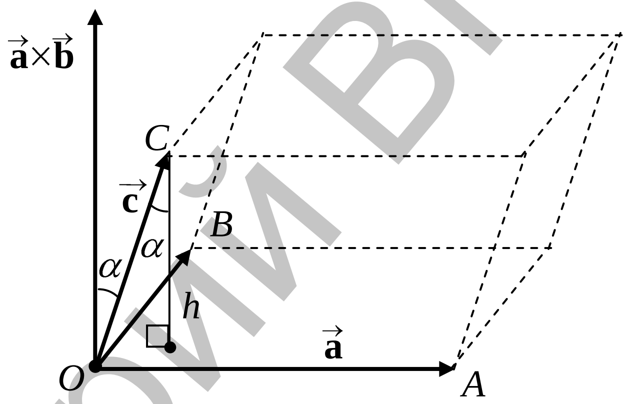


*fig.*7

<u>*First case*</u>. The triple $(\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}})$ is the right one. Then $\beta=\alpha$ (fig.7) and $\cos\alpha=\cos\beta>0$. Hence

$$V = S_{\text{осн}}\cdot h = |\vec{\mathbf{a}}\times\vec{\mathbf{b}}|\,|OC|\cos\alpha = $$
$$= |\vec{\mathbf{a}}\times\vec{\mathbf{b}}|\,|\vec{\mathbf{c}}|\cos\angle(\vec{\mathbf{a}}\times\vec{\mathbf{b}},\vec{\mathbf{c}}) =$$
$$= (\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}} > 0.$$

<u>*Second case*</u>. The triple $(\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}})$ is the left one. Then $\beta=\pi-\alpha>\pi$, $\cos\beta<0$ (fig. 8) and

$$\cos\alpha = -\cos\beta = |\cos\beta| \ \Rightarrow$$

$$V = |\vec{\mathbf{a}}\times\vec{\mathbf{b}}|\,|OC|\cos\alpha =$$
$$= -|\vec{\mathbf{a}}\times\vec{\mathbf{b}}|\,|\vec{\mathbf{c}}|\cos\angle(\vec{\mathbf{a}}\times\vec{\mathbf{b}},\vec{\mathbf{c}}) =$$
$$= -(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}}\,,$$

And we have $(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}}<0$, because $\cos\beta<0$. Hence

$$V = |(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}}\,|.$$

This formula is true in the first case as well. ■

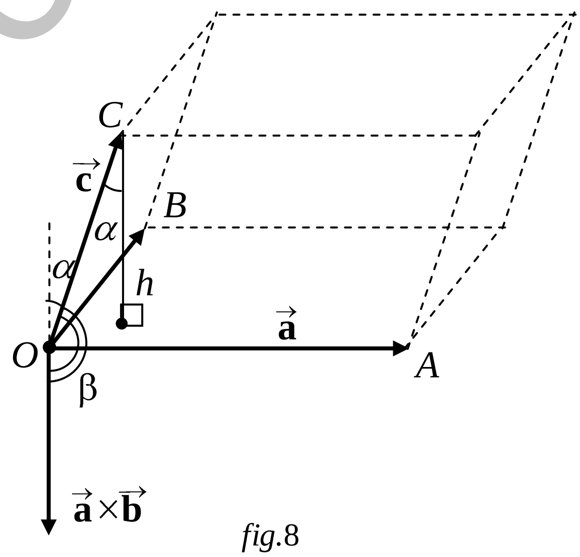


*fig.*8

Corollary. **1.** $\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}$ are coplanar $\Leftrightarrow$ $\vec{\mathbf{a}}\,\vec{\mathbf{b}}\,\vec{\mathbf{c}}=0$;

2. the triple $(\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}})$ is the right one $\Leftrightarrow$ $\vec{\mathbf{a}}\,\vec{\mathbf{b}}\,\vec{\mathbf{c}}>0$;

3. the triple $(\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}})$ is the left $\Leftrightarrow$ $\vec{\mathbf{a}}\,\vec{\mathbf{b}}\,\vec{\mathbf{c}}<0$.

Proof. **1.** $(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}}=0$ $\Leftrightarrow$ $\vec{\mathbf{a}}\times\vec{\mathbf{b}}\perp\vec{\mathbf{c}}$ $\Leftrightarrow$ $\overrightarrow{OC}$ belongs to the basis of the parallelepiped.

Items 2 and 3 have been proved above. ■

Properties of the mixed product.

1. *The mixed product doesn't depend on grouping of factors*: $(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}} = \vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\times\vec{\mathbf{c}})$;

2. *The mixed product doesn't change while cyclic replacement of cofactors*: $\vec{\mathbf{a}}\,\vec{\mathbf{b}}\,\vec{\mathbf{c}} = \vec{\mathbf{c}}\,\vec{\mathbf{a}}\,\vec{\mathbf{b}} = \vec{\mathbf{b}}\,\vec{\mathbf{c}}\,\vec{\mathbf{a}}$.

3. *The mixed product changes the sign while replacement of any two cofactors*: $\vec{\mathbf{a}}\,\vec{\mathbf{b}}\,\vec{\mathbf{c}} = -\vec{\mathbf{b}}\,\vec{\mathbf{a}}\,\vec{\mathbf{c}} = -\vec{\mathbf{a}}\,\vec{\mathbf{c}}\,\vec{\mathbf{b}} = -\vec{\mathbf{c}}\,\vec{\mathbf{b}}\,\vec{\mathbf{a}}$.

4. $(\lambda\vec{\mathbf{a}})\vec{\mathbf{b}}\vec{\mathbf{c}} = \vec{\mathbf{a}}\,(\lambda\vec{\mathbf{b}})\,\vec{\mathbf{c}} = \vec{\mathbf{a}}\,\vec{\mathbf{b}}\,(\lambda\vec{\mathbf{c}}) = \lambda(\vec{\mathbf{a}}\,\vec{\mathbf{b}}\,\vec{\mathbf{c}})$.

5. $(\vec{\mathbf{a}}+\vec{\mathbf{b}})\,\vec{\mathbf{c}}\,\vec{\mathbf{d}} = \vec{\mathbf{a}}\,\vec{\mathbf{c}}\,\vec{\mathbf{d}} + \vec{\mathbf{b}}\,\vec{\mathbf{c}}\,\vec{\mathbf{d}}$.

Proof. **1.** Using the computation formulas for vector and scalar products we get

$$(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}\right)\cdot(c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}) =$$

$$= c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

This expression represents by itself the decomposition of the third order determinant across the first line. We replace the first line of the determinant on the third place:

$$(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

From the other side,

$$\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\times\vec{\mathbf{c}}) = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\cdot\left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\mathbf{k}\right) =$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \blacksquare$$

It is just this property that allows us to use the notation $\vec{\mathbf{a}}\,\vec{\mathbf{b}}\,\vec{\mathbf{c}}$ without multiplication signs and brackets. Simultaneously we have proved the formula

$$\vec{\mathbf{a}}\,\vec{\mathbf{b}}\,\vec{\mathbf{c}} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad (12)$$

All the other properties of the mixed product are the implications of the analogous properties of the determinant.

§6. *The spherical and the cylindrical coordinate systems*

Let the Cartesian coordinate system $Oxyz$ is given in the space and let $M(x, y, z)$ be an arbitrary point. We drop a perpendicular MM_o on the plane Oxy. Then it is obvious, that $|MM_o| = |z|$. Let's denote $\rho = |OM|$, $\psi = \angle M_oOM$; and we consider that $\psi > 0$, if $z > 0$, and $\psi < 0$, if $z < 0$ (fig. 9).

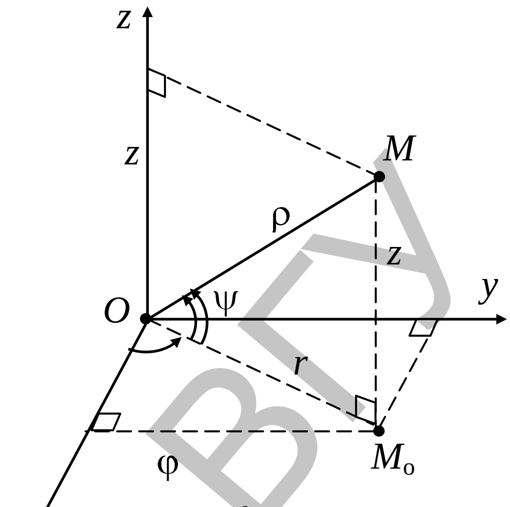


fig.9

Let (r, φ) be the polar coordinates of the point M_o on the plane Oxy. Then the triple (r, φ, ψ) is called *the spherical coordinates of the point* M, a and the triple (r, φ, z) is called *the cylindrical coordinates*. It is obvious, that $0 \le \rho < +\infty$, $-\pi/2 \le \psi \le \pi/2$. If $\psi = \pm\pi/2$, then the point M belongs to the axis Oz, $M_o = O$ and in this case the angle φ is considered to be undefined.

Now we need to find formulas, which connect the Cartesian, the spherical and the cylindrical coordinates of the same point M. From the rectangular triangle ΔOMM_o we find that

$$\begin{cases} r = \rho\cdot\cos\psi, \\ z = \rho\cdot\sin\psi. \end{cases} \qquad (13) \qquad\qquad \begin{cases} \rho = \sqrt{r^2 + z^2}, \\ \psi = \arcsin\dfrac{z}{\rho}. \end{cases} \qquad (13')$$

These formulas we can consider, as a transition from the spherical coordinates to the cylindrical ones and backwards; and the angle φ is common for these coordinate systems. We proved earlier the formulas

$$\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi. \end{cases} \qquad (14) \qquad\qquad \begin{cases} r = \sqrt{x^2 + y^2}, \\ \cos\varphi = x/r, \ \ \sin\varphi = y/r. \end{cases} \qquad (14')$$

These formulas connect the polar and the Cartesian coordinates on the plain. At the same time we can consider them, as the formulas, which connect the cylindrical and the Cartesian coordinates in the space.

If we substitute (13) in (14), then we get the formulas of transition from the spherical coordinates to the Cartesian ones. If we substitute (14′) in (13′), then we get the formulas of transition from the Cartesian coordinates to the spherical ones.

$$\begin{cases} x = \rho\cos\varphi\cdot\cos\psi, \\ y = \rho\sin\varphi\cdot\cos\psi, \\ z = \rho\cdot\sin\psi. \end{cases} \qquad (15) \qquad\qquad \begin{cases} \rho = \sqrt{x^2 + y^2 + z^2}, \\ \varphi = \pm\arccos\dfrac{x}{\sqrt{x^2 + y^2}}, \\ \psi = \arcsin(z/\rho). \end{cases} \qquad (15')$$

The sign in the second formula in (15′) should be chosen the same as the sign of y.

§7. *Examples of solving problems*

1. *It is given that* $|\vec{\mathbf{m}}| = 10$, $|\vec{\mathbf{n}}| = 3$, $\alpha = \angle(\vec{\mathbf{m}}, \vec{\mathbf{n}}) = 30^{\circ}$. *Vectors* $\vec{\mathbf{a}} = \vec{\mathbf{m}} - 3\vec{\mathbf{n}}$ *and* $\vec{\mathbf{b}} = \vec{\mathbf{m}} + 5\vec{\mathbf{n}}$ *are plotted from one point. Find the area of the triangle spanned by this vectors. Find the length of the median coming from the same point.*

Решение. The area of the parallelogram spanned by the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is equal to the modulus of its vector product. The area of the triangle spanned by this vectors is the half of this value: $S_{\Delta} = \frac{1}{2}|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|$. Using the definition and the properties of the vector product we can find

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |(\vec{\mathbf{m}} - 3\vec{\mathbf{n}}) \times (\vec{\mathbf{m}} + 5\vec{\mathbf{n}})| = |\vec{\mathbf{m}} \times \vec{\mathbf{m}} + 5\vec{\mathbf{m}} \times \vec{\mathbf{n}} - 3\vec{\mathbf{n}} \times \vec{\mathbf{m}} - 15\vec{\mathbf{n}} \times \vec{\mathbf{n}}| =$$

$$= |\vec{\mathbf{o}} + 5\vec{\mathbf{m}} \times \vec{\mathbf{n}} + 3\vec{\mathbf{m}} \times \vec{\mathbf{n}} + 15\vec{\mathbf{o}}| = 8|\vec{\mathbf{m}} \times \vec{\mathbf{n}}| = 8|\vec{\mathbf{m}}| \cdot |\vec{\mathbf{n}}| \cdot \sin\alpha =$$

$$= 8 \cdot 10 \cdot 3 \cdot \frac{1}{2} = 120 .$$

$$S_{\Delta} = \frac{1}{2}|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = 60 .$$

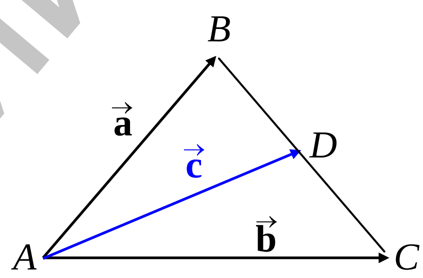


*fig.*10

If AD is the median in ΔABC, then $\vec{AD} = \frac{1}{2}(\vec{AB} + \vec{AC})$ (fig. 10). Let $\vec{\mathbf{c}} = \vec{AD}$. In our task $\vec{\mathbf{c}} = \frac{1}{2}(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = \vec{\mathbf{m}} + \vec{\mathbf{n}}$.

We are to find the length of this vector. The very first theorem after the definition of scalar product states: the scalar square of a vector $\vec{\mathbf{c}}^2 = \vec{\mathbf{c}} \cdot \vec{\mathbf{c}}$ is equal to the square of its length $|\vec{\mathbf{c}}|^2$. Thus

$$|\vec{\mathbf{c}}|^2 = \vec{\mathbf{c}} \cdot \vec{\mathbf{c}} = (\vec{\mathbf{m}} + \vec{\mathbf{n}})^2 = \vec{\mathbf{m}}^2 + 2\vec{\mathbf{m}} \cdot \vec{\mathbf{n}} + \vec{\mathbf{n}}^2 =$$

$$= |\vec{\mathbf{m}}|^2 + 2|\vec{\mathbf{m}}| \cdot |\vec{\mathbf{n}}| \cdot \cos\alpha + |\vec{\mathbf{n}}|^2 =$$

$$= 100 + 2 \cdot 10 \cdot 3 \cdot \frac{\sqrt{3}}{2} + 9 = 109 + 30\sqrt{3}.$$

Thus $|\vec{\mathbf{c}}| = \sqrt{109 + 30\sqrt{3}}$.

Answer: $S_{\Delta} = 60$, length of the median is equal $\sqrt{109 + 30\sqrt{3}}$.

We emphasize that one can't use the designation $\vec{\mathbf{m}}^2$ instead of $\vec{\mathbf{m}} \times \vec{\mathbf{m}}$; $\vec{\mathbf{m}}^2$ means $\vec{\mathbf{m}} \cdot \vec{\mathbf{m}}$. Pay attention, that while solving the problem we have used the property $\vec{\mathbf{n}} \times \vec{\mathbf{m}} = -\vec{\mathbf{m}} \times \vec{\mathbf{n}}$.

2. *Prove that vectors* $\vec{\mathbf{a}}(10, 11, 2)$ *and* $\vec{\mathbf{b}}(10,-10, 5)$ *plotted from one point, can be considered as the edges of a cube and find the third edge of the cube plotted from the same point.*

Решение. Vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ can be the edges of a cube if and only if they are perpendicular and have the equal length. Let's check:

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = 10\cdot 11 + 11\cdot(-10) + 2\cdot 5 = 0 \Rightarrow \vec{\mathbf{a}} \perp \vec{\mathbf{b}},$$

$$|\vec{\mathbf{a}}| = \sqrt{10^2 + 11^2 + 2^2} = 15,$$

$$|\vec{\mathbf{b}}| = \sqrt{10^2 + (-10)^2 + 5^2} = 15.$$

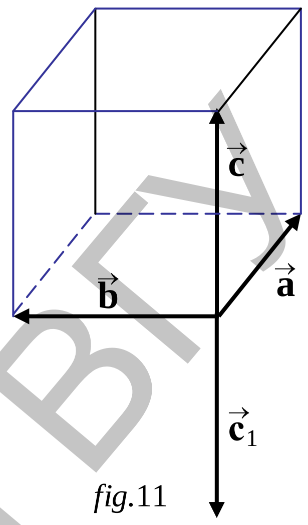


*fig.*11

A vector $\vec{\mathbf{c}}$ can define the third edge of a cube, if and only if it is perpendicular to $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, and has the same length. According to the definition of the vector product the vector $\vec{\mathbf{a}}\times\vec{\mathbf{b}}$ is perpendicular to $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. Let's see what length does it have:

$$|\vec{\mathbf{a}}\times\vec{\mathbf{b}}| = |\vec{\mathbf{a}}|\cdot|\vec{\mathbf{b}}|\cdot\sin\angle(\vec{\mathbf{a}},\vec{\mathbf{b}}) = 15\cdot 15\cdot \sin 90^{\circ} = 225.$$

Vector $\vec{\mathbf{c}}$ must have the length 15. Thus, $\vec{\mathbf{c}} = \dfrac{1}{15}\vec{\mathbf{a}}\times\vec{\mathbf{b}}$. First,we find

$$\vec{\mathbf{a}}\times\vec{\mathbf{b}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & 11 & 2 \\ 10 & -10 & 5 \end{vmatrix} = 75\mathbf{i} - 30\mathbf{j} - 210\mathbf{k},$$

Then $\vec{\mathbf{c}}(5, -2,-14)$. It is obvious, that the vector $\vec{\mathbf{c}}_1 = -\vec{\mathbf{c}}$ is also satisfies the conditions of the problem.

***Answer*:** $\vec{\mathbf{c}}(5, -2,-14)$, $\vec{\mathbf{c}}_1(-5, 2,14)$.

6. *Coordinates of vertexes a triangular pyramid* $SABC$ *are given*: $A(4, 0, 1)$, $B(5,-1, 1)$, $C(4, 7,-5)$, $S(7, 5, 2)$. *Find the volume of the pyramid, the area of the basis* ABC *and the height* (*use the vector and the mixed product*). *Find the angle* $\angle BAC$. *Specify, which vector is perpendicular to the basis. Draw this pyramid in the Cartesian coordinate system*.

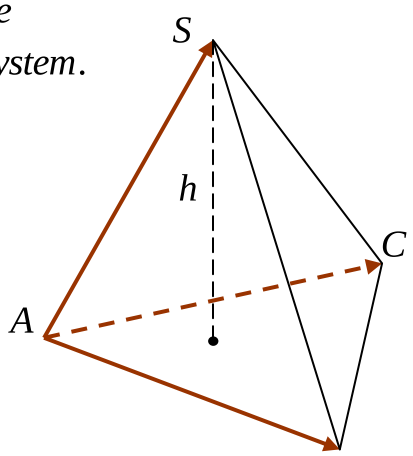


*fig.*12

Решение. First we find coordinates of three vectors, plotted from one vertex, which define the edges of the pyramid:

$$\overrightarrow{AB}\,(1,-1, 0),\quad \overrightarrow{AC}\,(0, 7,-6),\quad \overrightarrow{AS}\,(3, 5, 1).$$

Modulus of mixed product of these vectors is equal to the volume of the parallelepiped spanned by these vectors. The volume of the pyramid is one sixth of the volume of the parallelepiped:

$$V=\frac{1}{6}|\overrightarrow{AB}\overrightarrow{AC}\overrightarrow{AS}|.$$

Mixed product can be calculated as follows:

$$\overrightarrow{AB}\cdot\overrightarrow{AC}\cdot\overrightarrow{AS}=\begin{vmatrix}1&-1&0\\0&7&-6\\3&5&1\end{vmatrix}.$$

But there is another way. To calculate the area of the base, we need vector product $\overrightarrow{AB}\times\overrightarrow{AC}$. Therefore, to calculate the mixed product is easier to use its definition: $\overrightarrow{AB}\cdot\overrightarrow{AC}\cdot\overrightarrow{AS}=(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AS}$. Moreover, the probability of an arithmetic error is much less. One can use the both ways.

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&0\\0&7&-6\end{vmatrix}=\begin{vmatrix}-1&0\\7&-6\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&0\\0&-6\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&-1\\0&7\end{vmatrix}\mathbf{k}=6\mathbf{i}+6\mathbf{j}+7\mathbf{k}.$$

$$S_{\Delta ABC}=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\sqrt{36+36+49}=\frac{11}{2}.$$

$$(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AS}=6\cdot3+6\cdot5+7\cdot1=55.\qquad V=\frac{1}{6}|(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AS}|=\frac{55}{6}.$$

On the other hand, $V=\frac{1}{3}S_{\Delta ABC}\cdot h$. Thus

$$h=\frac{3V}{S_{\Delta ABC}}=\frac{55/2}{11/2}=5.$$

According to the definition of the vector product $\vec{\mathbf{a}}\times\vec{\mathbf{b}}$ is perpendicular to $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. Hence $\vec{\mathbf{h}}=\overrightarrow{AB}\times\overrightarrow{AC}$ is perpendicular to the basis of the pyramid; $\vec{\mathbf{h}}(6,6,7)$.

The angle between two vectors $\vec{\mathbf{a}}(a_1,a_2,a_3)$ and $\vec{\mathbf{b}}(b_1,b_2,b_3)$ can be calculated by the formula

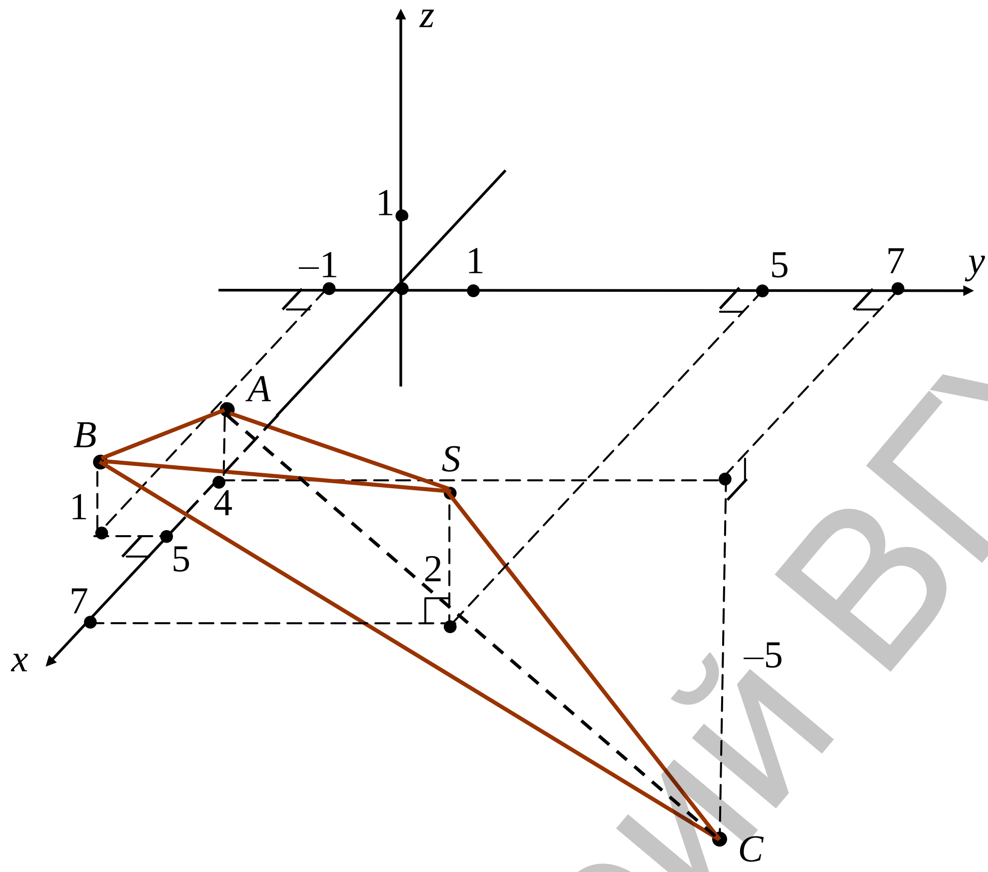
$$\cos\angle(\vec{\mathbf{a}},\vec{\mathbf{b}})=\frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{|\vec{\mathbf{a}}||\vec{\mathbf{b}}|}=\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}.$$

The angle $\angle BAC$ is the angle between two vectors $\overrightarrow{AB}(1,-1,0)$ и $\overrightarrow{AC}(0,7,-6)$. Thus

$$\cos\angle BAC=\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}=\frac{1\cdot0+(-1)\cdot7+0\cdot(-6)}{\sqrt{1^2+(-1)^2+0^2}\sqrt{0^2+7^2+(-6)^2}}=\frac{-7}{\sqrt{2}\sqrt{85}}.$$

$$\angle BAC=\arccos\frac{-7}{\sqrt{170}}=\pi-\arccos\frac{7}{\sqrt{170}}.$$

Let’s make the drawing of the pyramid in the Cartesian coordinate system $Oxyz$.



For example, let's explain how the point $S(7, 5, 2)$ has been drawn. We have drawn the line from the point 5 on the axis Oy parallel to Ox and the line from the point 7 on the axis Ox parallel to Oy. We have got the intersection point. Then we have drawn the segment upwards from this point parallel to Oz, which is equal to 2.

Answer: $V = \dfrac{55}{6}$, $S_{\Delta ABC} = \dfrac{11}{2}$, $h = 5$, $\vec{\mathbf{h}}(6, 6, 7)$.

6. *Spherical coordinates of a point are given*: $A(4\sqrt{3}, 150°, -60°)$. *Find its Cartesian coordinates*.

Solution. We use formulas

$$\begin{cases} x = \rho\cos\varphi\cdot\cos\psi, \\ y = \rho\sin\varphi\cdot\cos\psi, \\ z = \rho\cdot\sin\psi. \end{cases} \qquad (15)$$

We substitute here our data: $\rho = 4\sqrt{3}$, $\varphi = 150°$, $\psi = 60°$.

$$\begin{cases} x = 4\sqrt{3}\cos 150°\cdot\cos(-60°) = 4\sqrt{3}\,\dfrac{\sqrt{3}}{2}\cdot\dfrac{1}{2} = 3, \\ y = 4\sqrt{3}\sin 150°\cdot\cos(-60°) = 4\sqrt{3}\,\dfrac{1}{2}\cdot\dfrac{1}{2} = \sqrt{3}, \\ z = 4\sqrt{3}\cdot\sin(-60°) = 4\sqrt{3}\cdot\left(-\dfrac{\sqrt{3}}{2}\right) = 6. \end{cases}$$

Tasks for the practical classes

Practical classes 1, 2.

Vector and mixed product. Spherical and cylindrical coordinate systems

1. Let $|\vec{\mathbf{a}}|=3$, $|\vec{\mathbf{b}}|=5$, $\angle(\vec{\mathbf{a}},\vec{\mathbf{b}})=60^{\circ}$. Find:
   а) $|\vec{\mathbf{a}}\times\vec{\mathbf{b}}|$; б) $|(\vec{\mathbf{a}}+\vec{\mathbf{b}})\times(\vec{\mathbf{a}}-\vec{\mathbf{b}})|$; в) $|(3\vec{\mathbf{a}}+\vec{\mathbf{b}})\times(\vec{\mathbf{a}}-3\vec{\mathbf{b}})|$.
2. Let $|\vec{\mathbf{a}}|=3$, $|\vec{\mathbf{b}}|=5$, $|\vec{\mathbf{a}}\times\vec{\mathbf{b}}|=72$. Find $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$.
3. What can we say about the vectors $\vec{\mathbf{a}}$ и $\vec{\mathbf{b}}$, if the vectors $\vec{\mathbf{p}}=3\vec{\mathbf{a}}+\vec{\mathbf{b}}$ and $\vec{\mathbf{q}}=\vec{\mathbf{a}}-3\vec{\mathbf{b}}$ are collinear?
4. Calculate the area of a triangle spanned on the vectors $\vec{\mathbf{p}}=\vec{\mathbf{a}}-2\vec{\mathbf{b}}$ and $\vec{\mathbf{q}}=3\vec{\mathbf{a}}+2\vec{\mathbf{b}}$, plotted from one point, if $|\vec{\mathbf{a}}|=|\vec{\mathbf{b}}|=6$, $\angle(\vec{\mathbf{a}},\vec{\mathbf{b}})=45^{\circ}$.
5. Find sine of the angle between the vectors $\vec{\mathbf{a}}(11,10,2)$ и $\vec{\mathbf{b}}(2,2,1)$.
6. Make sure that the vectors $\vec{\mathbf{a}}(2,1,2)$ и $\vec{\mathbf{b}}(-2,2,1)$, plotted from one point can be taken as the cube edges and find the third edge, plotted from the same vertex.
7. Let $\overrightarrow{AB}(6,0,2)$, $\overrightarrow{AC}(1,5;0;1)$. Find the distance between parallel sides of a parallelogram $ABCD$.
8. Calculate the mixed product of vectors and determine their mutual position $\vec{\mathbf{a}}(13,12,11)$, $\vec{\mathbf{b}}(24,23,22)$, $\vec{\mathbf{c}}(35,34,33)$.
9. Coordinates of the vertexes of the triangle pyramid are given: $A(2,3,1)$, $B(4,1,-2)$, $C(6,3,7)$, $D(-5,-4,8)$. Find
   i) the volume of the pyramid; ii) the area of the base;
   iii) Find the altitude drawn from the vertex D.
10. Cartesian coordinates of the points are given: $A(-8,-4,1)$, $B(-2,-2,-1)$, $C(0,-4,3)$. Find there spherical coordinates (ρ,φ,ψ).
11. Find the Cartesian coordinates of the point, that belongs to the sphere of radius 4, if it has latitude 45° and longitude 330°.
12. Find the cylindrical coordinates of the points if their rectangular coordinates are given: $A(3,-4,5)$, $B(1,-1,-1)$, $C(-6,0,8)$.

Home task.

1. The diagonals of a parallelogram are defined by the vectors $\vec{\mathbf{p}}=3\vec{\mathbf{m}}+\vec{\mathbf{n}}$ and $\vec{\mathbf{q}}=\vec{\mathbf{m}}-5\vec{\mathbf{n}}$, plotted from one point. Find the area of a parallelogram if $|\vec{\mathbf{m}}|=|\vec{\mathbf{n}}|=1$, $\angle(\vec{\mathbf{a}},\vec{\mathbf{b}})=45^{\circ}$.

Tasks for the independent solving

1. *Coordinates of vertexes a triangular pyramid $SABC$ are given.*

i) *Find the equation of the basis plane ABC and the equation of the height SD.*

ii) *Find the value of the height.*

iii) *Find coordinates of the point D и точки F, which is symmetric to S with respect to the basal plane.*

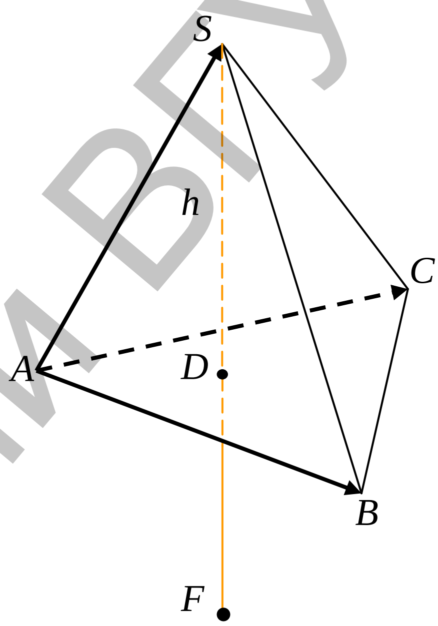


*fig.*25

1. $A(-1, 1, 2)$, $B(-5, 4, -2)$, $C(-1, 2, 3)$, $S(-8, -5, 4)$.
2. $A(0, 2, 2)$, $B(0, 4, 3)$, $C(1, 4, 2)$, $D(7, -1, 7)$.
3. $A(1, 1, 2)$, $B(1, 2, 4)$, $C(4, 1, 4)$, $S(2, -7, 3)$.
4. $A(-1, 1, -2)$, $B(-1, -2, -1)$, $C(1, -2, 0)$, $S(5, -2, -12)$.
5. $A(-5, 1, 2)$, $B(-5, -2, 6)$, $C(-4, 4, -2)$, $S(2, 12, 4)$.
6. $A(-5, 1, 2)$, $B(-5, -2, 6)$, $C(-4, 4, -2)$, $S(2, 12, 4)$.
7. $A(-6, 0, 1)$, $B(-6, -3, 5)$, $C(-5, 3, -3)$, $S(1, 9, -1)$.
8. $A(1, 0, -1)$, $B(2, 0, 4)$, $C(4, 2, 3)$, $S(10, -11, -8)$.
9. $A(-1, 3, 0)$, $B(-1, -1, 2)$, $C(0, 5, -2)$, $S(7, 2, 6)$.
10. $A(1, 4, 2)$, $B(7, 6, 3)$, $C(3, 4, 3)$, $S(6, -7, -7)$.
11. $A(2, 1, 4)$, $B(3, -1, 2)$, $C(3, 7, 6)$, $S(-7, 6, -7)$.
12. $A(1, -1, 0)$, $B(2, 1, 2)$, $C(1, 1, 1)$, $S(3, -2, 7)$.
13. $A(-1, 1, 1)$, $B(-1, 3, 2)$, $C(0, 3, 1)$, $S(6, -2, 6)$.
14. $A(-1, -1, 0)$, $B(-1, 0, 2)$, $C(2, -1, 2)$, $S(0, -9, 1)$.
15. $A(2, -1, 1)$, $B(3, -1, 2)$, $C(-2, -5, 4)$, $S(4, -8, -5)$.

2. *Spherical coordinates of a point are given:* $A(4\sqrt{3}, 150°, -60°)$. *Find its Cartesian coordinates.*

1. $A(4, 135°, 60°)$;
2. $A(8, 120°, -45°)$;
3. $A(6, 150°, 60°)$;
4. $A(4, 135°, -30°)$;
5. $A(4\sqrt{2}, 225°, 60°)$;
6. $A(2\sqrt{2}, 240°, 45°)$;
7. $A(\sqrt{2}, 225°, -30°)$;
8. $A(2\sqrt{2}, 150°, -30°)$;
9. $A(2\sqrt{2}, 135°, -30°)$;
10. $A(4\sqrt{2}, 300°, 45°)$;
11. $A(4\sqrt{2}, 45°, -60°)$;
12. $A(2\sqrt{3}, 120°, 30°)$;
13. $A(8\sqrt{3}, 210°, 60°)$;
14. $A(2\sqrt{2}, 210°, 45°)$.
15. $A(8, 135°, 45°)$.

Chapter 2. Straight lines and plains in the space

§1. *Equation of a surface and a curve in the space.*

Definition. Let the Cartesian coordinate system is given in the space. Let Φ be some surface in the space and $F(x, y, z)$ be a function of three variables. We say that

$$F(x, y, z) = 0 \qquad (1)$$

is *the equation of the surface* Φ *in implicit form*, if coordinates of an arbitrary point $M \in \Phi$ satisfies (16), and vice versa, each triple (x, y, z) of numbers, satisfying (1), defines a point $M(x, y, z)$ on the surface.

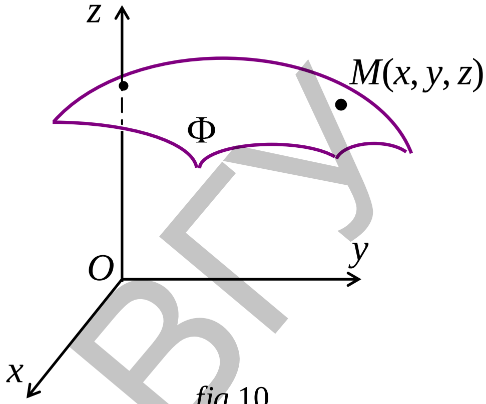


fig.10

We lay emphasis, that while making the equation of a surface one must check the implication in both directions.

***Example* 1.** Let's make the equation of the sphere S^2 with radius $R > 0$ and with the center at the point $O'(a, b, c)$. Let $M(x, y, z)$ be an arbitrary point of the sphere. Then

$$R = |O'M| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} \Leftrightarrow$$

$$\Leftrightarrow (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2. \qquad (2)$$

Backwards, if coordinates of a point $M(x, y, z)$ satisfies (17), then $|O'M| = R$ and $M \in S^2$.

If we manage to express one variable from the equation (16), then we get the equation in *the explicit form*: $z = f(x, y)$. It is impossible to rewrite the equation (2) in the explicit form.

A curve in the space, can't be defined by one equation. But there are some exceptional cases. For example, the equation $x^2 + y^2 = 0$ defines the straight line – axis Oz. As usual, a curve in the space is defined by the system of two equations

$$\begin{cases} F_1(x, y, z) = 0, \\ F_2(x, y, z) = 0. \end{cases} \qquad (3)$$

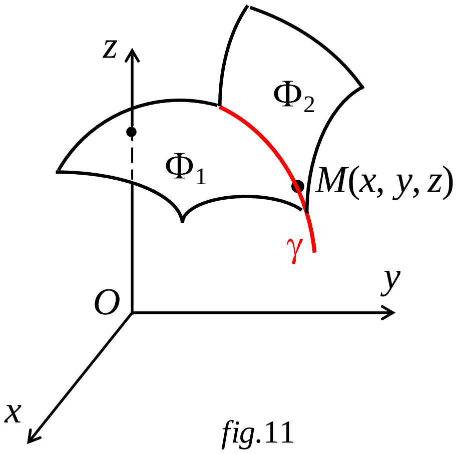


fig.11

Each equation separately defines a surface.
If coordinates of the point satisfies the system, then this point belongs to both surfaces at the same time, i.e. $M \in \Phi_1 \cap \Phi_2$. So system (3) determines the line of intersection of two surfaces.

***Example* 2.** The system of equations

$$\begin{cases} x^2+y^2+z^2=R^2, \\ z=0. \end{cases} \qquad (4)$$

determines a circle in the plain Oxy. The first equation determines the sphere with the center in the origin, and the second one determines the plane Oxy. The intersection of the sphere and the plane is the circle γ.

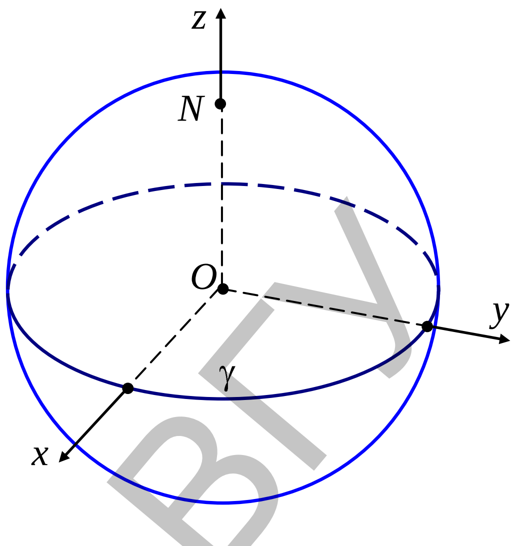


fig.12

Assume that a point moves along the curve γ in the space. Then its coordinates changes with time:

$$\begin{cases} x=x(t), \\ y=y(t), \\ z=z(t). \end{cases} \qquad (4)$$

The parameter t varies within the certain limits: $t\in I$, where I is an interval of the numerical line. We say, that (4) are *parametric equations of the curve* γ, if the following condition is fulfilled: a point $M(x, y, z)$ belongs to γ if and only if one can find such $t\in I$, that all the equalities (4) are true at the same time. And be sure the interval of changing of the parameter t has to be added to system (4).

***Example* 3.** The following equations

$$\begin{cases} x=a\cos t, \\ y=a\sin t, \\ z=bt. \end{cases} \qquad (5)$$

determines the spiral line (the helix). A point rotates around the axis Oz and simultaneously rises up.

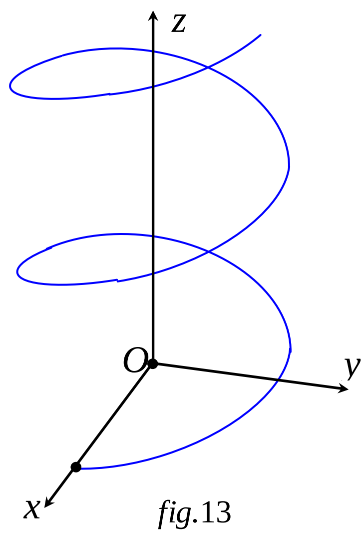


fig.13

§2. *Equations of the plain*

A plain π in the space can be determined

i) by means of a point $M_0\in\pi$ and a non-null vector $\vec{\mathbf{n}}\perp\pi$; then we can write, that $\pi=\{M\,|\,\overrightarrow{M_0M}\perp\vec{\mathbf{n}}\}$ (fig. 14); the vector $\vec{\mathbf{n}}$ is called *the normal vector*;

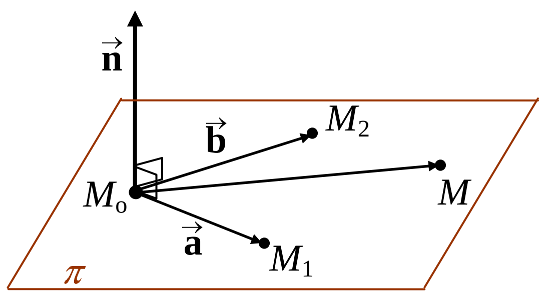


fig.14

ii) by means of a point $M_0\in\pi$ and two non-collinear vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, which are parallel to π;

iii) by means of three points $M_o, M_1, M_2 \in \pi$, which doesn't belong to a straight line.

***Theorem* 1. 1.** *Suppose that a plain* π, *passes through a point* $M_o(x_o, y_o, z_o)$, *and it is perpendicular to a vector* $\vec{\mathbf{n}}(A, B, C)$. *Then it is determined in the Cartesian coordinate system by the equation*

$$A(x - x_o) + B(y - y_o) + C(z - z_o) = 0. \tag{6}$$

2. *Let* $\vec{\mathbf{a}}(a_1, a_2, a_3)$ *and* $\vec{\mathbf{b}}(b_1, b_2, b_3)$ *be non-collinear vectors. If a plain* π, *passes through a point* $M_o(x_o, y_o, z_o)$, *and it is parallel to the vectors* $\vec{\mathbf{a}}$ *and* $\vec{\mathbf{b}}$, *then it is determined in the Cartesian coordinate system by the equation*

$$\begin{vmatrix} x-x_o & y-y_o & z-z_o \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0 \tag{7}$$

Proof. **1.** Let $M(x, y, z)$ be an arbitrary point of the plain. Then $\overrightarrow{M_oM} \perp \vec{\mathbf{n}} \Leftrightarrow \overrightarrow{M_oM} \cdot \vec{\mathbf{n}} = 0$. Therefore $\overrightarrow{M_oM}(x - x_o, y - y_o, z - z_o)$ and $\vec{\mathbf{n}}(A, B, C)$. So the least equality in coordinates is written as (6).

Backwards. suppose that coordinates of the point $M(x, y, z)$ satisfy (6). Then $\overrightarrow{M_oM} \perp \vec{\mathbf{n}}$ and it means, that $M \in \pi$.

2. Let $M(x, y, z)$ be an arbitrary point of the plain. Then $\overrightarrow{M_oM}$ is coplanar to the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and this implies, that the mixed product of these three vectors is equal to zero: $\overrightarrow{M_oM}\vec{\mathbf{a}}\vec{\mathbf{b}} = 0$. The least equality in coordinates is written as (7).

Backwards, suppose that coordinates of the point $M(x, y, z)$ satisfy (7). Then the equality $\overrightarrow{M_oM}\,\vec{\mathbf{a}}\,\vec{\mathbf{b}} = 0$ is true. Thus three vectors $\overrightarrow{M_oM}, \vec{\mathbf{a}}, \vec{\mathbf{b}}$ are coplanar, and it means, that $M \in \pi$. ■

***Corollary* 1.1.** *Let* $M_o(x_o, y_o, z_o)$, $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ *be arbitrary points, which aren't belong to one straight line. If a plain* π, *passes through the points* M_o, M_1, M_2, *then it is determined in the Cartesian coordinate system by the equation*

$$\begin{vmatrix} x-x_o & y-y_o & z-z_o \\ x_1-x_o & y_1-y_o & z_1-z_o \\ x_2-x_o & y_2-y_o & z_2-z_o \end{vmatrix} = 0 \tag{8}$$

Proof. Let a plane passes through the points $M_o(x_o, y_o, z_o)$, $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$, which aren't belong to one straight line. Then the vectors

$\overrightarrow{M_oM_1}(x_1 - x_o, y_1 - y_o, z_1 - z_o)$ and $\overrightarrow{M_oM_2}(x_2 - x_o, y_2 - y_o, z_2 - z_o)$ are not collinear and they are parallel to the plane π. We substitute their coordinates in (7) instead of coordinates of $\vec{\mathbf{a}}$ и $\vec{\mathbf{b}}$, and we get (8). ■

***Corollary* 1.2.** *Assume that a plane* π *cats off the coordinate axes non-null segments* a, b, c. *Then it is determined in the Cartesian coordinate system by the equation*

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \tag{9}$$

(*it is supposed, that* a, b, c *can be negative*).

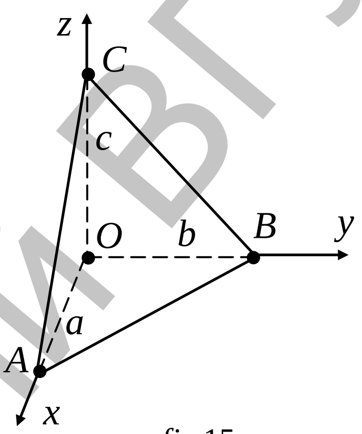


fig.15

Proof. The assumption means, that the plain passes through the points $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$ (fig. 15). We substitute their coordinates in (8):

$$\begin{vmatrix} x-a & y-0 & z-0 \\ 0-a & b-0 & 0-0 \\ 0-a & 0-0 & c-0 \end{vmatrix} = 0$$

We uncover the determinant and we get

$$bc(x-a)+acy+abz=0.$$

We divide this equation on abc:

$$\frac{x-a}{a}+\frac{y}{b}+\frac{z}{c}=0$$

And this equation is equivalent to (9). ■

Equation (9) is called *the equation of the plane in segments*.

***Corollary* 1.3.** *Any plain can be determined by the equation of the form*

$$Ax+By+Cz+D=0\,, \tag{10}$$

which is called general equation of the plane. Conversely, arbitrary equation (10) *defines a plane, if at least one of the coefficients* A, B, C *is not equal to zero.*

Proof. Any plane can be determined by means of a point and a normal vector, i.e. it can be determined by the equation (6). We uncover parenthesis and denote $D = -Ax_o - By_o - Cz_o = \text{const}$. Then we get equation (10).

Conversely, let some set π is defined by equation (6). Let $M_o(x_o, y_o, z_o)$ be an arbitrary point of this set. Then its coordinates satisfy equation (10):

$$Ax_o + By_o + Cz_o + D = 0.$$

Consequently $D = -Ax_o - By_o - Cz_o$. We substitute this value in (10) and we get (6). We know, that equation (6) determines a plain. ■

It is important to note that A, B, C in (10) are the same as in (6). So, the geometric sense of these coefficients in (10) is the following: A, B, C are *coordinates of a normal vector of the plane.*

§3. *Equation of a plain in the normal form. The distance from a point to a plain*

Определение. We say, that *the general equation of the plain*

$$Ax + By + Cz + D = 0, \tag{10}$$

has the normal form, if $A^2 + B^2 + C^2 = 1$. It is equivalent to the fact, that the normal vector $\vec{\mathbf{n}}(A, B, C)$ is the unit one.

If equation (10) is not in the normal form, we divide this equation on the number $\mu = \sqrt{A^2 + B^2 + C^2}$:

$$\frac{A}{\mu}x + \frac{B}{\mu}y + \frac{C}{\mu}z + \frac{D}{\mu} = 0,$$

Then there will be fulfilled $(A/\mu)^2 + (B/\mu)^2 + (C/\mu)^2 = 1$, i.e. we will get an equation in the normal form. We accept the following two results without proof.

Theorem 2. *Let a plain* π *is determined by the equation* (10) *in the normal form. Then the distance from the point* $M(x_1, y_1, z_1)$ *to the plane can be calculated by the formula*

$$h = |Ax_1 + By_1 + Cz_1 + D|. \tag{11}$$

Corollary. *If a plain is determined by an arbitrary general equation, then*

$$h = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}. \tag{11′}$$

If we omit the modulus sign in (11′), then we get the value, which is called the *deviation of a point from the plain*. The distance can't be negative but the deviation can. The sign of deviation is determined by the sign of the numerator. This sign determines the position of a point relative to the plain. Let's remind, that a plain divides the space on two semi-spaces. If two points M, N belong to one semi-space, then the segment MN doesn't intersect the plane (fig.16). If two points M, N belong to different semi-spaces, then the segment MN intersects the plane (fig.17).

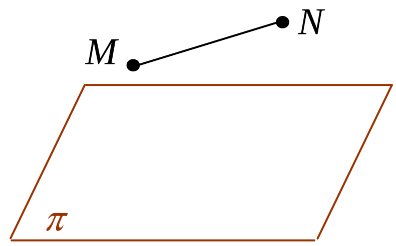


*fig.*16

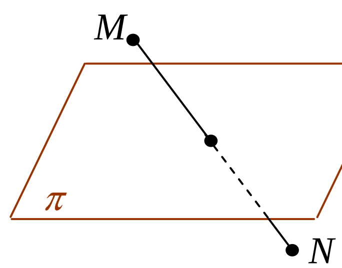


fig. 17

***Theorem* 3.** *Assume that a plain* π *is determined by a general equation and let* $M(x_1, y_1, z_1)$, $N(x_2, y_2, z_2)$ *be arbitrary points. If the values*

$$Ax_1 + By_1 + Cz_1 + D \;\; \text{and} \;\; Ax_2 + By_2 + Cz_2 + D$$

have the same sign, then the points are located in one semi-space, and this values have different signs, then the points are located in different semi-spaces in respect to the plain.

§4. *Mutual location of two plains in the space*

Let two plains in the space are determined by their general equations:

$$\pi_1:\; A_1x + B_1y + C_1z + D_1 = 0\,,$$
$$\pi_2:\; A_2x + B_2y + C_2z + D_2 = 0\,.$$

Then we can make the conclusion, that $\vec{\mathbf{n}}_1(A_1, B_1, C_1)$ and $\vec{\mathbf{n}}_2(A_2, B_2, C_2)$ are normal vectors to π_1 and π_2.

***Theorem* 4. 1.** $\pi_1 \,||\, \pi_2 \;\Leftrightarrow\; \dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{C_2} \neq \dfrac{D_1}{D_2}\,.$ (12)

2. $\pi_1 = \pi_2 \;\Leftrightarrow\; \dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{C_2} = \dfrac{D_1}{D_2}\,.$ (13)

3. $\pi_1 \perp \pi_2 \;\Leftrightarrow\; A_1A_2 + B_1B_2 + C_1C_2 = 0.$ (14)

4. *the angle between* π_1 and π_2 *is calculated by the formula*

$$\cos\alpha = \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{|\vec{\mathbf{n}}_1||\vec{\mathbf{n}}_2|} = \frac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}\,. \tag{15}$$

Proof. **1,2.** It is obvious, that two plains are parallel or coincide if and only if their normal vectors $\vec{\mathbf{n}}_1(A_1, B_1, C_1)$ and $\vec{\mathbf{n}}_2(A_2, B_2, C_2)$ are collinear (fig. 18). According to the second criterion of colinearity it is equivalent $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{C_2}$.
We denote the common ratio as λ:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \lambda. \tag{16}$$

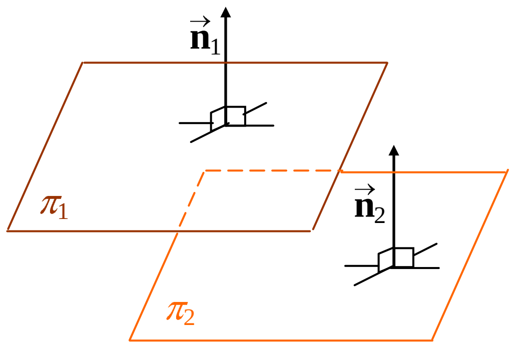


*fig.*18

In this case the plains coincide $\Leftrightarrow$ they have a common point $M_o(x_o, y_o, z_o)$, i.e. if and only if the following equalities are true at the same time:

$$A_1x_o + B_1y_o + C_1z_o + D_1 = 0,$$
$$A_2x_o + B_2y_o + C_2z_o + D_2 = 0.$$

We multiply the second equality on λ and subtract it from the first one:

$$(A_1 - \lambda A_2)x_o + (B_1 - \lambda B_2)y_o + (C_1 - \lambda C_2)z_o + (D_1 - \lambda D_2) = 0.$$

According to (16) all the expressions in brackets are equal to zero. So we get $D_1 - \lambda D_2 = 0 \iff D_1/D_2 = \lambda$. We unite the least equality with (16) and we get (13).

If the plains are parallel, then (16) is true and (13) doesn't, it means, that (12) takes place.

3, 4. Recall, that intersection of two planes forms two pairs of vertical dihedral angles, and the smaller angle is called the angle between two planes. Thus the angle α between two planes lies within the range $0 \le \alpha \le \pi/2$ and $\cos\alpha \ge 0$.

Let $\beta = \angle(\vec{\mathbf{n}}_1, \vec{\mathbf{n}}_2)$. Than it is obvious that either $\beta = \alpha$, or it is adjacent to α: $\alpha = \pi - \beta$ (you can see only the second case on the fig.19).

In the first case we have

$$\cos\alpha = \cos\beta = \frac{\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2}{|\vec{\mathbf{n}}_1||\vec{\mathbf{n}}_2|},$$

and in the second case we have

$$\cos\alpha = \cos(\pi - \beta) = -\cos\beta = |\cos\beta| = \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{|\vec{\mathbf{n}}_1||\vec{\mathbf{n}}_2|}.$$

The last formula is true in the first case also. ■

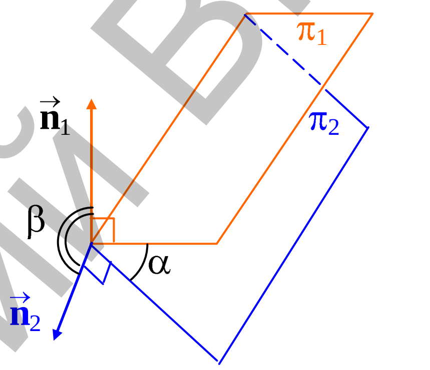


*fig.*19

§5. *Equation of a straight line in the space.*

A straight line in the space can be determined

i) by means of a point $A_o \in l$ and nonzero vector $\vec{\mathbf{a}} || l$; this vector is called *the directing vector of the line* and we can write that

$$l = \{M \mid \overrightarrow{M_oM} \,||\, \vec{\mathbf{a}}\}; \qquad (*)$$

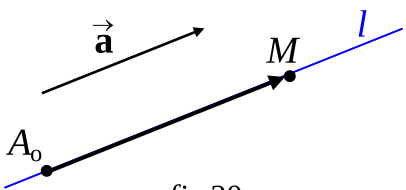


*fig.*20

ii) as the intersection of two planes $l = \pi_1 \cap \pi_2$; in this case l is determined by the system of two equations (see §1); Equivalently, given a point $M_o \in l$ and two vectors $\vec{\mathbf{n}}_1$ and $\vec{\mathbf{n}}_2$ perpendicular to l ($\vec{\mathbf{n}}_1$ is not collinear to $\vec{\mathbf{n}}_2$) (fig.21).

It is impossible to define a straight line in the space by means of one normal vector. The infinite number of lines passes through the given point perpendicular to the given vector.

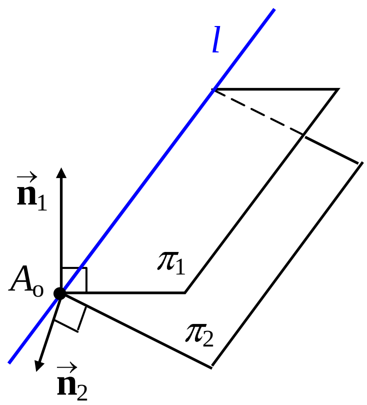


*fig.*21

***Theorem* 5. 1.** *A straight line l, which passes through a point $M_o(x_o, y_o, z_o)$ in parallel with a vector $\vec{\mathbf{a}}(a_1, a_2, a_3)$ can be defined*

i) *by the canonical equation*

$$\frac{x-x_o}{a_1}=\frac{y-y_o}{a_2}=\frac{z-z_o}{a_3}, \tag{17}$$

ii) *by parametric equations*

$$\begin{cases} x=x_o+a_1 t, \\ y=y_o+a_2 t, \\ z=z_o+a_3 t, \quad t\in\mathbf{R}, \end{cases} \tag{18}$$

We can rewrite these equations in vector form: $\vec{\mathbf{r}}=\vec{\mathbf{r}}_o+t\vec{\mathbf{a}},\ t\in\mathbf{R}$, *wher* $\vec{\mathbf{r}}_o=\overrightarrow{OM_o}$ *is the radius-vector of the point* M_o.

2. *A straight line, which passes through a point $M_o(x_o, y_o, z_o)$, perpendicular to vectors $\vec{\mathbf{n}}_1(A_1, B_1, C_1)$ and $\vec{\mathbf{n}}_2(A_2, B_2, C_2)$ is defined by the system of equations*

$$\begin{cases} A_1(x-x_o)+B_1(y-y_o)+C_1(z-z_o)=0, \\ A_2(x-x_o)+B_2(y-y_o)+C_2(z-z_o)=0. \end{cases} \tag{19}$$

Proof **1.** Proof of this item is the same as the proof of the analogous result for the straight line on the plane with only difference that we must use thee coordinates instead of two ones.

2. The first of equations (19) defines the plane π_1, which passes through the point M_o perpendicular to the vector $\vec{\mathbf{n}}_1$, and the second equation defines the plane π_2, which passes through the point M_o perpendicular to the vector $\vec{\mathbf{n}}_2$. The intersection of these two planes defines our straight line. ■

Corollary. *A straight line, which passes through two points $M_o(x_o, y_o, z_o)$ and $M_1(x_1, y_1, z_1)$, can be defined by the equation*

$$\frac{x-x_o}{x_1-x_o}=\frac{y-y_o}{y_1-y_o}=\frac{z-z_o}{z_1-z_o}, \tag{20}$$

§6. *Mutual location of a straight line and a plane in the space.*

Let a plane π be defined by a general equation and a straight line l be defined by a canonical equation

$$\pi:\ Ax+By+Cz+D=0, \qquad l: \frac{x-x_o}{a_1}=\frac{y-y_o}{a_2}=\frac{z-z_o}{a_3}.$$

Then we can immediately note, that $\vec{\mathbf{n}}(A, B, C)$ is the normal vector for the plane π, $\vec{\mathbf{a}}(a_1, a_2, a_3)$ is the directing vector for the straight line l and the point $M_o(x_o, y_o, z_o)\in l$.

***Теорема* 6. 1.** $l \in \pi \;\Leftrightarrow\; \begin{cases} Aa_1 + Ba_2 + Ca_3 = 0, & (21.1) \\ Ax_o + By_o + Cz_o + D = 0. & (21.2) \end{cases}$

2. $l \| \pi \;\Leftrightarrow\; \begin{cases} Aa_1 + Ba_2 + Ca_3 = 0, & (21.1) \\ Ax_o + By_o + Cz_o + D \neq 0. & (21.3) \end{cases}$

3. $l \perp \pi \;\Leftrightarrow\; \dfrac{A}{a_1} = \dfrac{B}{a_2} = \dfrac{C}{a_3}.$ (22)

4. *The angle between* l *and* π *can be calculated by the formula*

$$\sin\alpha = \frac{|\vec{\mathbf{n}} \cdot \vec{\mathbf{a}}|}{|\vec{\mathbf{n}}|\,|\vec{\mathbf{a}}|} = \frac{|Aa_1 + Ba_2 + Ca_3|}{\sqrt{A^2+B^2+C^2}\,\sqrt{a_1^2+a_2^2+a_3^2}}. \qquad (23)$$

Proof. **1,2.** It is obvious, that in both cases $l \| \pi$ or $l \in \pi$ must be fulfilled $\vec{\mathbf{a}} \perp \vec{\mathbf{n}} \Leftrightarrow \vec{\mathbf{n}} \cdot \vec{\mathbf{a}} = 0$. The last equality in coordinates is just (21.1). If additionally (21.2) is fulfilled, than $M_o(x_o, y_o, z_o) \in \pi$, and it means that the whole line lies in the plane. If (32.3) is fulfilled, than $M_o \notin \pi$, and it means that $l \notin \pi$.

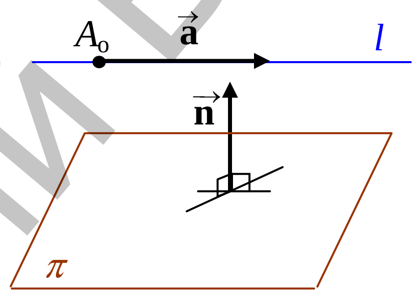


fig.22

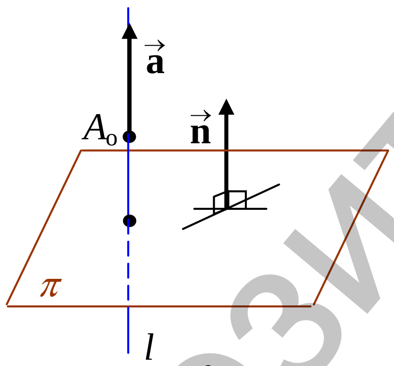


fig.23

3. It is obvious, that $l \perp \pi \Leftrightarrow \vec{\mathbf{a}} \| \vec{\mathbf{n}}$, and (33) is just the condition for this vectors to be collinear.

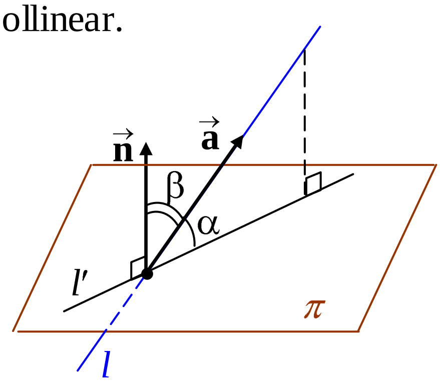


fig.24 a)

4. Recall that the angle between the line and the plane is the angle between the line and its projection on the plane. Therefore if α is the angle between l и π, then $0 \le \alpha \le \pi/2$, и $\sin\alpha \ge 0$.

Denote $\beta = \angle(\vec{\mathbf{a}}, \vec{\mathbf{n}})$. Then two cases are possible: $\alpha = \pi/2 - \beta$ or $\alpha = \beta - \pi/2$. Both cases are shown on drawings 13 a) and 13 b).

In the first case we have

$$\sin\alpha = \cos\beta = \frac{\vec{\mathbf{n}} \cdot \vec{\mathbf{a}}}{|\vec{\mathbf{n}}|\,|\vec{\mathbf{a}}|},$$

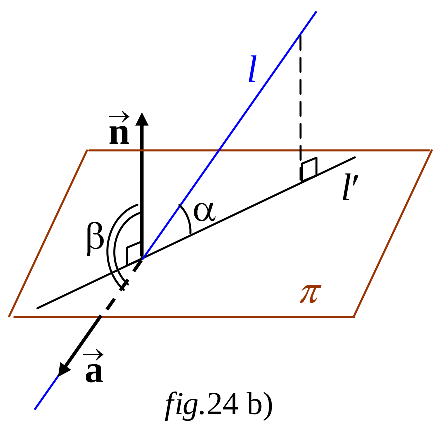


fig.24 b)

In the second case we have

$$\sin\alpha = -\cos\beta = |\cos\beta| = \frac{|\vec{\mathbf{n}}\cdot\vec{\mathbf{a}}|}{|\vec{\mathbf{n}}|\,|\vec{\mathbf{a}}|}\ .$$

This formula is also true in the first case. ■

§7. *Examples of solving problems*

1. *Coordinates of vertexes a triangular pyramid* $SABC$ *are given*: $A(-3, 7, 1)$, $B(-1, 9, 2)$, $C(-3, 6, 6)$ $S(6,-5,-2)$.

i) *Find the equation of the basis plane* ABC *and the equation of the height* SD.

ii) *Find the value of the height.*

iii) *Find coordinates of the point* D *и точки* F, *which is symmetric to* S *with respect to the basal plane.*

Solution. **i)** First, we find coordinates of two vectors parallel to the basal plane $\pi = ABC$:

$$\vec{\mathbf{a}} = \overrightarrow{AB}(2, 1, 1),\ \vec{\mathbf{b}} = \overrightarrow{AC}(0,-1, 5).$$

The equation of a plane passing through the point $M(x_o, y_o, z_o)$ parallel to the two non-collinear vectors $\vec{\mathbf{a}}(a_1, a_2, a_3)$, $\vec{\mathbf{b}}(b_1, b_2, b_3)$ is the following:

$$\begin{vmatrix} x-x_o & y-y_o & z-z_o \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0.$$

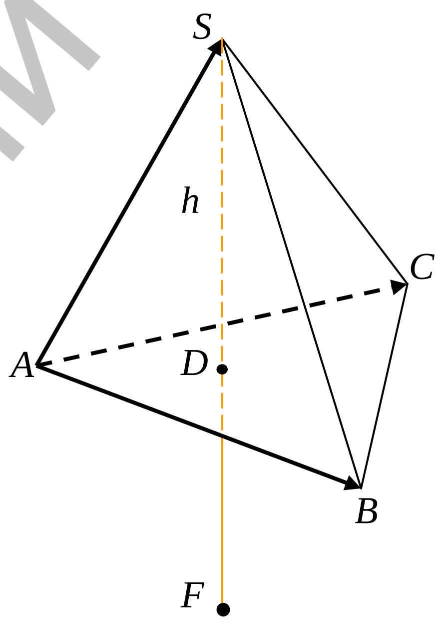


*fig.*25

We substitute into this equation our data:

$$\begin{vmatrix} x+3 & y-7 & z-1 \\ 2 & 2 & 1 \\ 0 & -1 & 5 \end{vmatrix} = 0.$$

We expand the determinant by the first raw:

$$11(x-3) - 10(y-7) - 2(z-1) = 0,$$

$$11x - 10y - 2z + 105 = 0,$$

ii) We can calculate the height by the as the distance between the point S and the basal plane by the formula

$$h = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}\ . \qquad (11')$$

In our case

$$h = \frac{|11\cdot 6 - 10\cdot(-5) - 2\cdot(-5) + 105|}{\sqrt{11^2 + (-10)^2 + (-2)^2}} = 15.$$

iii) From the equation of the plane we find that the vector $\vec{\mathbf{n}}(11,-10,-2)$ is the normal vector of the plane. This vector is the directing vector of the straight line $h=SD$. The parametric equation of the straight line passing through the point $M_o(x_o, y_o, z_o)$ with the directing vector $\vec{\mathbf{a}}(a_1, a_2, a_3)$ is the following:

$$\begin{cases} x = x_o + a_1 t, \\ y = y_o + a_2 t, \\ z = z_o + a_3 t. \end{cases}$$

In our case, we get the equation:

$$SD: \begin{cases} x = \ 6 + 11t, \\ y = -5 - 10t, \\ z = -2 - 2t. \end{cases} \qquad (*)$$

Let's find the base of the height. This is the point of intersection of the straight line h with the plane π. To do this, we need to solve simultaneously the equations of h и π. We substitute x, y, z from the equation of SD in the equation of π:

$$11(6+11t)-10(-5-10t)-2(-2-2t)+105=0,$$
$$66+121t+50+100t+4+4t+105=0,$$
$$225y=-225, \quad t=-1.$$

This corresponds to the point D. So we denote it as t_D. Then we substitute this t_D in equation $(*)$ and find coordinates $D(-5, 5, 0)$.

The physical meaning of the parametric equation of the line is following: it sets a uniform motion. In our case initial point is S, and the velocity vector is $\vec{\mathbf{n}}$. The segment SF is twice as long as the segment SD. Thus we need twice more time for passing this segment. If it takes t_D time for passing from S to H, than it takes $t_F = 2t_D = -2$ time for passing from S до F. We substitute this meaning in (*), and find $F(-16, 15; 2)$.

We can calculate the height as the length of the segment CD:

$$|CD| = \sqrt{(-5-6)^2 + (5-(-5))^2 + (0-(-2))^2} = 15.$$

We got the same result as before.

***Answer*:** ABC: $11x - 10y - 2z + 105 = 0$, $D(-5, 5, 0)$, $S'(-16, 15; 2)$, $\quad SH: \begin{cases} x = \ 6 + 11t, \\ y = -5 - 10t, \\ z = -2 - 2t. \end{cases}$

2. *The equations of the straight line* l *and of the plain* π *are given*:

$$l: \ \frac{x-6}{1} = \frac{y}{-1} = \frac{z-2}{2}, \quad \pi: 5x - 2y + 4z + 7 = 0.$$

Make sure, that l *и* π *intersects and make the equation of the projection* l' *of the line* l *on the plane. Find the angle between* l *and* π *.*

Solution. From the equation of the straight line we can find the directing vector $\vec{\mathbf{a}}(1,-1,2)$ and a point $M_o(6,0,2)$ on the line. From the equation of the plane we can find the normal vector of the plane: $\vec{\mathbf{n}}(5,-2,4)$. It is obvious that the necessary condition for l being parallel to π is $\vec{\mathbf{a}}\perp\vec{\mathbf{n}}$ i.e. $\vec{\mathbf{n}}\cdot\vec{\mathbf{a}}=0$. Let's check:

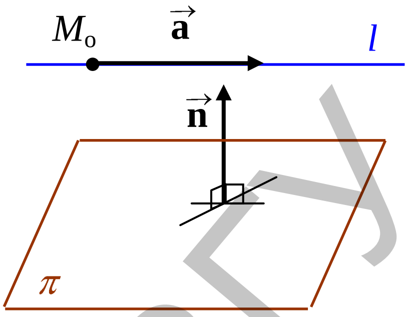


fig.26

$$\vec{\mathbf{n}}\cdot\vec{\mathbf{a}}=5\cdot1-2\cdot(-1)+4\cdot2=15\neq0.$$

Thus l intersects π. The angle between l and π can be found by the formula:

$$\sin\alpha=\frac{|\vec{\mathbf{n}}\cdot\vec{\mathbf{a}}|}{|\vec{\mathbf{n}}|\cdot|\vec{\mathbf{a}}|};$$

$$|\vec{\mathbf{a}}|=\sqrt{1^2+(-1)^2+2^2}=\sqrt{6},\ |\vec{\mathbf{n}}|=\sqrt{5^2+(-2)^2+4^2}=\sqrt{45}=3\sqrt{5}.$$

$$\sin\alpha=\frac{15}{3\sqrt{5}\cdot\sqrt{6}}=\frac{\sqrt{30}}{6}.$$

Let N be the projection of the point M_o on the plane and P be the intersection point: $P=l\cap\pi$. Then $l'=NP$ is the projection of the straight line l. First, we find coordinates of the point P. To do this, we rewrite the equation of a line l in the parametric form:

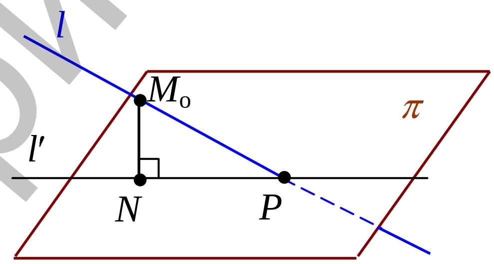


fig.29

$$l:\begin{cases}x=6+t,\\ y=-t,\\ z=2+2t,\end{cases}$$

and solve it simultaneously with the equation of the plane π. We substitute x, y, z from the equation of l in the equation of π:

$$5(6+t)-2(-t)+4(2+2t)+7=0,$$
$$30+5t+2t+8+8t+7=0,$$
$$15t=-45,\quad t=-3.$$

Then we substitute this t in the equation of l and find coordinates $B(3,3,4)$. Now we make the equation of the perpendicular $h=M_oN$. For this line the vector $\vec{\mathbf{n}}$ is the directing vector. Thus h has the equation

$$h:\begin{cases}x=6+5t,\\ y=-2t,\\ z=2+4t,\end{cases}$$

We solve it simultaneously with the equation of π in order to find coordinates of the point N:

$$5(6+5t)-2(-2t)+4(2+4t)+7=0,$$
$$30+25t+4t+8+16t+7=0,$$
$$45t=-45, \quad t=-1.$$

We substitute this t in the equation of h and find $N(1, 2,-2)$. Then we find the directing vector of the straight line $l': \overrightarrow{NP}(2, 1,-2)$ and compose its equation:

$$\frac{x-1}{2}=\frac{y-2}{1}=\frac{z+2}{-2}.$$

8. *Vertexes of the triangle ΔABC are given: $A(9, 5, 1)$, $B(-3, 8, 4)$, $C(9,-13,-8)$. AD is the height.*

i) *Find coordinates of the point D.*

ii) *Calculate $h=|AD|$ and the area of the triangle.*

iii) *Check your answer by calculation $S_{\Delta ABC}$ by means of vector product.*

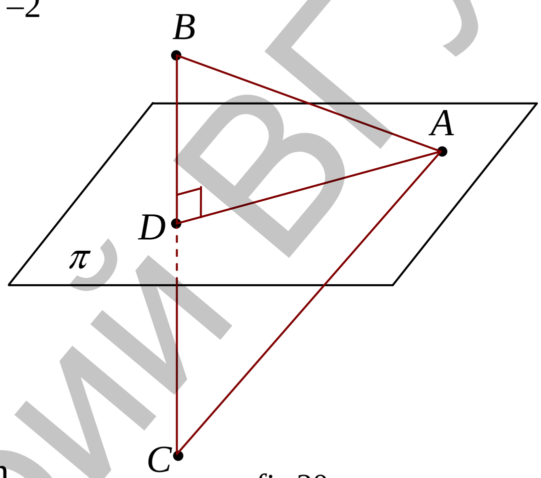


*fig.*30

Solution. **i)** Let π be the plane, which passes through the point A perpendicular to the straight line BC. Then the height AD belongs to this plane. So we can find D as interjection: $D=\pi\cap BC$. Vector $\overrightarrow{BC}$ is the normal vector for the plane π. We find its coordinates: $\overrightarrow{BC}(12,-21,-12)$. We see, that all the coordinates of $\overrightarrow{BC}$ can be easily divided on 3. Thus we can take $\vec{\mathbf{n}}=\frac{1}{3}\overrightarrow{BC}$ as the normal vector; $\vec{\mathbf{n}}(4,-7,-4)$. Equation of a plane, which passes through the point $M_o(x_o, y_o, z_o)$ perpendicular to the vector $\vec{\mathbf{n}}(a, b, c)$, has the form:

$$a(x-x_o)+b(y-y_o)+c(z-z_o)=0.$$

For our plane π we have the equation:

$$4(x-9)-7(y-5)-4(z-1)=0,$$
$$4x-7y-4z+3=0.$$

Now we make the equation of the straight line BC. The vector $\vec{\mathbf{n}}$ is its directing vector:

$$BC: \begin{cases} x=-3+4t, \\ y=\ \ 8-7t, \\ z=\ \ 4-4t, \end{cases} \qquad (*)$$

We have $D=\pi\cap BC$. For finding coordinates of the point D we must solve simultaneously equations of π и BC. We substitute x, y, z from the equation of BC in the equation of π:

$$4(-3+4t)-7(8-7t)-4(4-4t)+3=0,$$
$$-12+16t-56+49t-16+16t+3=0,$$
$$81t=81, \quad t=1.$$

We substitute this t in the equation of BC and find $D(1, 1, 0)$.

ii) We calculate $h=|AD|$ by the formula of distance between points:

$$h=\sqrt{(1-9)^2+(1-5)^2+(0-1)^2}=9.$$

And we find

$$BC=\sqrt{12^2+(-21)^2+(-12)^2}=3\sqrt{4^2+(-7)^2+(-4)^2}=27.$$

Finally,

$$S_{\Delta ABC}=\frac{1}{2}BC\cdot h=\frac{1}{2}27\cdot 9=\frac{243}{2}.$$

iii) We must use the formula $S_{\Delta ABC}=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$. First, we find the vector $\overrightarrow{AB}\times\overrightarrow{AC}$, and then we find its modulus.

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -12 & 3 & 3 \\ 0 & -18 & -9 \end{vmatrix}=-27\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix}=-27(-\mathbf{i}+4\mathbf{j}-8\mathbf{k}).$$

(We used the property of the determinant: the common factor of elements in one line can be taken forward before the sign of the determinant).

$$S_{\Delta ABC}=\frac{1}{2}\cdot 27\sqrt{(-1)^2+4^2+(-8)^2}=\frac{243}{2}.$$

This answer coincides with the previous one.

We can use another method. The point D is the closest point to A among all the points of the line BC. Thus we can use the methods of differential calculus. Let $M(t)$ be an arbitrary point of the straight line BC. Its coordinates are defined by the system $(*)$:

$$M(-3+4t,\ 8-7t,\ 4-4t).$$

We find the square of the distance between A and $M(t)$:

$$h^2(t)=(9+3-4t)^2+(5-8+7t)^2+(1-4+4t)^2$$
$$=(12-4t)^2+(-3+7t)^2+(-3+4t)^2=$$
$$=144-96t+16t^2+9-42t+49t^2+9-24t+16t^2=$$
$$=81t^2-162t+162.$$

We must find the minimal value of this function. Thus we find the derivative of the function $h^2(t)$ and equate the derivative to zero:

$$\frac{\mathrm{d}}{\mathrm{d}t}h^2(t) = 162t - 162; \quad \frac{\mathrm{d}}{\mathrm{d}t}h^2(t) = 0 \quad \Rightarrow \quad t = 1.$$

We substitute this value of t in the equation of BC and find that $D(1, 1, 0)$ is the closest point to A among all the points of the line BC.

9. *Explore the relative position of the following pairs of planes (intersect, parallel, equal). If the planes intersect, find the angle between them, if they are parallel – find the distance between them.*

i) π_1: $2y + \mathrm{z} + 5 = 0$, π_2: $5x + 4y - 2\mathrm{z} + 11 = 0$.

ii) π_1: $\sqrt{2}x - y + 2\sqrt{2}\mathrm{z} + 8 = 0$, π_2: $2x - \sqrt{2}y + 4\mathrm{z} - 15 = 0$.

Solution. **i)** If two planes π_1 and π_2 are defined by their general equations

$$A_1x + B_1y + C_1\mathrm{z} + D_1 = 0, \quad A_2x + B_2y + C_2\mathrm{z} + D_2 = 0,$$

then

$$\pi_1 \,||\, \pi_2 \quad \Leftrightarrow \quad \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \neq \frac{D_1}{D_2},$$

$$\pi_1 = \pi_2 \quad \Leftrightarrow \quad \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}.$$

In our case $\frac{0}{5} \neq \frac{1}{4} \neq \frac{5}{-2}$, thus two planes are not parallel and they don’t coincide. It means that they intersect. The angle between two planes can be calculated by the formula

$$\cos\alpha = \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{|\vec{\mathbf{n}}_1||\vec{\mathbf{n}}_2|},$$

where $\vec{\mathbf{n}}_1$ и $\vec{\mathbf{n}}_2$ are the normal vectors to these planes. In our case

$$\vec{\mathbf{n}}_1(0, 2, 1), \ \vec{\mathbf{n}}_2(5, 4, -2), \quad \vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2 = 0 \cdot 5 + 2 \cdot 4 + 1 \cdot (-2);$$

$$|\vec{\mathbf{n}}_1| = \sqrt{0^2 + 2^2 + 1^2} = \sqrt{5}, \ |\vec{\mathbf{n}}_2| = \sqrt{5^2 + 4^2 + 2^2} = 3\sqrt{5}.$$

Thus, $\cos\alpha = \dfrac{|6|}{\sqrt{5} \cdot 3\sqrt{5}} = \dfrac{2}{5}$.

***Ответ*:** $\alpha = \arccos\frac{2}{5}$.

ii) We check if the planes are parallel:

$$\frac{\sqrt{2}}{2} = \frac{-2}{-2\sqrt{2}} = \frac{2\sqrt{2}}{4} \neq \frac{8}{-15}$$

It is true. Thus, $\pi_1 \,||\, \pi_2$. The distance from the point M to the plane defined by general equation $Ax + By + Cz + D = 0$ can by calculated by the formula

$$h = \frac{|Ax + By + Cz + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

We choose any point $M \in \pi_1$. To do this, we choose any three coordinates that satisfy the equation of π_1. The most simple choice is: $M(0, 8, 0)$. The distance between M до π_2 is the distance between π_1 и π_2:

$$h = \frac{|2\cdot 0 - \sqrt{2}\cdot 8 + 4\cdot 0 - 12|}{\sqrt{2^2 + (\sqrt{2})^2 + 4^2}} = \frac{4(2\sqrt{2} - 3)}{\sqrt{22}}.$$

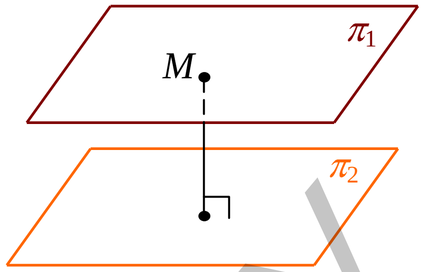


*fig.*31

Tasks for the practical classes

Practical classes 3,4.

The equation of a plane in space. Mutual position of planes. The distance from the point to the plane.

1. Write the equation of a plane that passes

а) through the point $M_o(1,0,2)$ and is parallel to the vectors $\vec{\mathbf{a}}(1,2,3)$, $\vec{\mathbf{b}}(0,3,1)$.

б) through the point $M_o(31,0,1)$ and through the axis Ox;

в) through the point $M_o(1,2,-2)$ perpendicular to the vector $\vec{\mathbf{n}}(2,-1,3)$.

2. Coordinates of the vertexes of a triangle pyramid are given: $A(2,1,0)$, $B(1,3,5)$, $C(6,3,4)$, $D(0,-7,8)$. Write the equation of a plane that passes through the edge AB and the middle of the edge CD.

3. Make the equation of the plane, if the point $A(1,-1,3)$ is the basis of the perpendicular from the origin to this plane.

4. Calculate the volume of the tetrahedron bounded by the coordinate planes and the plane $3x - 5y + 15z - 30 = 0$.

5. Find out which of the following pairs of planes intersect, are parallel or identical. If the planes intersect, find the angle between them.

а) $x - y + 3z + 1 = 0$ и $2x - y + 5z - 2 = 0$;

б) $2x - y + 2z + 4 = 0$ и $4x + 2y + 4z + 8 = 0$;

в) $3x + 2y - z + 2 = 0$ и $6x + 4y - 2z + 1 = 0$;

г) $2x + y - 2z + 6 = 0$ и $2x + 2y - z + 8 = 0$.

6. Make the equation of a plane passing through the axis Oz and forming with the plane $2x + y - \sqrt{5}z - 7 = 0$ the angle equal to $60°$.

7. Coordinates of the vertexes of a triangle pyramid are given: $A(0,6,4)$, $B(3,5,3)$, $C(-2,11,5)$, $D(1,-1,4)$. Calculate the length of its height, drawn from the point A to the face BCD.

8. Find the distance between two parallel planes $x-2y-2z+7=0$ and $2x-4y-4z+17=0$.

9. Write the equations of the planes, that are parallel to the plane $\pi: 2x-2y-z-6=0$ if the distance between these planes and π is equal $d=7$.

10. The planes $3x-y+7z-4=0$ and $5x+3y-5z+2=0$ are the faces of the dihedral angles. Write the equations of the planes, that bisect this dihedral angles.

11. Find the coordinates of the center and the radius of the sphere inscribed in the tetrahedron bounded by the coordinate planes and the plane $11x-10y-2z-57=0$.

Home task.

1. Make the equation of a plane passing through the point $A(3,5,-7)$ and cutting off on the coordinate axes segments of equal size.

2. Find the distance between two parallel planes $6x+2y-4z+15=0$ and $9x+3y-6z+10=0$.

Practical classes 5.

Equation of a straight line in the space. Relative position of a line and a plane. Mutual position of the two lines in space. Distance between straight lines.

1. Write a canonical and parametric equations of the straight line passing through the point $M_0(0,6,4)$ if it is parallel to the vector $\vec{\mathbf{a}}_1(3,-2,-2)$.

2. Coordinates of the vertexes of a triangle are given: $A(3,5,7)$, $B(1,2,3)$, $C(3,0,1)$. Make a parametric equation of its median drawn from the point A.

3. Coordinates of the vertexes of a triangle are given: $A(1,2,-7)$, $B(2,2,-7)$, $C(3,4,-5)$. Make a parametric equation of its bisectrix of the internal angle at the vertex A.

4. Write parametric equations of the straight line $x+y+2z-3=0$, $x-y-z-1=0$.

5. AD is the height of the triangle ΔABC with the vertexes $A(9,5,1)$, $B(-3,8,4)$, $C(9,-13,-8)$.
i) Find coordinates of the point D;
ii) make the equation of the line AD;
iii) find $h=|AD|$.

6. Prove that the following lines intersect and find the coordinates of the points of intersection: $x=-3t$, $y=2+3t$, $z=1$; $x=1+5t'$, $y=1+13t'$, $z=1+10t'$.

7. Prove that the following lines are crossed and find the distance between them: $x=3-6t$, $y=-1+4t$, $z=t$; $x=-2+3t'$, $y=4$, $z=3-t'$.

8. Find out the relative position of the next line and the plane. If they intersect, find the angle between them:

а) $x=2+4t,\ y=-1+t,\ z=2-t;\quad 4x+y-z+13=0;$

б) $x=2-3t,\ y=-1+t,\ z=-2t;\quad x+y-z+3=0;$

в) $x=t,\ y=-8-4t,\ z=-3-3t;\quad x+y-z+5=0;$

г) $\dfrac{x-7}{5}=\dfrac{y-4}{1}=\dfrac{z-5}{4};\ 3x-y+2z-5=0.$

9. Find the projection of the point $A(2,11,-5)$ on the plane $x+4y-2z+7=0$ and find coordinates of the symmetrical point.

Home task.

1. Find the projection of the point $A(6,-5,5)$ on the plane $2x-3y+z-4=0$ and find coordinates of the symmetrical point.

Tasks for independent solving

1. *Coordinates of vertexes a triangular pyramid $SABC$ are given.*

i) *Find the equation of the basis plane ABC and the equation of the height SD.*

ii) *Find the value of the height.*

iii) *Find coordinates of the point D и точки F, which is symmetric to S with respect to the basal plane.*

1. $A(1,0,-1)$, $B(2,0,4)$, $C(4,2,3)$, $S(10,-11,-8)$;
2. $A(-1,3,0)$, $B(-1,-1,2)$, $C(0,5,-2)$, $S(7,2,6)$;
3. $A(1,4,2)$, $B(7,6,3)$, $C(3,4,3)$, $S(6,-7,-7)$;
4. $A(-1,1,2)$, $B(-5,4,-2)$, $C(-1,2,3)$, $S(-8,-5,4)$;
5. $A(0,2,2)$, $B(0,4,3)$, $C(1,4,2)$, $S(7,-1,7)$;
6. $A(1,1,2)$, $B(1,2,4)$, $C(4,1,4)$, $S(2,-7,3)$;
7. $A(-1,1,-2)$, $B(-1,-2,-1)$, $C(1,-2,0)$, $S(5,-3,-14)$;
8. $A(-5,1,2)$, $B(-5,-2,6)$, $C(-4,4,-2)$, $S(2,12,4)$;
9. $A(-2,0,1)$, $B(-1,2,1)$, $C(-3,0,4)$, $S(5,-3,0)$;
10. $A(-6,0,1)$, $B(-6,-3,5)$, $C(-5,3,-3)$, $S(1,10,-4)$;
11. $A(0,2,1)$, $B(0,-2,3)$, $C(1,4,-1)$, $S(8,1,7)$;
12. $A(0,5,1)$, $B(6,7,2)$, $C(2,5,2)$, $S(5,-6,-8)$;

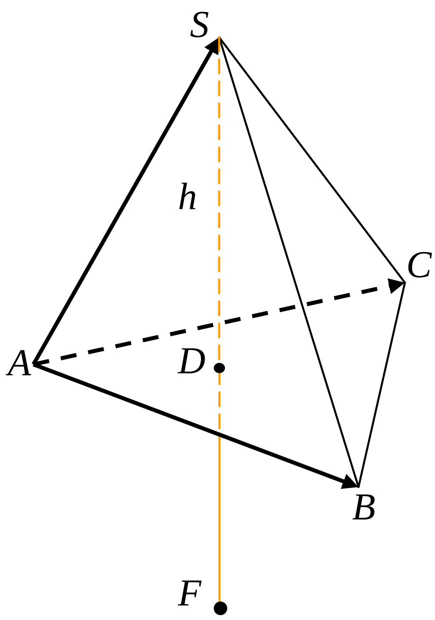


*fig.*25

CHAPTER 3. SURFASES OF THE SECOND ORDER

§1. *Cylindrical surfaces*

Definition. We say that a surface is *ruled surface*, if through each point on the surface passes at least one straight line, which belongs to the surface.

Definition. We say, that a surface is *a cylindrical surface*, if it consists of the set of parallel lines, which pass through each point of a curve γ. The lines are called *ruling lines*, and the curve is called *the directing curve*.

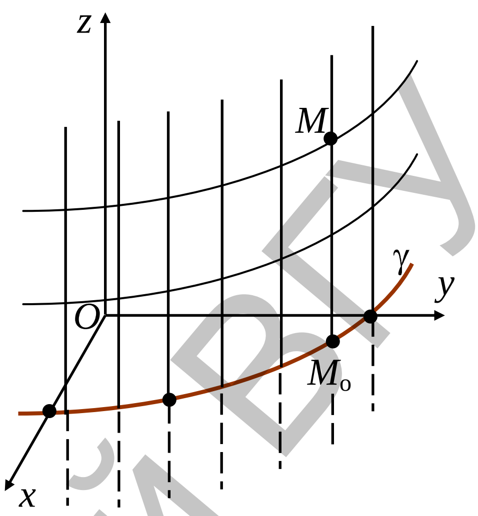


fig.32

Let Φ be a cylindrical surface. We choose the Cartesian coordinate system in such way that the axis Oz is parallel to ruling lines. Also we can consider, that the directing line γ belongs to the plane Oxy. Let

$$\varphi(x, y) = 0 \qquad (1)$$

be the equation of the directing line γ in the plane Oxy (in the space is defined by the system of two equations: $\varphi(x, y) = 0$ и $z = 0$). Let $M(x, y, z)$ be an arbitrary point on the surface Φ. Then its projection on the plane Oxy is the point $M_0(x, y, 0)$ and this point belongs to the curve γ. That is why its coordinates satisfy to equation (1). But it means, that the coordinates of the point M also satisfy this equation, because the coordinates x and y are the same as for M_0 and there is no z in equation (1).

Backwards, let coordinates of the point $M(x, y, z)$ satisfy (1). Then the coordinates of the point $M_0(x, y, 0)$ also satisfy this equation. Thus $M_0\in\gamma$. Moreover M и M_0 lies on one line, which is parallel to the axis Oz. Therefore $M\in\Phi$.

So, we have found out, that (1) is the equation of the surface Φ. In other words, we have proved the following result. *The equation of a cylindrical surface coincides with the equation of its directing line* γ *which lies in the plane* Oxy, *if the ruling lines are parallel to* Oz. Analogously, if ruling lines are parallel to Oy, then the equation of a cylindrical surface coincides with the equation of the directing line in the plane Oxz.

Backwards. If there is no z in the equation of a surface, we can say that this surface is cylindrical and its ruling lines are parallel to Oz.

Example. Let a surface be defined by the equation $y^2 = 2z$. There is no x in this equation. Thus this surface is a cylindrical one and its ruling lines are parallel to. The directing line lies in the plane Oyz and its equation is

$$\begin{cases} y^2 = 2z, \\ x = 0. \end{cases}$$

Thus the ruling line is parabola. The name of the surface is "parabolic cylinder". It is drawn on the fig. 33.

Since the equation of the cylindrical surface coincides with the equation of the directing curve, then the list of the cylindrical surfaces of the second order coincides with the list of directing curves of the second order.

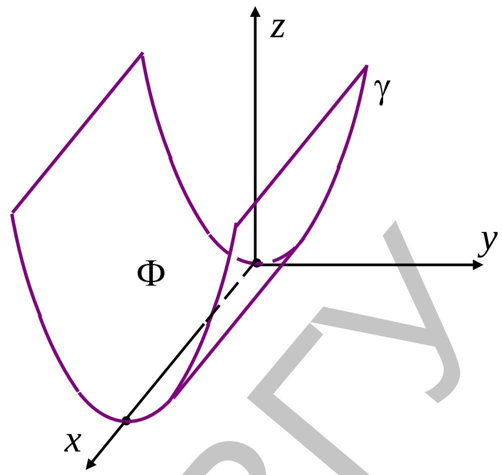


fig.33

1. Elliptical cylinder	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
2. Imaginary elliptical cylinder (∅)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$
3. Hyperbolic cylinder	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
4. Parabolic cylinder	$y^2 = 2px$
5. A pair of intersecting planes	$a^2x^2-b^2y^2=0$
6. A pair of imaginary planes that intersect in the real line	$a^2x^2+b^2y^2=0$
7. A pair of parallel planes	$x^2=a^2$
8. A pair of matching planes (or the double plane)	$x^2=0$
9. A pair of imaginary parallel planes (∅)	$x^2=-a^2$

Exercise. Determine for yourself what the surface is shown in each of the following figures.

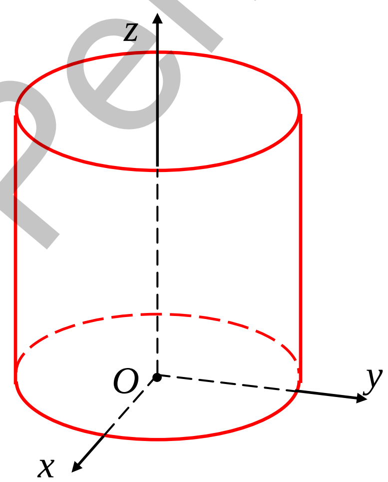


fig.34

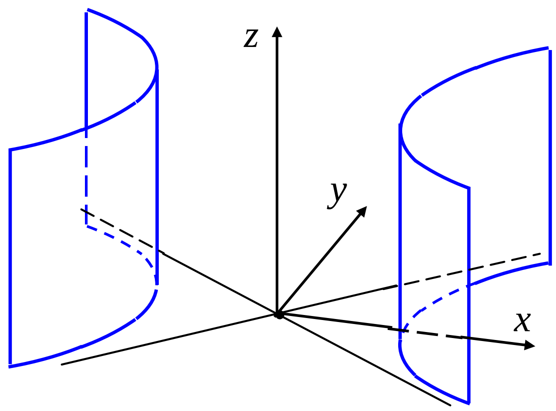


fig.35

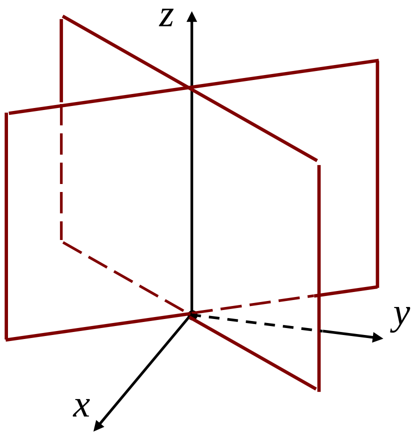


fig.36

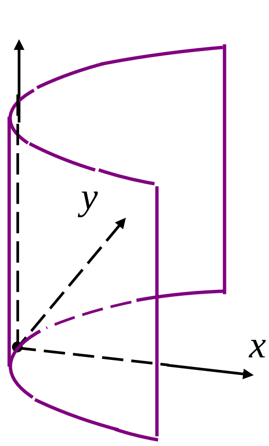


fig.37

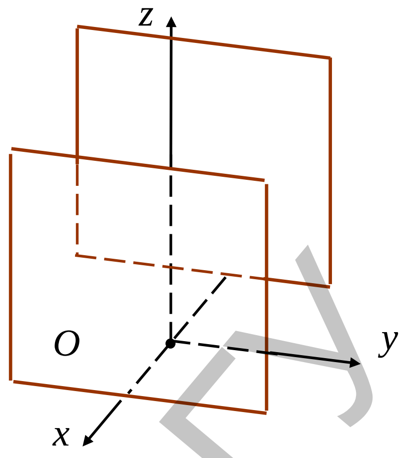


fig.38

§2. *Conical surfaces*

Definition. We say that a surface is a conical surface, if it consists of the straight lines that passes through one point O and through each point of some curve γ. The lines are called the ruling lines, the point O is called the vertex and the curve is called the directing curve (fig. 39).

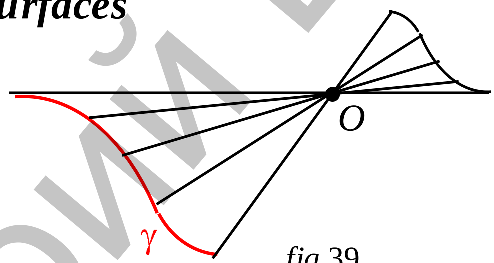


fig.39

Let's choose the Cartesian coordinate system so that the initial point coincides with the vertex of a conical surface Φ. Suppose, that the directing curve is ellipse and it is located in the plane $z = c$. Then its equation is

$$\gamma: \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \\ z = c \end{cases} \qquad (*)$$

Let $M(x_1, y_1, z_1)$ be an arbitrary point of the surface Φ. Then the whole straight line OM must lay on the surface. Its parametric equations:

$$OM: \begin{cases} x = x_1 t, \\ y = y_1 t, \\ z = z_1 t. \end{cases}$$

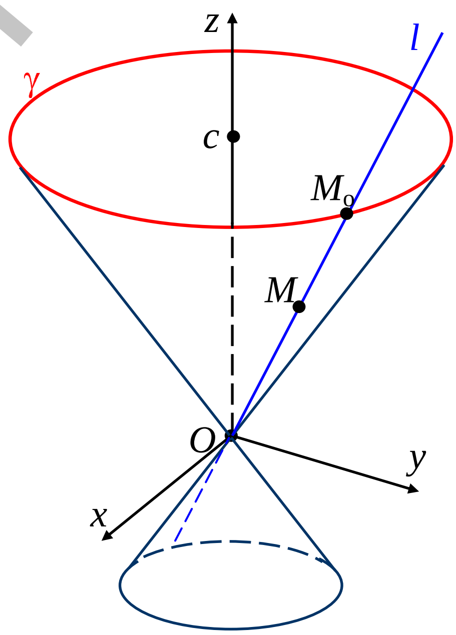


fig.40

Suppose that OM intersects γ at the point $M_o(x_o, y_o, c)$. Then the coordinates (x_o, y_o, c) must satisfy the equation of the straight line OM:

$$\begin{cases} x_o = x_1 t, \\ y_o = y_1 t, \\ c = z_1. \end{cases} \quad \Rightarrow \quad \begin{cases} x_o = x_1 c/z_1, \\ y_o = y_1 c/z_1, \\ t = c/z_1. \end{cases}$$

Now we substitute these coordinates in the equation of the ellipse:

$$\frac{(x_1 c/z_1)^2}{a^2} + \frac{(y_1 c/z_1)^2}{b^2} = 1.$$

We transfer 1 to the left and multiply this equation on z_1/c:

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - \frac{z_1^2}{c^2} = 0. \qquad (2)$$

Backwards, let coordinates of a point $M(x_1, y_1, z_1)$ satisfies equation (2). Arbitrary point on the straight line OM has coordinates $(x_1 t, yt, z_1 t)$. We substitute this coordinates in equation (2):

$$\frac{(x_1 t)^2}{a^2} + \frac{(y_1 t)^2}{b^2} - \frac{(z_1 t)^2}{c^2} = 0, \quad t^2\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - \frac{z_1^2}{c^2}\right) = 0, \quad t^2 \cdot 0 = 0.$$

We got a true equality. In means, that the entire straight line lies on the surface. If we substitute $z = c$ in (2), we get

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

This equation of an ellipse. Therefore, (2) is the equation of conical surface. We omit indexes and , we finally obtain the canonical equation of the cone.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

Analogously, if , the directing curve is a hyperbola,

$$\begin{cases} \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1, \\ z = c, \end{cases}$$

then we get the equation of conical surface

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0.$$

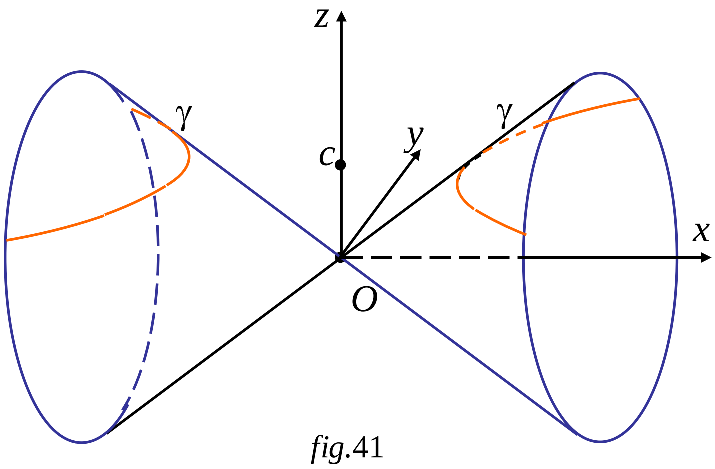


fig.41

It is the surface of the same type. The only difference is the following: its axis is Ox instead of Oz.

Let the directing curve is parabola

$$\begin{cases} x^2 = 2py, \\ z = c. \end{cases}$$

Then we get the surface of the same tipe but it is located as it is shown on fig. 42. We accept it without proof.

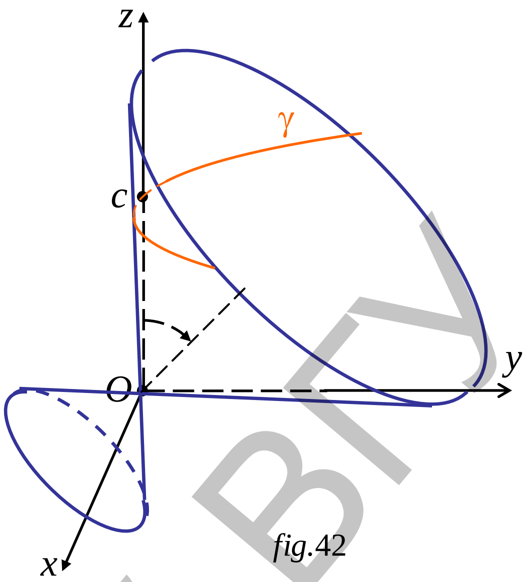


fig.42

If the directing curve is a pair of straight lines, then conical surface is a pair of planes. This planes must intersect, because both of them passes through the initial point of the coordinate system (see fig.43 and 44).

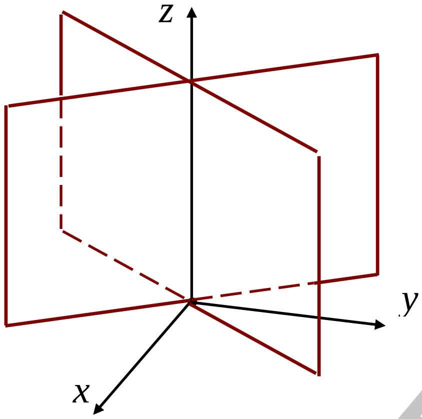


fig.43

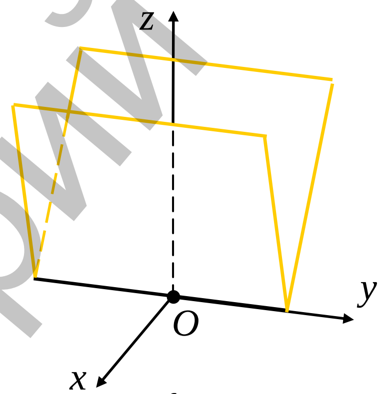


fig.44

So, we found out, that there are 4 types of conical surfaces:

1. The cone with the canonical equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$

2. A pair of intersecting planes $a^2x^2-b^2y^2=0$.

3. A pair of imaginary planes that intersect in the real line $a^2x^2+b^2y^2=0$.

4. A pair of matching planes (or the double plane) $x^2=0.$

§3. *Surface of revolution*

Let a curve γ be located in the plane Oyz. We rotate it around the axis Oz. Then we get a surface Φ, which is called a surface of revolution. Each point of the curve γ describes a circle with the center on the axis Oz. This circle is called parallel.

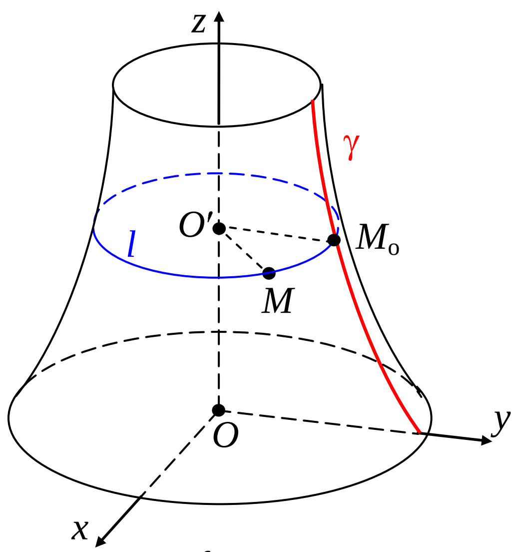


fig.44

Let

$$\varphi(y,z)=0 \qquad (3)$$

be the equation of the curve γ in the plane Oyz. Then it is defined in the space by the system

$$\begin{cases} \varphi(y,z)=0, \\ x=0. \end{cases}$$

We accept without proof, that the equation of the surface of revolution is

$$\varphi(\sqrt{x^2+y^2}, z)=0. \tag{4}$$

In other words, if we want to make the equation of the surface of revolution around Oz from the equation of the curve γ, we should replace y by the expression $\sqrt{x^2+y^2}$ and we should remain z without changing.

Conversely, if the equation of the surface contains the expression $\sqrt{x^2+y^2}$, and there is no more x and y in this equation, then we can say that our surface is the surface of revolution around Oz.

***Example* 1.** Let γ be a circle in the plane Oyz with radius b and with the center $A(0,\ a,\ 0)\in Oy$, $a>b>0$. If we rotate the circle around Oz, then we get a surface, which is called the torus (fig. 45). Let's make its equation.

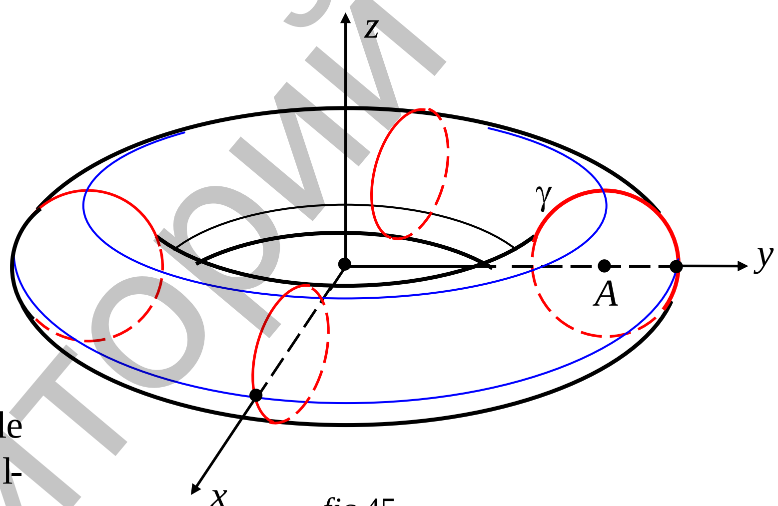


*fig.*45

The equation of the circle γ in the plain Oyz is the following:

$$(y-a)^2+z^2=b^2.$$

We remain z without changing, replace y by the expression $\sqrt{x^2+y^2}$:

$$(\sqrt{x^2+y^2}-a)^2+z^2=b^2.$$

It is the equation of the torus.

***Пример* 2.** Let the surface Φ be defined by the equation

$$x^2+z^2=2y.$$

We can rewrite it as follows:

$$(\sqrt{x^2+z^2})^2=2y.$$

Coordinates x and z are included in the equation only in the term $\sqrt{x^2+z^2}$. Thus our equation is a surface of revolution around Oy.

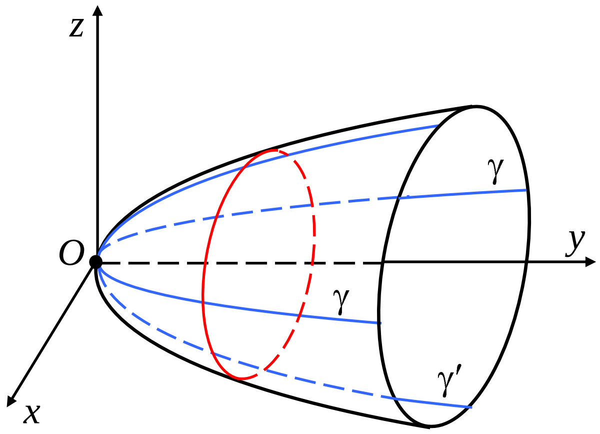


*fig.*46

In order to obtain the equation of the curve, which is rotated, we replace $\sqrt{x^2+z^2}$ by x and we get the equation of the curve γ in the plane Oxy: $x^2=2y$. This curve is defined in the space by the system

$$\begin{cases} x^2=2y, \\ z=0. \end{cases}$$

Thus this curve is a parabola and the surface is called a paraboloid of revolution.

§4. *Ellipsoid.*

Definition. *Ellipsoid* is a surface which has a canonical equation of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \qquad (5)$$

It is drawn on figure 47.

The cross section of this surface by any plane is ellipse.

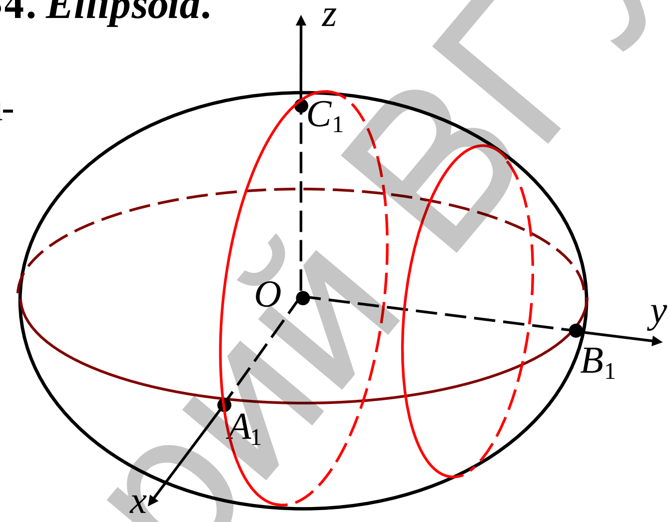


*fig.*47

Geometric properties of the ellipsoid.

1. From equation (5) we get

$$\frac{x^2}{a^2}\le 1,\ \frac{y^2}{b^2}\le 1,\ \frac{z^2}{c^2}\le 1.$$

Thus $|x|\le c$, $|y|\le b$, $|z|\le c$. It means that the ellipsoid is contained in a parallelepiped, which is defined by these inequalities.

2. Coordinate axes intersect the ellipsoid at the points:

$$Ox:\ A_1(a,0,0),\ A_2(-a,0,0), \qquad Oy:\ B_1(0,b,0),\ B_2(0,-b,0),$$
$$Oz:\ C_1(0,0,c),\ C_2(0,0,-c).$$

This points are called the vertexes of the ellipsoid.

3. The coordinate axes are axes of symmetry of the ellipsoid, the coordinate planes are the planes of symmetry and origin of coordinates is the center of symmetry.

4. If $a=b$, the ellipsoid is the surface of revolution around Oz. Indeed, in this case, its equation can be rewritten as follows:

$$\frac{(\sqrt{x^2+y^2})^2}{a^2}+\frac{z^2}{c^2}=1.$$

Analogously, if $a=c$ the ellipsoid is the surface of revolution around Oy, and if $b=c$ it is the surface of revolution around Ox.

If $a=b=c$ the ellipsoid is the sphere:

$$x^2+y^2+z^2=a^2 \qquad (**).$$

Arbitrary ellipsoid can be obtained from the sphere (**) by uniform compression (expansion) along two mutually perpendicular directions.

§5. *One-sheet and two-sheet hyperboloid.*

Определение. One-sheet and two-sheet hyperboloids are the surfaces with the canonical form of the equation respectively

$$\Phi_1:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad (6) \qquad\qquad \Phi_2:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1. \qquad (7)$$

If the plane $z = h$ is parallel to the coordinate plane Oxy, then it intersects both surfaces by an ellipse, but if $h<c$, the cross-section of Φ_2 is the empty set. If the plane $x=h$ is parallel to the coordinate plane Oyz, then it intersect both surfaces by a hyperbola, but only for $h=a$ the cross-section of Φ_1 is the is the pair of intersecting straight lines. Analogously, the plane $y=h$, which is parallel to the coordinate plane Oxz intersect both surfaces by a hyperbola, but only for $h=b$ the cross-section of Φ_1 is the is the pair of intersecting straight lines.

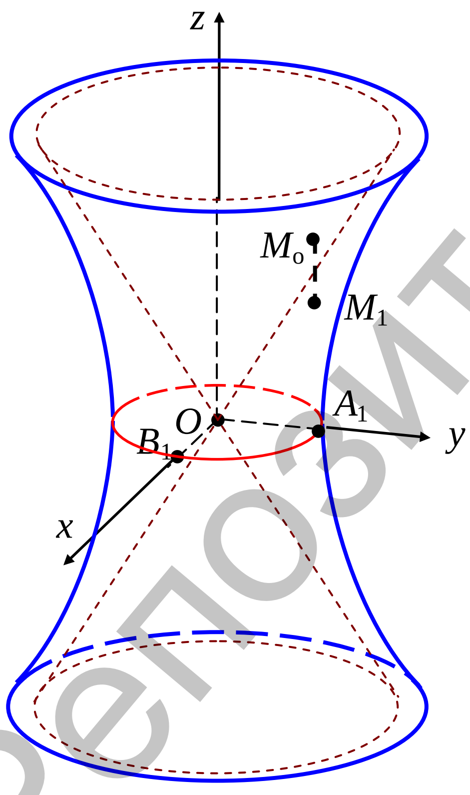


*fig.*48

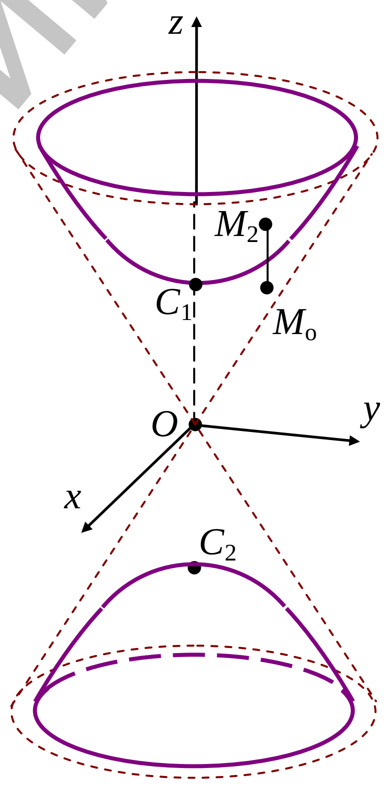


*fig.*49

<u>*Other geometric properties of the hyperboloids*</u>.

1. The coordinate axes intersect Φ_1 at the points

Ox: $A_1(a,0,0)$, $A_2(-a,0,0)$; Oy: $B_1(0,-b,0)$, $B_2(0,b,0)$.

This points are called the vertexes. The axis Oz doesn't intersect Φ_1. The axis Oz intersect Φ_2 at the points $C_1(0, 0, c)$, $C_2(0, 0, -c)$. They are also called the vertexes. The axes Ox and Oy don't intersect Φ_2.

2. The coordinate axes are axes of symmetry of the hyperboloids, the coordinate planes are the planes of symmetry and origin of coordinates is the center of symmetry.

3. If $a = b$, both hyperboloids the surfaces of revolution around Oz.

4. Let Φ_o be the cone, which is defined by the equation

$$\Phi_o:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$

Both hyperboloids asymptotically approach to the cone at infinity.

5. We have already seen that a pair of lines can be obtained in cross-section of Φ_1. It turns out that Φ_1 is a ruled surface. A pair of lines lying on the surface passes through every point of Φ_1 (fig. 50).

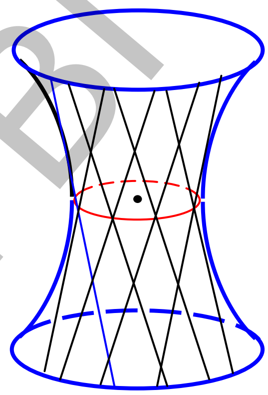


fig.50

§6. *Elliptic and hyperbolic paraboloid.*

Определение. *Elliptic and hyperbolic paraboloids* are the surfaces which have the canonical form of the equation respectively

$$\Phi_3:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=2z \qquad (8) \qquad\qquad \Phi_4:\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=2z. \qquad (9)$$

If the plane $z = h$ is parallel to the coordinate plane Oxy, then it intersects Φ_3 by an ellipse and it intersects Φ_4 by a hyperbola. Only for $h = 0$ the cross-section of Φ_1 is the is the pair of intersecting straight lines. If the plane $x = h$ is parallel to the coordinate plane Oyz, then it intersect both surfaces by a parabola. And the same is true for the planes $y = h$, which are parallel to the coordinate plane Oyz.

Elliptic paraboloid is shown on fig.51 and hyperbolic paraboloid is shown on fig.52. Cross-sections, that are parabolas are signed as $\gamma_1, \gamma'_1, \gamma_2, \gamma'_2$. The ellipse is signed as γ_3 and hyperbola is signed as γ_4.

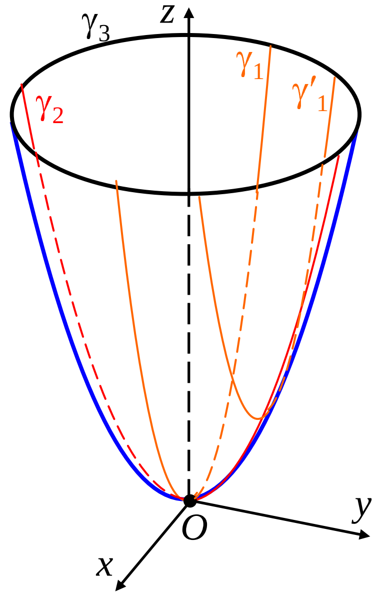


fig.51

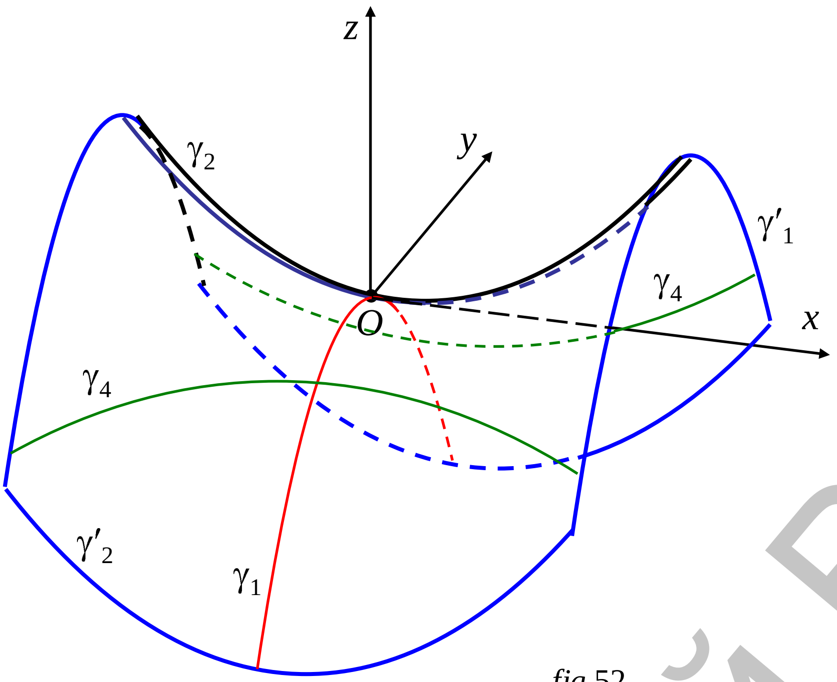


fig.52

<u>*Other geometric properties of the paraboloids*</u>.

1. The coordinate axes intersect both surfaces only at the point $O(0, 0, 0)$, which is called the vertex.

2. Only the axis Oz is the axis of symmetry of the paraboloids, only the planes Oxz и Oyz are the planes of symmetry for both surfaces (except the case $a=b$ for Φ_3).

4. If $a=b$ Φ_3 is the surfaces of revolution around Oz.

5. We have already seen that a pair of lines can be obtained in cross-section of Φ_4. It turns out that Φ_4 is a ruled surface. A pair of lines lying on the surface passes through every point of Φ_1.

§7. *Classification of the surfaces of the second order.*

Определение. *Surface of the second order* is the locus of points in space whose coordinates satisfy the equation of the form:

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xy + 2a_{23}yz +$$
$$+ 2a_1x + 2a_2y + 2a_3z + c = 0, \qquad (10)$$

where among the coefficients a_{ij}, $i, j = 1, 2, 3$ there is at least one non-zero coefficient. The expression in the first line is called *the quadratic part of the equation*, c is called *the constant term*, $2a_1x + 2a_2y + 2a_3z$ is called *the linear part of the equation*.

The quadratic part of the equation (8) is a quadratic form. Its coefficients form a matrix

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

It is proved in the course in linear algebra that any matrix of the quadratic form with the help of rotation of the coordinate axes can be reduced to the diagonal form

$$\mathbf{A}'=\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Then in the new Cartesian coordinate system $Ox'y'z'$ with the same origin the quadratic part of the equation (10) takes the form:

$$\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2. \qquad (11)$$

This form is called the diagonal one. In the new coordinate system we have the equation:

$$\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + 2b_1 x' + 2b_2 y' + 2b_3 z' + c = 0 \qquad (12)$$

Then we select the perfect squares. We will study this procedure in details on the practical classes. After this procedure we get the equation of the form

$$\lambda_1 (x'')^2 + \lambda_2 (y'')^2 + \lambda_3 (z'')^2 = -c'.$$

If $c' \neq 0$ we can divide this equation on $|c'|$.

As the final result we will get one of the following equations:

Surface	Canonical form of equation
1. Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$
2. Imaginary ellipsoid (Ø)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1,$
3. Imaginary cone (точка)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0,$
4. Two-sheet hyperboloid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1,$
5. One-sheet hyperboloid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$
6. Cone	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0,$
7. Elliptic paraboloid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z,$

8. Hyperbolic paraboloid	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z,$
9. Imaginary elliptical cylinder	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$
10. Imaginary elliptical cylinder	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1,$
11. A pair of imaginary planes intersecting on the real line	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0,$
12. Hyperbolic cylinder	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$
13. A pair of intersecting planes	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0,$
14. Parabolic cylinder	$x^2=2py$
15. A pair of parallel planes	$x^2=a^2$
16. A pair of matching planes (double plane)	$x^2=0$
17. A pair of imaginary parallel planes	$x^2=-a^2$

§8. *Examples of solving problems*

1. *With the translation of origin lead the equation of the second order to the canonical form. Determine the type of curve and represent it in the original coordinate system*:

$$4x^2+z^2-24x+8y+2z+5=0.$$

Solution. We select the perfect squares:

$$4(x^2-6x+9)-36+(z^2+2z+1)-1+8y+5=0,$$

$$4(x-3)^2+(z+1)^2+8y-32=0,$$

$$4(x-3)^2+(z+1)^2=-8(y-4).$$

Then we make the change of coordinates:

$$\begin{cases} x'=x-3 \\ y'=y-4, \\ z'=z+1. \end{cases}$$

It is equivalent to the transfer of the origin to the point $O'(3, 4,-1)$. After the change of coordinates we get the equation

$$4x'^2+z'^2=-8y', \quad \Leftrightarrow \quad (x')^2+\frac{(z')^2}{4}=-8y',$$

Это уравнение задает эллиптический параболоид, ось которого будет $O'y'$. In the section of the plane $y'=-8$ we get the ellipse

$$\frac{(x')^2}{16}+\frac{(z')^2}{64}=1.$$

It has semiaxes 4 and 8. We must take it into account while drawing.

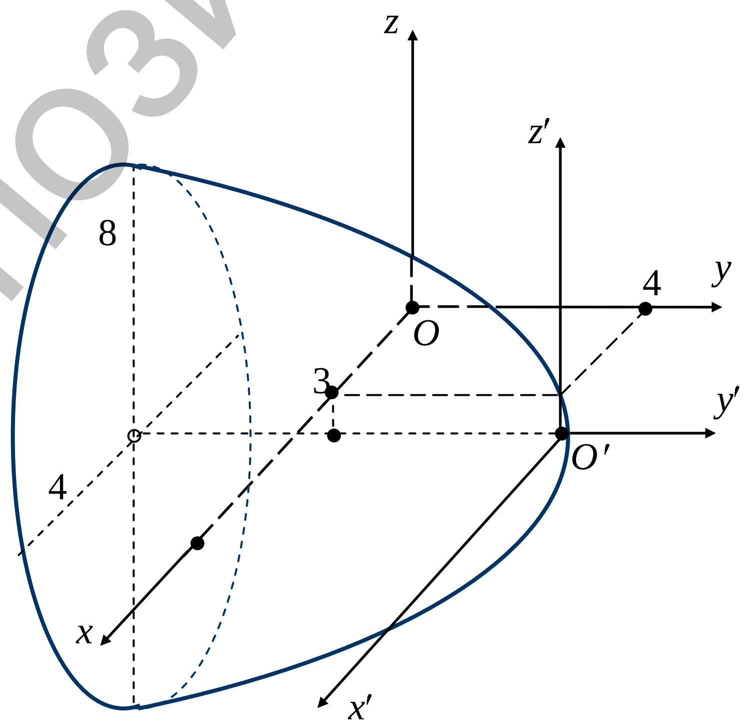


fig.53

3. *Find the equation of the surface obtained by rotating of the curve*

$$\begin{cases} z=2y-2, \\ x=0. \end{cases}$$

i) around Oz; ii) around Oy. *Identify the type of the surface. Draw it in the Cartesian coordinate system.*

Solution. This system of equations defines a straight line *l*, which lies in the plane *Oyz*. The first equation is the equation of *l* in this plane.

i) If we rotate this curve around *Oz*, we remain z without change and we replace y on the term $\sqrt{x^2+y^2}$. Then we get the equation

$$z=2\sqrt{x^2+y^2}-2, \Rightarrow$$

$$x^2+y^2-\frac{(z+2)^2}{4}=0.$$

This equation defines a cone, that has the axis Oz, and the vertex of the cone is the point $O'(0,0,-2)$.

Next, we draw the image of the cone.

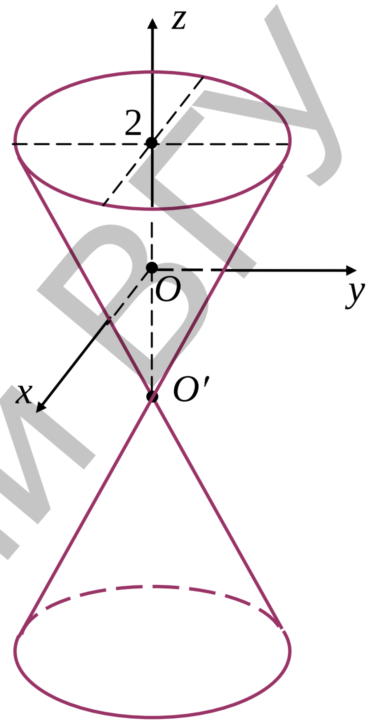


*fig.*54

1) If we substitute $z=2$ in the equation of the cone, then we get $x^2+y^2=4$. Thus the cross-section of the cone by the plane $z=2$ is the circle with radius 2. Through the point $z=2$ on the axis Oz we draw auxiliary lines, parallel to the axes Ox and Oy; then we plot from this point the segments of length 2 and we get 4 points. Next we draw an ellipse through this points. This ellipse is the image of a circle. We choose the scale on the axis *Ox* two times less than on the axes Oy и *Oz*.
2) We draw an ellipse equal to the first one with the center $z=-6$ on the axis *Oz*.
3) We draw the tangent lines to both ellipses through the point O'. *It should be emphasized that the point of tangency in do not coincide with the vertices of the ellipse*.
4) The part of the lower ellipse lying between the points of tangency we draw with the dashed line.

ii) Similarly, in order to obtain the equation of surface of revolution around Oy, we in the equation of the line remain y without change, and we replace z by the term $\sqrt{x^2+z^2}$. Then we get the equation

$$x^2-4(y-1)^2+z^2=0,$$
$$\frac{x^2}{4}-(y-1)^2+\frac{z^2}{4}=0.$$

This equation defines a cone with the vertex $O'(0, 1, 0)$ and with the axis Oy. While drawing this surface we take into consideration, that the cross-section by the plane $y=3$ is a circle of radius 4.

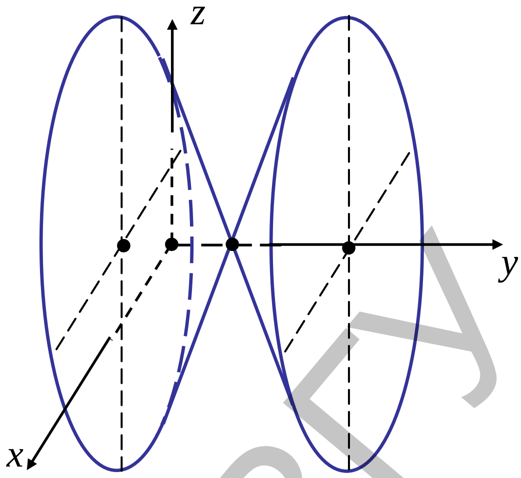


*fig.*53

4. *Find out, if the following equation*

$$-x^2+\frac{y^2}{4}+\frac{z^2}{4}=1$$

defines a surface of revolution. If yes, by rotation of what kind of curve (write an equation) around which axis it is received? Draw this surface.

Solution. This surface is a one-sheet hyperboloid. In the equation of the surface we can identify the expression $\sqrt{y^2+z^2}$:

$$-x^2+\frac{(\sqrt{y^2+z^2})^2}{4}=1,$$

and there is no x and z anywhere else in the equation. Therefore, we conclude immediately that this equation defines a surface of revolution around Oy. In order to determine what curve rotates, we replace $\sqrt{y^2+z^2}$ by y and we get the equation of the curve in the plane $z=0$:

$$-x^2+\frac{y^2}{4}=1$$

To define this curve in space, we need to write the system of equations

$$\begin{cases} -x^2+\frac{y^2}{4}=1, \\ z=0. \end{cases}$$

We can also replace $\sqrt{y^2+z^2}$ by z, and then we get the equation of the curve γ', that lies in the plane $y=0$:

$$\begin{cases} -x^2+\frac{y^2}{4}=1, \\ y=0. \end{cases}$$

5. *Make equation of each point of which is equidistant from the plane* $x = -a$ *and from the point* $F(a, 0, 0)$.

Solution. Let $M(x, y)$ be an arbitrary point on the surface. Then

$$|MF| = \sqrt{(x-a)^2 + y^2 + z^2},$$

and the distance from M to the plane is equal $|x + a|$. According to the statement of the problem

$$\sqrt{(x-a)^2 + y^2 + z^2} = |x + a|.$$

We make a square of both sides:

$$x^2 - 2ax + a^2 + y^2 + z^2 = x^2 + 2ax + a^2$$

$$y^2 + z^2 = 4ax \Leftrightarrow \frac{y^2}{2a} + \frac{z^2}{2a} = 2ax.$$

This equation defines an elliptic paraboloid whose axis is Ox.

6. *Find the point of intersection of the ellipsoid* $\frac{x^2}{9} + \frac{y^2}{36} + \frac{z^2}{81} = 1$ *and the straight line* $\frac{x-1}{1} = \frac{y-4}{-6} = \frac{z+6}{12}$.

Solution. We rewrite the equation of the line in parametric form

$$\begin{cases} x = 1 + t, \\ y = 4 - 6t, \\ z = -6 + 12t. \end{cases}$$

Then we substitute these equalities in the equation of the ellipsoid:

$$\frac{(1+t)^2}{9} + \frac{4(2-3t)^2}{36} + \frac{9(-2+4t)^2}{81} = 1,$$

$$1 + 2t + t^2 + 4 - 12t + 9t^2 + 4 - 16t + 16t^2 = 9,$$

$$26t^2 - 26t = 0 \quad \Leftrightarrow \quad 26t(t-1) = 0.$$

We have two solutions: $t_1 = 0$, $t_2 = 1$. Next, we substitute these solutions in the equation of the line, and we find to points $M_1(1, 4, -6)$, $M_2(2, -2, 6)$.

Answer: $M_1(1, 4, -6)$, $M_2(2, -2, 6)$.

7. *Determine which curve is obtained in the cross section of the surface* $\frac{x^2}{9} + \frac{y^2}{16} - \frac{z^2}{4} = 1$

i) *by the plane* $y = 2z$; **ii)** *by the plane* $y = 2z + 2$.

Solution. **i)** This surface is a one-sheet hyperboloid. We substitute $y = 2z$ in the equation of the surface:

$$\frac{x^2}{9}+\frac{(2z)^2}{16}-\frac{z^2}{4}=1 \quad \Leftrightarrow \quad x^2=9.$$

It is the equation of the projection of the curve on the coordinate plane *Oxz*. It determines a pair of parallel lines. Consequently, our curve is also a pair of parallel lines.

ii) We substitute $y=2z+4$ in the equation of the surface:

$$\frac{x^2}{9}+\frac{(2z+2)^2}{16}-\frac{z^2}{4}=1 \quad \Leftrightarrow \quad \frac{x^2}{9}+\frac{z^2+4z+4-z^2}{4}=1 \quad \Leftrightarrow \quad x^2=9z.$$

It is the equation of the projection of the curve on the coordinate plane *Oxz*. It determines a parabola. Consequently, our curve is also a parabola.

Tasks for independent solving

1. *With the translation of origin lead the equation of the second order to the canonical form. Determine the type of curve and represent it in the original coordinate system*:

1. $5x^2+4y^2+20x-8y+20z-16=0$,
2. $5y^2+2z^2+10x-10y+8z-7=0$,
3. $4x^2+2z^2-24x+12y-4z+14=0$,
4. $2x^2+6y^2+8x+12y-6z+11=0$,
5. $7x^2+2z^2+14x-14y-12z-3=0$,
6. $3x^2+7y^2+18x+14y-21z-8=0$,
7. $4x^2+2y^2+24x+8y-6z-4=0$,
8. $5x^2+4z^2+20x-20y+8z-16=0$,
9. $2y^2+6z^2+12x+8y-24z-4=0$,
10. $5x^2+4y^2-20x+8y-20z-16=0$,
11. $2x^2+5z^2-10x+10y+8z-7=0$,
12. $4x^2+3z^2-24x+12y+6z+15=0$

2. *Find the equation of the surface obtained by rotating of the given curve around the given axis. Identify the type of the surface. Draw it in the Cartesian coordinate system.*

1. $\begin{cases} \dfrac{x^2}{16}-\dfrac{y^2}{25}=1, \\ z=0, \end{cases}$ around Ox;

2. $\begin{cases} 5x+3z=0, \\ y=0, \end{cases}$ around Oz;

3. $\begin{cases} 2x-3y=0, \\ z=0, \end{cases}$ around Oy;

4. $\begin{cases} y^2=2z-6, \\ x=0, \end{cases}$ around Oz;

5. $\begin{cases} \dfrac{x^2}{16}-\dfrac{y^2}{9}=1, \\ z=0, \end{cases}$ around Oy;

6. $\begin{cases} \dfrac{y^2}{2}-\dfrac{z^2}{9}=1, \\ x=0, \end{cases}$ around Oz;

7. $\begin{cases} \dfrac{x^2}{4}+\dfrac{z^2}{9}=1, \\ y=0, \end{cases}$ around Ox;

8. $\begin{cases} x^2=4y+8, \\ z=0, \end{cases}$ around Oy;

9. $\begin{cases} x+2z=1, \\ y=0, \end{cases}$ around Oz;

10. $\begin{cases} 2x-3y=0, \\ z=0, \end{cases}$ around Ox;

11. $\begin{cases} \dfrac{x^2}{9}+\dfrac{y^2}{4}=1, \\ z=0, \end{cases}$ around Oy;

12. $\begin{cases} \dfrac{y^2}{9}-z^2=1, \\ x=0, \end{cases}$ around Oz.

Tasks for the practical classes

Practical classes 6.
Cylindrical and conical surfaces of the second order.
The sphere. The ellipsoid.

1. Which figures are determined in the Cartesian coordinate system by the following equations:

i) $\frac{x^2}{16}+\frac{y^2}{25}=1$; ii) $\frac{x^2}{16}-\frac{y^2}{25}=0$; iii) $y^2=8z$?

2. Make an equation of a figure, which is obtained

а) by the rotation of the straight line $\begin{cases} z-2=0, \\ y=0. \end{cases}$ around Ox;

б) by the rotation of the straight line $\begin{cases} x-2y=0, \\ z=0. \end{cases}$ Ox;

в) by the rotation of the same line around Oy.

3. Which figure is determined in the Cartesian coordinate system by the equation

$$-9x^2+36y^2+4z^2-18x+144y-8z+139=0?$$

4. Find coordinates of the center and the radius of the sphere

$$x^2+y^2+z^2+2x-6y+8z+10=0.$$

Draw it in the Cartesian coordinate system.

5. Find coordinates of the center and the radius of the circle

$$\begin{cases} x^2+y^2+z^2-12x+4y+24=0, \\ 2x+2y+z+1=0. \end{cases}$$

6. Find coordinates of the center and the semiaxes of the ellipsoid

$$4x^2+9y^2+6z^2-6x+16y-36z+49=0.$$

Draw it in the Cartesian coordinate system.

7. Find the semiaxes and the vertexes of the ellipse

$$\begin{cases} \frac{x^2}{12}+\frac{y^2}{4}+\frac{z^2}{3}=1, \\ x-2=0. \end{cases}$$

8. Find the point of intersection of the ellipsoid and the straight line

$\frac{x^2}{81}+\frac{y^2}{36}+\frac{z^2}{9}=1$; $\frac{x-3}{3}=\frac{y-4}{-6}=\frac{z+2}{4}$.

Home task.

1. Make an equation of a figure, which is obtained by the rotation of the ellipse

$\begin{cases} \frac{x^2}{4}+\frac{z^2}{9}=1, \\ y=0. \end{cases}$ **i)** around Ox; **ii)** around Oz.

Practical classes 7.

Hyperboloids. Paraboloids.

1. Write an equation of a figure obtained by rotating a hyperbola

$\begin{cases} x^2 - \dfrac{y^2}{4} = 0, \\ z = 0. \end{cases}$ **i)** around the axis Ox; **ii)** around the axis Oy.

Determine the type of the surface. Schematically draw it in a Cartesian coordinate system.

2. Write an equation of a figure obtained by rotating a parabola

$\begin{cases} x^2 = -2z, \\ z = 0. \end{cases}$ around the axis Oz.

3. Find the point of intersection of the figure of the second order and the straight line:

$$\frac{x^2}{4} + y^2 - \frac{z^2}{9} = -1; \quad \frac{x-3}{1} = \frac{y-1}{1} = \frac{z-6}{3}.$$

Name the type of the figure.

4. Find out, what figure is determined by the following system of equations

$\begin{cases} \dfrac{x^2}{16} + \dfrac{y^2}{16} - \dfrac{z^2}{4} = 1, \\ x = 2z. \end{cases}$

5. Find out what the figure given by the equation

$$2x^2 + 3y^2 + 6x - 18y - 12z + 47 = 0.$$

Schematically draw it in the Cartesian coordinate system and find coordinates of the vertex.

6. Ratio of distances from each point of a figure to the point $F(0, 0, 2)$ and to the plane $z = 1$ is equal $\sqrt{2}$. Write an equation of the figure.

Home task.

1. Find out what the figure given by the equation

$$2x^2 - 3y^2 + 12x + 12y - 12z - 42 = 0.$$

Schematically draw it in the Cartesian coordinate system and find coordinates of the vertex.

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