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MATHEMATICAL ANALYSIS

Study guide

for the self-organized work of the students of the specialty "Applied Informatics"

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This study guide is intended for self-organized work of the first-year students of the Mathematical Faculty taught with a specialization in "Applied Informatics". Theoretical material is outlined; the examples of problems solution are presented. Problems for solving in practical classes and individual assignments are also attached.

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Content

Introduction

This study guide is intended for self-organized work of the students of the Faculty of Mathematics taught with a specialization "Applied Informatics". It combines the lecture notes on the subject "Mathematical analysis" (section "Analytical Geometry on the plane"), examples of problems solving, problems for the solution at the practical classes and individual tasks for a self-sustaining solutions. The volume of material calculated on the basis of the theoretical training time available taking into account students' self-organized work. This study guide can not completely replace the lecture course on mathematical analysis. Many results are given without proofs, and some topics are omitted.

Number of options for an individual practice assignment is to be chosen in accordance with the serial number of the student in the teacher's records. Before solving the problem, examine the example of its solution. In case if there are unforeseen difficulties, you should consult the teacher.

Icon ■ in the text signifies the completion of the proof.

§1. *Coordinate system on the line and on the plane*

Let l be a line. Let's choose a point $O \in l$, which we will call the initial point, and the direction, which we will call *positive*. The direction opposite to positive is called *the negative one*. The positive direction usually is represented on the right and is denoted by the arrow. In positive direction from the point O we lay aside a segment OE, which is considered as a unit one. This segment defines the scale. The line with the segment OE and the direction is called *an axis*. We subscribe the point E by the number 1. Now we can tell about the distance on the axis.

Let M be an arbitrary point on the line. Let $x = |OM|$ (distance between O and M), if M belongs to the positive direction, and $x = -|OM|$, if M belongs to the negative direction. Then x is called *the coordinate of the point* M *on the axis*.

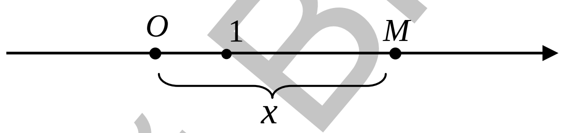


fig.1

Let two perpendicular axes are chosen on the plane and let O be their point of intersection. We call O *the initial point of the coordinate system* and the axes are called *the coordinate axes*. We denote them as Ox и Oy. In addition we require, that $90°$ rotation, which matches the positive direction of Ox with the positive direction of Oy, should come about counterclockwise. Then we say that these axes together with the point O forms the *Cartesian coordinate system* Oxy on the plane.

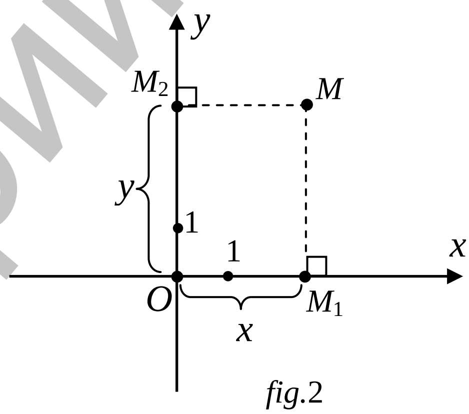


fig.2

Let M be an arbitrary point on the plane. Let's draw two lines $MM_1 \perp Ox$ and $MM_2 \perp Oy$. Let the point M_1 has a coordinate x on the axis Ox, and the point M_2 has a coordinate y on the axis Oy. Then we say, that the point M has coordinates (x, y) and we write $M(x, y)$. The coordinate x is called *the abscissa* of the point M, and the coordinate y is called *the ordinate* of the point M. The pair of numbers (x, y) is called *the Cartesian coordinates of the point* M. The points M_1 and M_2 are called *the projections of the point* M *on the coordinate axes*.

Coordinate axes divide the plane on four right angles and we call these angles *the coordinate angles* or *the quadrants*. The numbers of the quadrants are shown on the figure 3.

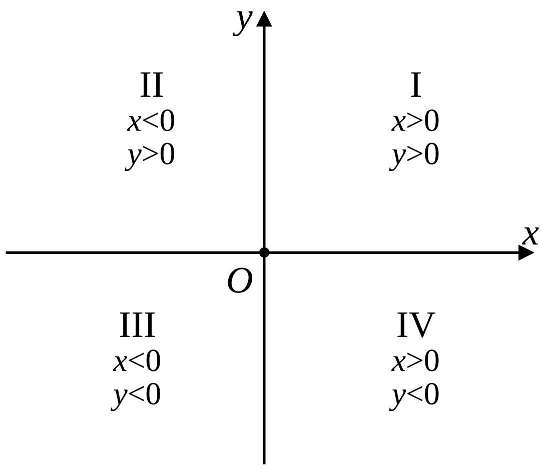


fig.3

The signs of x and y are written there as well.

Let an arbitrary ray OP with the unit segment OE be given on the plane. We will call it *a polar axis*, and we will call the point O *the initial point* or *the pole*. Let M be an arbitrary point on the plane. Denote $r = |OM|$, and let φ be the angle between the rays OP и OM. Moreover, if the rotation from the ray OP to the ray OM comes about counterclockwise, we consider that φ is positive, and if this rotation comes about clockwise, we consider that φ is negative.

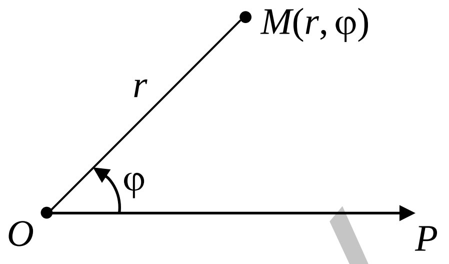


*fig.*4

Definition. The pair (r, φ) is called *the polar coordinates of the point M*, and a combination of the point O and of the axis OP is called *the polar coordinate system on the plane*.

It is obvious, that $0 \le r < +\infty$, and for the angle φ we need to make arrangement, that $0 \le \varphi < 2\pi$, or that $-\pi < \varphi \le \pi$. In the case $r = 0$, we consider, that φ is not defined.

Now suppose, that two coordinate systems are given on the plane at the same time: one of them is a Cartesian one and another is a polar. Suppose, that they have the same initial point O and the positive direction of Ox matches with the polar axis OP. Then from the triangles ΔOMM_1 and ΔOMM_2 we obtain

$$\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi. \end{cases} \qquad (1)$$

$$\begin{cases} r = \sqrt{x^2 + y^2}, \\ \cos\varphi = x/r, \ \sin\varphi = y/r. \end{cases} \qquad (2)$$

We emphasize, that the values $\sin\varphi$ or $\cos\varphi$ separately do not allow us to define the angle φ. Therefore in the formulas (2) we should write both equalities $\cos\varphi = x/r$, $\sin\varphi = y/r$.

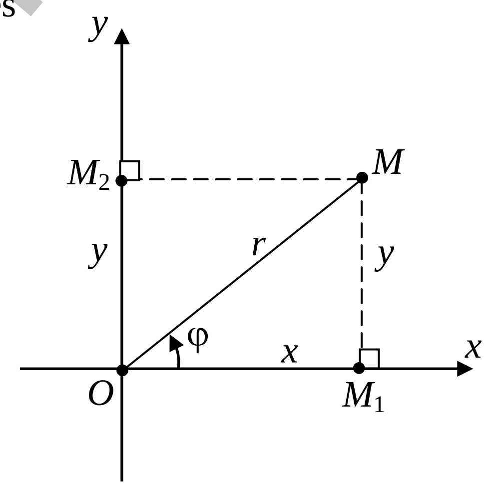


*fig.*5

§2. *Notion of the vector*

Definition. *A segment* AB is called *a directed segment*, if it is indicated, which of the points A or B is its beginning point and which is the end point of this segment. If A is the beginning point and B is the end point, we denote this directed segment as $\overrightarrow{AB}$, and on the drawing we indicate its end by the arrow (fig. 6).

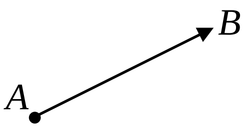


*fig.*6

Definition. Let $\overrightarrow{AB}$ and $\overrightarrow{A_1B_1}$ be directed segments. We say, that $\overrightarrow{AB}$ and $\overrightarrow{A_1B_1}$ are co-directed, if the rays AB and A_1B_1 are co-directed; in this case we write $\overrightarrow{AB}\uparrow\uparrow\overrightarrow{A_1B_1}$. We say, that $\overrightarrow{AB}$ and $\overrightarrow{A_1B_1}$ are counter-directed, if the rays AB and A_1B_1 are counter-directed; in this case we write $\overrightarrow{AB}\uparrow\downarrow\overrightarrow{A_1B_1}$.

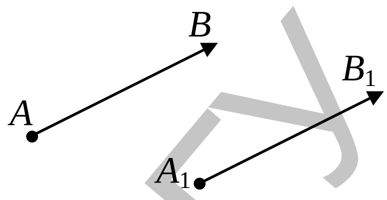


fig.7

Two directed segments $\overrightarrow{AB}$ and $\overrightarrow{A_1B_1}$ are called [equivallent]() or [equal](), if they can be matched by means of parallel translation (i.e. they are codirected and have the equal length. We write $\overrightarrow{AB}=\overrightarrow{A_1B_1}$.

Definition. *Vector* is a class of equivalent to each other directed segments. In other words, *each directed segment defines a vector and the equivalent segments define the same vector.*

We denote vectors by a small letter with an arrow on the top: $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}\ldots$ If the vector $\vec{\mathbf{a}}$ is determined by the directed segment $\overrightarrow{AB}$, we write $\vec{\mathbf{a}}=\overrightarrow{AB}$. In this case we say, that $\overrightarrow{AB}$ is the *vector* $\vec{\mathbf{a}}$, *laid aside from the point* A. On the drawing a vector can be represented by any directed segment, which determines this vector.

Definition. *Length of the vector* is the length of any directed segment, which determines this vector. We denote the length of the vector $\vec{\mathbf{a}}$ by $|\vec{\mathbf{a}}|$. *The direction of the vector* is the direction of any directed segment, which determines this vector.

In other words, the vector has the direction and the length, but it has no defined beginning point and no end point. To lay a vector aside from a point A means to indicate a directed segment $\overrightarrow{AB}=\vec{\mathbf{a}}$.

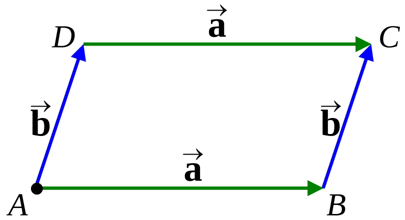


fig.8

Example. Let $ABCD$ be a parallelogram. Then $\overrightarrow{AB}=\overrightarrow{DC}$, and therefore this directed segments define the same vector $\vec{\mathbf{a}}$. Analogously $\overrightarrow{BC}$ и $\overrightarrow{AD}$ define the same vector $\vec{\mathbf{b}}$.

Definition. The vector, whose length is equal to zero, is called *the null vector*, and we denote this vector as $\vec{\mathbf{o}}$. This vector is defined by directed segment, whose beginning point and end point coincide. There is the only one null vector. A vector, whose length is equal to 1, is called *a unit vector*.

Definition. Two vectors $\vec{\mathbf{a}}$ и $\vec{\mathbf{b}}$ are called *co-directed*, if the directed segments defining these vectors are co-directed. We write $\vec{\mathbf{a}}\uparrow\uparrow\vec{\mathbf{b}}$. Two vectors $\vec{\mathbf{a}}$ и $\vec{\mathbf{b}}$ are called counter-directed, if the directed segments defining these vectors are counter-directed. We write $\vec{\mathbf{a}}\uparrow\downarrow\vec{\mathbf{b}}$. Two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are called collinear, if they are co-directed or counter-directed. We write $\vec{\mathbf{a}}\,||\,\vec{\mathbf{b}}$. We arrange that a direction of $\vec{\mathbf{o}}$ is not defined and it is collinear to each vector.

We underline once more that a vector and the directed segment are not the same. A vector is defined by a directed segment; i.e. in order to define a vector, one should point out a directed segment, which determines this vector. A vector itself has no the beginning point and the end point. We can put aside a vector from any point, and then we get a directed segment. This directed segment has the beginning and the end points. On practical classes we will call a directed segment by the word ‘vector’ for simplicity.

§3. *A sum and a difference of two vectors*

Definition. Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be arbitrary vectors. Let us choose an arbitrary point O and lay aside the vector $\vec{\mathbf{a}}$ from this point: $\vec{\mathbf{a}} = \overrightarrow{OA}$. Then we lay aside the vector $\vec{\mathbf{b}}$ from the point A: $\vec{\mathbf{b}} = \overrightarrow{AB}$. Let $\vec{\mathbf{c}}$ be the vector, which is defined by the directed segment $\overrightarrow{OB}$. Then we say, that the vector $\vec{\mathbf{c}}$ is *a sum of the vectors* $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. We write $\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}}$.

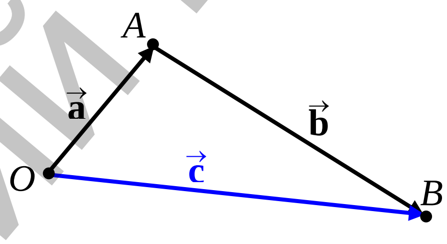


*fig.*9

This method to draw a sum of two vectors is called *the rule of triangle*. If one wants to draw a sum according to this rule, one should lay aside the second vector from the end point of the first vector.

Let a vector $\vec{\mathbf{a}}$ be defined by the directed segment $\overrightarrow{AB}$, and a vector $\vec{\mathbf{x}}$ be defined by the directed segment $\overrightarrow{BA}$. Then we say, that $\vec{\mathbf{x}}$ is *the opposite* to the vector $\vec{\mathbf{a}}$ and we write $\vec{\mathbf{x}} = -\vec{\mathbf{a}}$. According to the rule of triangle the vector $\vec{\mathbf{a}} + \vec{\mathbf{x}}$ is defined by the directed segment $\overrightarrow{AA}$, thus it is the null vector. So, the following equality is true: $\vec{\mathbf{a}} + (-\vec{\mathbf{a}}) = \vec{\mathbf{o}}$.

Properties of the sum operation.

$\forall\ \vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ are true

1. $\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}}$ (*commutativity*);

2. $(\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} = \vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}})$ (*assosiativity*);

3. $\vec{\mathbf{a}} + \vec{\mathbf{o}} = \vec{\mathbf{a}}$.

4. $\exists!\ \vec{\mathbf{x}}$ such that $\vec{\mathbf{a}} + \vec{\mathbf{x}} = \vec{\mathbf{o}}$.

The symbol ‘$\forall$’ means ‘for any’. The symbol ‘$\exists$’ means ‘exists’. The pair of symbols ‘$\exists!$’ means ‘exists and the unique’. Property 2 gives us an opportunity to write $\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}}$ without brackets.

Proof. We are going to prove only the first property. Let's lay aside the given vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ from an arbitrary point O: $\vec{\mathbf{a}} = \overrightarrow{OA}$, $\vec{\mathbf{b}} = \overrightarrow{OB}$. We get the triangle ΔOAB. Then we continue drawing in order to get the parallelogram $OACB$. Let $\vec{\mathbf{c}}$ be the vector, which is defined by the directed segment $\overrightarrow{OC}$.

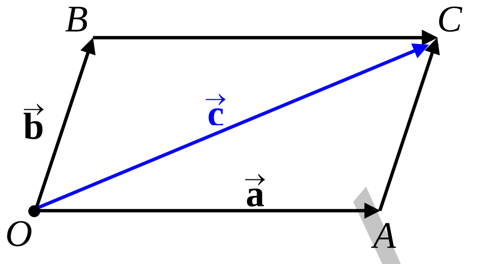


fig.10

From one side, it is obvious, that $\overrightarrow{AC} = \overrightarrow{OB}$, i.e. $\vec{\mathbf{b}} = \overrightarrow{AC}$. Thus, according to the rule of triangle we have $\vec{\mathbf{a}} + \vec{\mathbf{b}} = \overrightarrow{OC}$. From the other side, $\overrightarrow{BC} = \overrightarrow{OA}$, therefore $\vec{\mathbf{a}} = \overrightarrow{BC}$ and according to the rule of triangle $\vec{\mathbf{b}} + \vec{\mathbf{a}} = \vec{\mathbf{c}}$. ■

At the same time we got to know one more rule of drawing the sum of two given vectors: it is *the rule of parallelogram*. We lay aside the given vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ from the same point O, and we get a triangle ΔOAB. Then we continue drawing in order to get the parallelogram $OACB$. The diagonal OC of the parallelogram defines the sum $\vec{\mathbf{a}} + \vec{\mathbf{b}}$.

Definition. The vector $\vec{\mathbf{d}}$ is called *the difference* of the two given vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, if $\vec{\mathbf{b}} + \vec{\mathbf{d}} = \vec{\mathbf{a}}$. We write $\vec{\mathbf{d}} = \vec{\mathbf{a}} - \vec{\mathbf{b}}$.

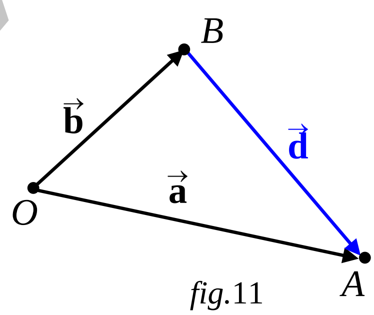


fig.11

How can we draw the difference of two vectors? Let's lay aside $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ from the same point O: $\vec{\mathbf{a}} = \overrightarrow{OA}$, $\vec{\mathbf{b}} = \overrightarrow{OB}$, and let $\vec{\mathbf{d}} = \overrightarrow{BA}$. Then according to the rule of triangle $\vec{\mathbf{b}} + \vec{\mathbf{d}} = \vec{\mathbf{a}}$. It means, that $\vec{\mathbf{d}} = \vec{\mathbf{a}} - \vec{\mathbf{b}}$.

Thus, if $OACB$ is a parallelogram and $\vec{\mathbf{a}} = \overrightarrow{OA}$, $\vec{\mathbf{b}} = \overrightarrow{OB}$, than the diagonal $\overrightarrow{OC}$ defines a sum, and the diagonal $\overrightarrow{BA}$ defines the difference of two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

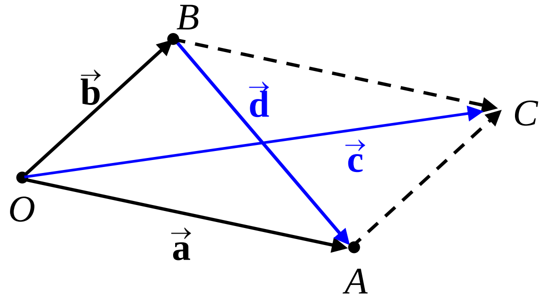


fig.12

§4. *A product of a vector and a number*

Definition. Let $\vec{\mathbf{a}}$ be non-null vector and λ be a number. A vector $\vec{\mathbf{b}}$ is said to *be a product of* $\vec{\mathbf{a}}$ *on* λ, if

1. $\vec{\mathbf{b}}\uparrow\uparrow\vec{\mathbf{a}}$, if $\lambda>0$, and $\vec{\mathbf{b}}\uparrow\downarrow\vec{\mathbf{a}}$, if $\lambda<0$;

2. $|\vec{\mathbf{b}}|=|\lambda|\cdot|\vec{\mathbf{a}}|$.

We write $\vec{\mathbf{b}}=\lambda\vec{\mathbf{a}}$.

In other words, vectors $2\vec{\mathbf{a}}$ и $-2\vec{\mathbf{a}}$ have the same length: twice more, than the vector $\vec{\mathbf{a}}$ has, but $2\vec{\mathbf{a}}$ has the same direction with $\vec{\mathbf{a}}$, and the vector $-2\vec{\mathbf{a}}$ has the opposite direction. If $\vec{\mathbf{a}}=\vec{\mathbf{o}}$, then $\lambda\vec{\mathbf{a}}=\vec{\mathbf{o}}$.

Both operations above: the sum of two vectors and the product of a vector on the number are called *linear operations*.

Examples **1.** Let A_1B_1 be the line joining midpoints of two sides in the triangle ΔABC, parallel to AB (we will call it 'medline' and it is not the same as 'median'). Let $\vec{\mathbf{a}}=\overrightarrow{AB}$, $\vec{\mathbf{b}}=\overrightarrow{A_1B_1}$. Then $\vec{\mathbf{b}}=\frac{1}{2}\vec{\mathbf{a}}$, because this vectors are co-directed and the length of the vector $\vec{\mathbf{b}}$ is twice lesser than the length of the vector $\vec{\mathbf{a}}$.

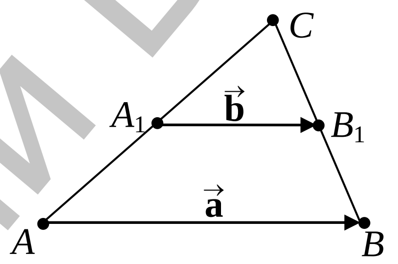


*fig.*13

2. Let AM be the median in the triangle ΔABC, and let $\vec{\mathbf{c}}=\overrightarrow{AM}$. Then we continue drawing and we get the parallelogram $ABCD$. Then M is the point of intersection of diagonals. Let $\overrightarrow{AB}=\vec{\mathbf{a}}$, $\overrightarrow{AC}=\vec{\mathbf{b}}$. Then $\overrightarrow{AD}=\vec{\mathbf{a}}+\vec{\mathbf{b}}$, and $\overrightarrow{AM}=\frac{1}{2}\overrightarrow{AD}$. It means, that the vector $\vec{\mathbf{c}}$, which is defined by the median $\overrightarrow{AM}$, is equal to semi-sum of two vectors defined by the sides of triangle $\overrightarrow{AB}$ and $\overrightarrow{AC}$: $\vec{\mathbf{c}}=\frac{1}{2}(\vec{\mathbf{a}}+\vec{\mathbf{b}})$.

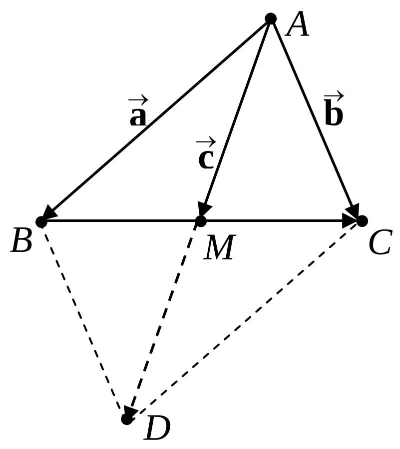


*fig.*14

Properties of the product of a vector to a number (without proof)**.**

5. $\lambda(\vec{\mathbf{a}}+\vec{\mathbf{b}})=\lambda\vec{\mathbf{a}}+\lambda\vec{\mathbf{b}}$; **6.** $(\lambda+\mu)\vec{\mathbf{a}}=\lambda\vec{\mathbf{a}}+\mu\vec{\mathbf{a}}$;

7. $\lambda(\mu\vec{\mathbf{a}})=(\lambda\mu)\vec{\mathbf{a}}$; **8.** $1\cdot\vec{\mathbf{a}}=\vec{\mathbf{a}}$.

The first property is demonstrated on the drawing in case if $\lambda>0$. On one side $\overrightarrow{OB_1}=\overrightarrow{OA_1}+\overrightarrow{A_1B_1}=\lambda\vec{\mathbf{a}}+\lambda\vec{\mathbf{b}}$ and on the other side $\overrightarrow{OB_1}=\lambda\overrightarrow{OB}=\lambda(\vec{\mathbf{a}}+\vec{\mathbf{b}})$.

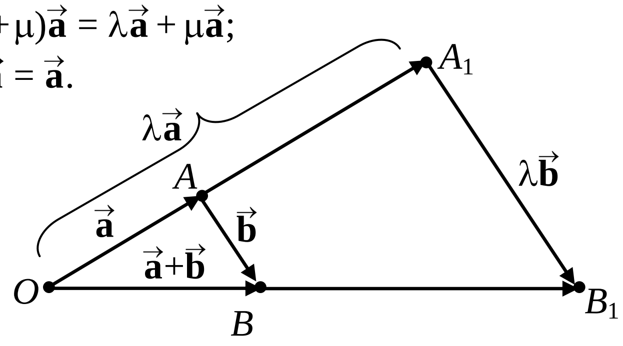


*fig.*15

***Theorem* 1** (*the first criterion of collinearity of two vectors*). *Non-null vectors* $\vec{\mathbf{a}}$ *are* $\vec{\mathbf{b}}$ *collinear, if and only if there exists such number* λ, *that* $\vec{\mathbf{b}} = \lambda\vec{\mathbf{a}}$.

We leave this theorem without proof, but we are going to submit a commentary. For example, if $|\vec{\mathbf{a}}|=2$, $|\vec{\mathbf{b}}|=5$ and $\vec{\mathbf{a}}\uparrow\uparrow\vec{\mathbf{b}}$, we will multiply $\vec{\mathbf{a}}$ to 2,5 in order to get $\vec{\mathbf{b}}$. And if $\vec{\mathbf{a}}\uparrow\downarrow\vec{\mathbf{b}}$, we will multiply $\vec{\mathbf{a}}$ to –2,5.

§5. *Coordinates of a vector*

Definition. Let a coordinate system is defined on a line and a vector $\vec{\mathbf{a}}$ is given on a line. We lay it aside from the initial point O: $\vec{\mathbf{a}}=\overrightarrow{OA}$. Denote $\vec{\mathbf{i}}=\overrightarrow{OE}$. We have $\vec{\mathbf{a}}\,||\,\vec{\mathbf{i}}$. According to the theorem 1 there exists a number x, such that $\vec{\mathbf{a}}=x\vec{\mathbf{i}}$. This number is called a coordinate of the vector $\vec{\mathbf{a}}$.

If A belongs to the positive direction, than $\vec{\mathbf{a}}\uparrow\uparrow\vec{\mathbf{i}}$ and we have $x>0$. The length of the vector $\vec{\mathbf{i}}$ is equal to 1. Therefore $|OA|=|\vec{\mathbf{a}}|=|x|\cdot|\vec{\mathbf{i}}|=x$. If A belongs to the negative direction, than $\vec{\mathbf{a}}\uparrow\downarrow\vec{\mathbf{i}}$ and we have $x<0$ and $|OA|=|\vec{\mathbf{a}}|=|x|\cdot|\vec{\mathbf{i}}|=-x$.

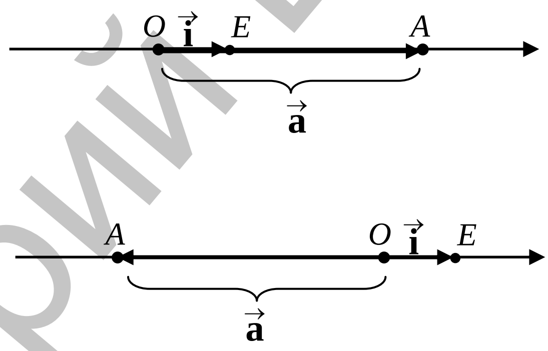
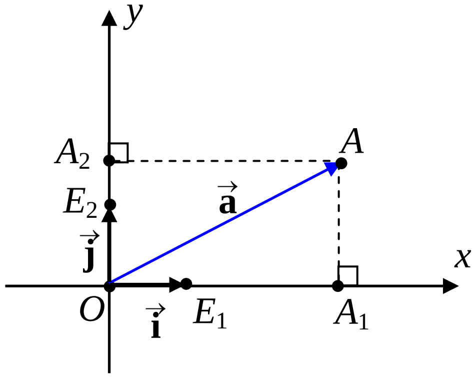


fig.16

In both cases we see, that the coordinate of the vector $\vec{\mathbf{a}}$ coincides with the coordinate of the point A.

Let a vector $\vec{\mathbf{a}}$ be given on the plane. We lay it aside from the initial point O of the Cartesian coordinate system: $\vec{\mathbf{a}} = \overrightarrow{OA}$ and let A_1 and A_2 be the projections of the point A on the coordinate axes. According to the rule of parallelogram



$$\vec{\mathbf{a}}=\overrightarrow{OA_1}+\overrightarrow{OA_2}.$$

fig.17

Let $E_1(1,0)$, $E_2(0,1)$ and $\vec{\mathbf{i}}=\overrightarrow{OE_1}$, $\vec{\mathbf{j}}=\overrightarrow{OE_2}$. We call these vectors the basis vectors or the orts. Together these vectors form the basis $\{\vec{\mathbf{i}}, \vec{\mathbf{j}}\}$. We have

$$|\vec{\mathbf{i}}|=|\vec{\mathbf{j}}|=1, \text{ and } \vec{\mathbf{i}}\perp\vec{\mathbf{j}}.$$

We see, that $\overrightarrow{OA_1}\,||\,\vec{\mathbf{i}}$ and $\overrightarrow{OA_2}\,||\,\vec{\mathbf{j}}$. Thus there exist such numbers x and y, that $\overrightarrow{OA_1}=x\vec{\mathbf{i}}$ and $\overrightarrow{OA_2}=y\vec{\mathbf{j}}$. So, we get

$$\vec{\mathbf{a}} = x\vec{\mathbf{i}} + y\vec{\mathbf{j}}. \qquad (3)$$

This equality is called a decomposition of the vector $\vec{\mathbf{a}}$ by basis $\{\vec{\mathbf{i}}, \vec{\mathbf{j}}\}$. The pair of numbers (x, y) is called the coordinates of the vector $\vec{\mathbf{a}}$. We write as follows: $\vec{\mathbf{a}}(x, y)$. We see, that x coincides with a coordinate of the point A_1 on the axis Ox and y coincides with a coordinate of the point A_2 on the axis Oy. Thus the point A has coordinates $A(x, y)$ just the same as the vector $\vec{\mathbf{a}}(x, y)$.

If $\vec{\mathbf{a}} = \overrightarrow{OA}$, we call $\vec{\mathbf{a}}$ *the radius vector* of the point A. And we emphasize once more that *coordinates of a point coincide with coordinates of its radius vector*.

According to the Pythagoras theorem for the triangle ΔOAA_1 we get

$$|OA| = \sqrt{|OA_1|^2 + |OA_2|^2} = \sqrt{x^2 + y^2}.$$

It means, that the length of a vector $\vec{\mathbf{a}}(x, y)$ can be calculated by the formula

$$|\vec{\mathbf{a}}| = \sqrt{x^2 + y^2}. \qquad (4)$$

Suppose, that two vectors are given on the plane: $\vec{\mathbf{a}}(x_1, y_1)$, $\vec{\mathbf{b}}(x_2, y_2)$. Then

$$\vec{\mathbf{a}} = x_1\vec{\mathbf{i}} + y_1\vec{\mathbf{j}}, \quad \vec{\mathbf{b}} = x_2\vec{\mathbf{i}} + y_2\vec{\mathbf{j}}.$$

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = (x_1 + x_2)\vec{\mathbf{i}} + (y_1 + y_2)\vec{\mathbf{j}}, \quad \lambda\,\vec{\mathbf{a}} = (\lambda x_1)\vec{\mathbf{i}} + (\lambda y_1)\vec{\mathbf{j}}.$$

So we can formulate the rule. *While adding vectors their coordinates are added; while multiplying a vector by a number its coordinates are multiplied by this number*. This rule implies that *while subtracting vectors their coordinates are subtracted*.

Suppose, that two points are given on the plane: $A(x_1, y_1)$, $B(x_2, y_2)$ and $\vec{\mathbf{c}} = \overrightarrow{AB}$. Consider vectors $\vec{\mathbf{a}} = \overrightarrow{OA}$, $\vec{\mathbf{b}} = \overrightarrow{OB}$. Then $\vec{\mathbf{c}} = \vec{\mathbf{b}} - \vec{\mathbf{a}}$ and $\vec{\mathbf{a}}(x_1, y_1)$, $\vec{\mathbf{b}}(x_2, y_2)$. Therefore

$$\vec{\mathbf{c}}(x_2 - x_1, y_2 - y_1). \qquad (5)$$

So we can formulate the rule: *to find the coordinates of the vector, one must subtract the coordinates of its end from the coordinates of its beginning*.

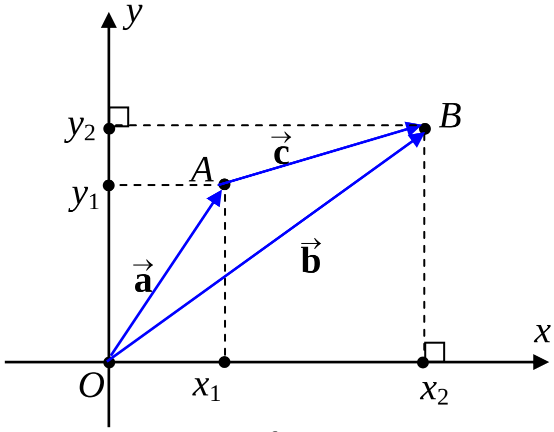


fig.18

This rule implies one more rule: *to find the coordinates of the vector's end one must add the coordinates of its beginning to the coordinates of the vector.*

The length of the vector $\vec{\mathbf{a}}$ coincides with the length of the segment AB. This value is also called the distance between two points A и B. Formula (5) implies the formula for the computation of the distance between two given points $A(x_1, y_1)$, $B(x_2, y_2)$:

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,. \tag{6}$$

Theorem 2 (*the second criterion of collinearity of two vectors*). *Two non-null vectors are collinear if and only if there's coordinates are proportional*:

$$\vec{\mathbf{a}}(x_1, y_1) \,||\, \vec{\mathbf{b}}(x_2, y_2) \;\Leftrightarrow\; \frac{x_1}{x_2} = \frac{y_1}{y_2}.$$

Proof. Let $\vec{\mathbf{a}}(x_1, y_1) \,||\, \vec{\mathbf{b}}(x_2, y_2)$. According to the theorem 1, there exists $\lambda \in \boldsymbol{R}$ such that $\vec{\mathbf{a}} = \lambda\vec{\mathbf{b}}$. The vector $\lambda\vec{\mathbf{b}}$ has the coordinates $(\lambda x_2, \lambda y_2)$. Vectors $\vec{\mathbf{a}}$ and $\lambda\vec{\mathbf{b}}$ are equal and it means, that their coordinates are equal: $x_1 = \lambda x_2$, $y_1 = \lambda y_2$. Hence $\frac{x_1}{x_2} = \lambda$ and $\frac{y_1}{y_2} = \lambda$ at the same time, so we have got $\frac{x_1}{x_2} = \frac{y_1}{y_2}$.

Backwards. Let the equality $\frac{x_1}{x_2} = \frac{y_1}{y_2}$ takes place. Denote the common ratio as λ: $\frac{x_1}{x_2} = \frac{y_1}{y_2} = \lambda$. These equalities imply $x_1 = \lambda x_2$, $y_1 = \lambda y_2$ and therefore $\vec{\mathbf{a}} = \lambda\vec{\mathbf{b}}$. ■

§6. *The angle between two vectors. Scalar product of two vectors*

Definition. Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be two non-null vectors. We lay them aside from an arbitrary point O: $\vec{\mathbf{a}} = \overrightarrow{OA}$, $\vec{\mathbf{b}} = \overrightarrow{OB}$. Then the angle $\alpha = \angle AOB$ between two rays OA and OB is called *the angle between two vectors* $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. We write $\alpha = \angle(\vec{\mathbf{a}}, \vec{\mathbf{b}})$.

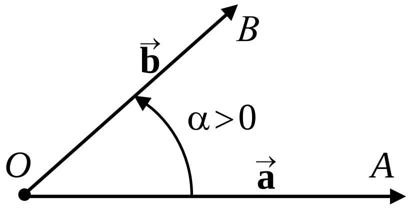


fig.19.1

For two vectors on the plane we can define the notion of *oriented angle between two vectors*. If the shortest rotation from the ray OA to the ray OB is performed counterclockwise (fig. 19.1), we consider that $\alpha > 0$. If this angle is performed clockwise, we consider that $\alpha < 0$ (fig. 19.2). According to such definition, $-\pi < \alpha \le \pi$. If $\alpha > 0$, then a pair of vectors $(\vec{\mathbf{a}}, \vec{\mathbf{b}})$ is called *the right pair*, and if $\alpha < 0$ it is called *the left pair*.

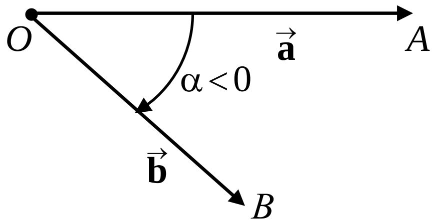


fig.19.2

Definition. A number

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \operatorname{Cos}\angle(\vec{\mathbf{a}}, \vec{\mathbf{b}}). \tag{7}$$

is called *a scalar product* of two vectors $\vec{\mathbf{a}}$ и $\vec{\mathbf{b}}$. A number $\vec{\mathbf{a}}^2 = \vec{\mathbf{a}} \cdot \vec{\mathbf{a}}$ is called *a scalar square* or simply *a square* of the vector $\vec{\mathbf{a}}$.

The definition immediately implies

$$\vec{\mathbf{a}}^2 = |\vec{\mathbf{a}}||\vec{\mathbf{a}}| \cos 0^\circ = |\vec{\mathbf{a}}|^2 \quad \Rightarrow \quad |\vec{\mathbf{a}}| = \sqrt{\vec{\mathbf{a}} \cdot \vec{\mathbf{a}}}\,.$$

It is also obvious, that the equality $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}=0$ takes place only in the following cases: 1. $|\vec{\mathbf{a}}|=0$, 2. $|\vec{\mathbf{b}}|=0$, 3. $\angle(\vec{\mathbf{a}},\vec{\mathbf{b}})=\pi/2$.

So we have proved the following theorem.

***Theorem* 3. 1.** *A scalar square of a vector is equal to a square of its length.*

2. *Non-null vectors* $\vec{\mathbf{a}}$ *and* $\vec{\mathbf{b}}$ *are perpendicular if and only if its scalar product is equal to zero* ($\vec{\mathbf{a}}\perp\vec{\mathbf{b}} \Leftrightarrow \vec{\mathbf{a}}\cdot\vec{\mathbf{b}}=0$).

From the definition we immediately get the formula, which makes possible to calculate the angle between two vectors:

$$\mathrm{Cos}\angle(\vec{\mathbf{a}},\vec{\mathbf{b}})=\frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{|\vec{\mathbf{a}}||\vec{\mathbf{b}}|}. \tag{8}$$

Properties of the scalar product. For any vectors $\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}$ and for any $\lambda\in\boldsymbol{R}$

1. $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}=\vec{\mathbf{b}}\cdot\vec{\mathbf{a}}$ (*commutativity*);

$$\left.\begin{array}{l}\textbf{2. }(\lambda\vec{\mathbf{a}})\cdot\vec{\mathbf{b}}=\lambda(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}});\\ \textbf{3. }\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}+\vec{\mathbf{c}})=\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}+\vec{\mathbf{a}}\cdot\vec{\mathbf{c}};\end{array}\right\}\quad(\textit{linearity})$$

4. $\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}\geq 0$, and $\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}=0\Leftrightarrow\vec{\mathbf{a}}=\vec{\mathbf{o}}$ (*positive definiteness*).

We admit these properties without a proof. Using these properties it is possible to open the brackets as if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are numbers. For example:

$$(2\vec{\mathbf{a}}+3\vec{\mathbf{b}})\cdot(\vec{\mathbf{a}}-5\vec{\mathbf{b}})=2\vec{\mathbf{a}}^2-7\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}-15\vec{\mathbf{b}}^2.$$

§7. *Formula for the computation of the scalar product and its implications*

Let's remind that we have denoted basis vectors of coordinate axes as $\vec{\mathbf{i}}$ and $\vec{\mathbf{j}}$: $\vec{\mathbf{i}}\,||\,Ox$ and $\vec{\mathbf{j}}\,||\,Oy$. And we have $|\vec{\mathbf{i}}|=|\vec{\mathbf{j}}|=1$, $\vec{\mathbf{i}}\perp\vec{\mathbf{j}}$. Therefore

$$\vec{\mathbf{i}}\cdot\vec{\mathbf{i}}=|\vec{\mathbf{i}}|^2=1,\ \vec{\mathbf{j}}\cdot\vec{\mathbf{j}}=|\vec{\mathbf{j}}|^2=1,\ \vec{\mathbf{i}}\cdot\vec{\mathbf{j}}=0.$$

Let $\vec{\mathbf{a}}(x_1,y_1)$, $\vec{\mathbf{b}}(x_2,y_2)$ be arbitrary vectors. We remind, that it means

$$\vec{\mathbf{a}}=x_1\vec{\mathbf{i}}+y_1\vec{\mathbf{j}},\ \ \vec{\mathbf{b}}=x_2\vec{\mathbf{i}}+y_2\vec{\mathbf{j}}.$$

Hence

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}=(x_1\vec{\mathbf{i}}+y_1\vec{\mathbf{j}})\cdot(x_2\vec{\mathbf{i}}+y_2\vec{\mathbf{j}})=x_1x_2\vec{\mathbf{i}}\cdot\vec{\mathbf{i}}+x_1y_2\vec{\mathbf{i}}\cdot\vec{\mathbf{j}}+y_1x_2\vec{\mathbf{j}}\cdot\vec{\mathbf{i}}+y_1y_2\vec{\mathbf{j}}\cdot\vec{\mathbf{j}}=$$
$$=x_1x_2\cdot 1+x_1y_2\cdot 0+y_1x_2\cdot 0+y_1y_2\cdot 1=x_1x_2+y_1y_2.$$

As a result we get a formula:

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}=x_1x_2+y_1y_2. \tag{9}$$

The first implication of this formula is, that the length of a vector $\vec{\mathbf{c}}(x,y)$ can be calculated as follows.

$$|\vec{\mathbf{c}}| = \sqrt{\vec{\mathbf{c}} \cdot \vec{\mathbf{c}}} = \sqrt{x^2 + y^2} \qquad (10)$$

We mentioned this formula above. The second implication is the formula for computation of the angle between two vectors:

$$\mathrm{Cos}\angle(\vec{\mathbf{a}}, \vec{\mathbf{b}}) = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}||\vec{\mathbf{b}}|} = \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}. \qquad (11)$$

Denote the angles between positive directions of the coordinate axes and a vector $\vec{\mathbf{a}}(x, y)$ as α and β (fig. 20). Consider the scalar products of the vector $\vec{\mathbf{a}}$ with the basis vectors.

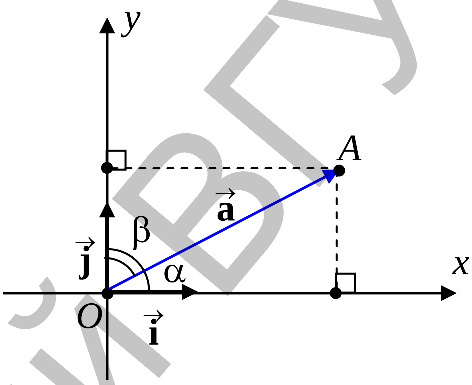


fig.20

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{i}} = x \cdot 1 + y \cdot 0 = x,$$
$$\vec{\mathbf{a}} \cdot \vec{\mathbf{j}} = x \cdot 0 + y \cdot 1 = y.$$

On the other side

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{i}} = |\vec{\mathbf{a}}||\vec{\mathbf{i}}|\ \mathrm{Cos}\angle(\vec{\mathbf{i}}, \vec{\mathbf{a}}) = |\vec{\mathbf{a}}|\mathrm{Cos}\,\alpha,$$
$$\vec{\mathbf{a}} \cdot \vec{\mathbf{j}} = |\vec{\mathbf{a}}||\vec{\mathbf{j}}|\mathrm{Cos}\angle(\vec{\mathbf{j}}, \vec{\mathbf{a}}) = |\vec{\mathbf{a}}|\mathrm{Cos}\,\beta.$$

We have $\beta = \frac{\pi}{2} - \alpha$. So $\mathrm{Cos}\,\beta = \mathrm{Sin}\,\alpha$. Thus we have the formulas

$$\begin{cases} x = |\vec{\mathbf{a}}|\mathrm{Cos}\,\alpha, \\ y = |\vec{\mathbf{a}}|\mathrm{Sin}\,\alpha. \end{cases} \qquad (12)$$

§8. *Partition of a segment in the prescribed ratio*

Definition. Let AB be a segment and a point C lies on AB. We say that C *divides the segment* AB *in the ratio* $\lambda_1 : \lambda_2$, if

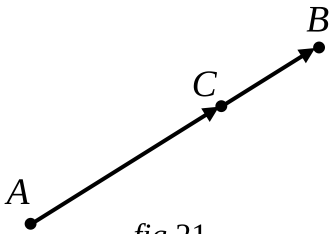


fig.21

$$\frac{|AC|}{|CB|} = \frac{\lambda_1}{\lambda_2} \Leftrightarrow \lambda_2|AC| = \lambda_1|CB|.$$

Taking into account that $\overrightarrow{AC} \uparrow\uparrow \overrightarrow{CB}$ (fig.21), we can rewrite the last equality as follows:

$$\lambda_2 \overrightarrow{AC} = \lambda_1 \overrightarrow{CB}. \qquad (13)$$

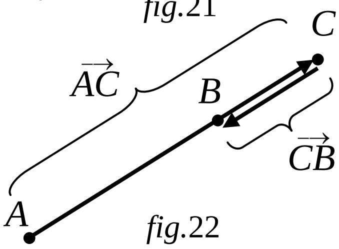


fig.22

Now we introduce the extension of our definition. We say that a point *divides the segment* AB *in the ratio* $\lambda_1 : \lambda_2$, if the equality (13) takes place. Such definition admits, that the point C can lie on the straight line AB, but not on the segment AB, if $\lambda_1 : \lambda_2 < 0$ (fig. 22). The number $\lambda = \lambda_1/\lambda_2$ ($\overrightarrow{AC} = \lambda\overrightarrow{CB}$) is called *the simple ratio of three points* A, B, C and we denote it as (AB, C).

Suppose, that we know the coordinates: $A(x_1, y_1)$, $B(x_2, y_2)$, and we know $\lambda = (AB, C)$. The task is to find the unknown coordinates $C(x, y)$. We are going to prove the following formulas on the practical classes:

$$x = \frac{x_1 + \lambda x_2}{1+\lambda}, \quad y = \frac{y_1 + \lambda y_2}{1+\lambda}. \tag{14}$$

Privately, if C divides the segment AB on two equal parts, then

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}. \tag{15}$$

So, the coordinates of the midpoint of the segment AB are the arithmetic means of the coordinates of the points A and B.

§9. *A square of a parallelogram and of a triangle*

Theorem 3. *Let ABCD be a parallelogram, and suppose, that we know coordinates of the vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{AD}$: $\overrightarrow{AB}(x_1, y_1)$, $\overrightarrow{AD}(x_2, y_2)$. *Then a square of the parallelogram (fig.23) may be calculated by the formula*

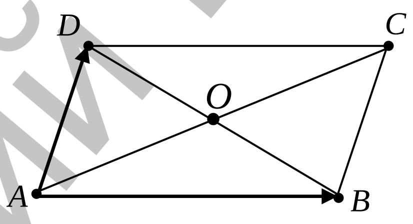


*fig.*23

$$S_{ABCD} = \mathrm{mod}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = |x_1y_2 - x_2y_1|. \tag{16}$$

Proof. From the school program it is known, that a square of the parallelogram may be calculated by the formula

$$S_{ABCD} = AB \cdot AD \cdot \mathrm{Sin}\angle BAD.$$

So,

$$(S_{ABCD})^2 = |AB|^2 \cdot |AD|^2 \cdot \mathrm{Sin}^2\angle BAD = |AB|^2 \cdot |AD|^2 \cdot (1 - \mathrm{Cos}^2\angle BAD) =$$

$$= |AB|^2 \cdot |AD|^2 - (|AB| \cdot |AD| \cdot \mathrm{Cos}\angle BAD)^2 = |AB|^2 \cdot |AD|^2 - (\overrightarrow{AB} \cdot \overrightarrow{AD})^2 =$$

$$= (x_1^2 + y_1^2)(x_2^2 + y_2^2) - (x_1x_2 + y_1y_2)^2 = (x_1y_2 - x_2y_1)^2.$$

Hence:

$$S_{ABCD} = \sqrt{(x_1y_2 - x_2y_1)^2} = |x_1y_2 - x_2y_1|.$$

Corollary. *Suppose, that we know coordinates of the vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$: $\overrightarrow{AB}(x_1, y_1)$, $\overrightarrow{AC}(x_2, y_2)$. *Then a square of the triangle ABC (fig.24) may be calculated using the formula*:

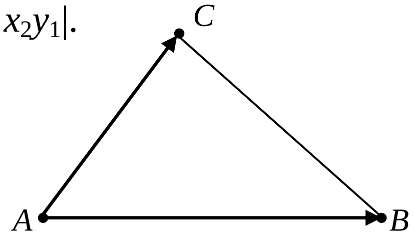


*fig.*24

$$S_{ABC} = \frac{1}{2}\,\mathrm{mod}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \frac{1}{2}|x_1y_2 - x_2y_1|. \tag{17}$$

§10. *An equation of a straight line*

From the school program it is known, that the equation

$$y = kx + b \tag{18}$$

($k=\text{const}$, $b=\text{const}$) defines a straight line on the plane, which isn't parallel to Oy. It means that a straight line consists of those and only those points, whose coordinates (x, y) satisfy to the equation (18).

Lets substitute $x=0$ to (18). Then we get $y=b$. So we have the geometric sense of the coefficient b: it is a segment, that the straight line cuts of the coordinate axis Oy (this segment can be negative).

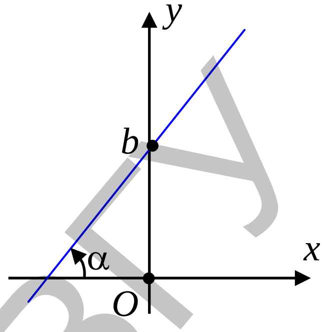


fig.25

Let's chose a direction on the straight line, which corresponds to increasing of ordinate y, and let's call this direction positive. The angle between the axis Ox and the positive direction of the line is called *the inclination angle of the line* (fig.25).

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be arbitrary points on the straight line l, and let $y_2 \geq y_1$. We substitute its coordinates in the equation (18):

$$y_1 = kx_1 + q, \quad y_2 = kx_2 + q.$$

Then we subtract the first equality from the second one:

$$y_2 - y_1 = k(x_2 - x_1).$$

Because $l \nparallel Oy$, we have $x_2 \neq x_1 \Rightarrow$

$$k = \frac{y_2 - y_1}{x_2 - x_1}. \tag{19}$$

Let S be the point with coordinates (x_2, y_1).

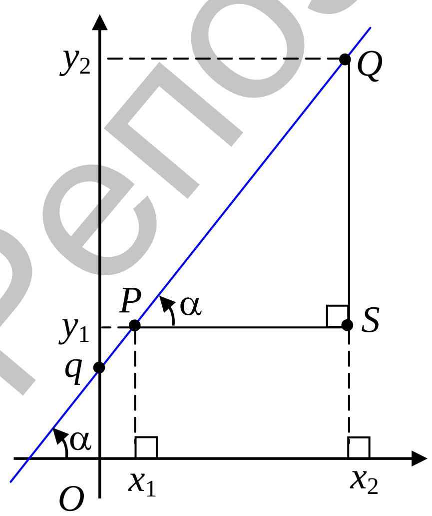


fig.26.1

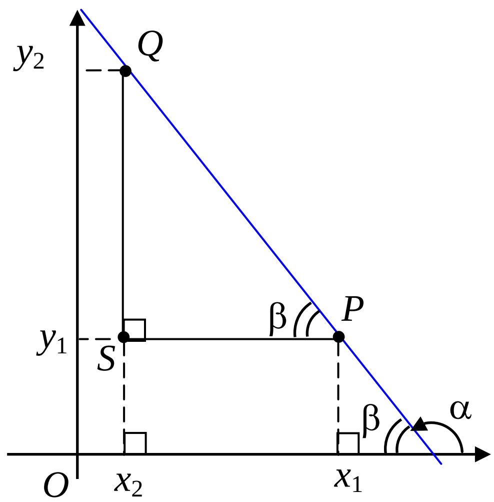


fig.26.2

Case 1: $x_2 > x_1$ (fig 26.1). Then $y_2 - y_1 = QS$, $x_2 - x_1 = PS$ and from ΔPQS we find $k = QS/PS = \operatorname{tg}\alpha$.

Case 2: $x_2 < x_1$ (fig 26.2). Then $y_2 - y_1 = QS$, $x_2 - x_1 = -PS$ and from ΔPQS we find $k = -QS/PS = = -\operatorname{tg}\beta$, where $\beta = \angle QPS$. But $\beta = \pi - \alpha \Rightarrow -\operatorname{tg}\beta = \operatorname{tg}\alpha$. Consequently, we have $k = QS/PS = \operatorname{tg}\alpha$, like in the first case.

Thus, we have proved that k is equal to $\operatorname{tg}\alpha$, where α is the inclination angle of the straight line. Therefore k is called *the angle coefficient*.

Suppose, that coordinates of two points on a straight line l are given: $P(x_1, y_1)$, $Q(x_2, y_2)$. How can we make the equation of the line? First, we find the angle coefficient

$$k = \frac{y_2 - y_1}{x_2 - x_1}.$$

Then we can write the equation

$$y - y_1 = k(x - x_1).$$

Finally we open the brackets and transfer y_1 to the right. The examples will be given in the paragraph 18.

§11. *Mutual location of two straight lines. The angle between two lines*

Definition. Suppose, that two lines l_1 and l_2 intersect at the point M. Then they form two pairs of vertical angles. Let θ_1 and θ_2 be their values. The smallest of two numbers θ_1 and θ_2 is called the value of the angle between l_1 and l_2 (fig.27).

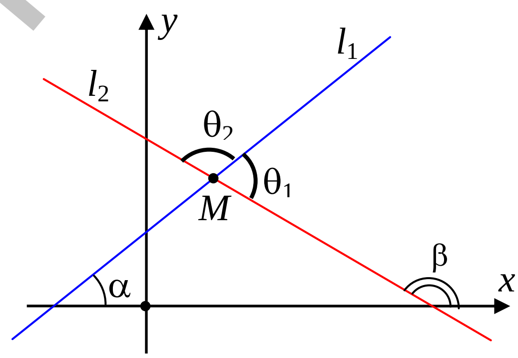


fig.27

Theorem 4. *Let two straight lines be given on the plane by its equations with angle coefficient*

$$l_1: y = k_1 x + b_1, \qquad l_2: y = k_2 x + b_2.$$

Then

1) *the angle between them may be calculated by the formula*

$$\operatorname{tg}\theta = \frac{|k_2 - k_1|}{|1 + k_1 k_2|}; \tag{20}$$

2) *the lines coincide if and only if* $k_1 = k_2$, $b_1 = b_2$;
3) *the lines are parallel if and only if* $k_1 = k_2$, $b_1 \neq b_2$;
3) *the lines are perpendicular if and only if* $k_2 = -1/k_1$.

For example, the straight lines l_1: $y = 2x + 5$ и l_2: $y = 2x - 3$ are parallel, and the straight lines l_1: $y = 2x + 5$ и l_3: $y = -0{,}5x - 3$ are perpendicular.

§12. *General equation of a straight line. The distance between a point and a straight line.*

We should note that not any straight line on the plain can be defined by the equation with the angle coefficient. If the line is parallel to the axis Oy, it has the equation $x = c$, $c = \text{const}$. Both kinds of the equations can be united by the equation of the form

$$ax + by + c = 0. \qquad (21)$$

This equation is called *the general equation of a straight line*. And vice versa, an arbitrary equation of the form (21) defines the straight line, if $a^2 + b^2 \neq 0$ (it means that at least one of the numbers a and b is not equal to zero).

It's easy to understand that the equations

$$ax + by + c = 0 \quad \text{and} \quad max + mby + mc = 0,\ m \neq 0$$

define the same straight line on the plane. In other words, the equations

$$a_1x + b_1y + c_1 = 0 \quad \text{and} \quad a_2x + b_2y + c_2 = 0$$

define the same straight line if and only if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

The same lines are parallel if and only if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}.$$

Let's remind that the distance from the point to the straight line is the length of the perpendicular which is dropped from this point to the line (fig.28).

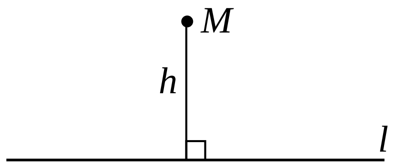


fig.28

***Theorem* 5.** *Let a straight line* l *be defined by its general equation* (21). *Then the distance from* $M(x, y)$ *to this line can be calculated by the formula*

$$h = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}. \qquad (22)$$

It is useful to know, that the vector $\vec{\mathbf{n}}(A, B)$ is perpendicular to the straight line defined by the equation (21).

§13. *Canonical and parametric equations of the straight line. The equation 'in segments'*

The straight line on the plane may be defined by the point $M_0 \in l$ and by the non null vector $\vec{\mathbf{a}} \,||\, l$. In this case $\vec{\mathbf{a}}$ is called *the directing vector of the line*.

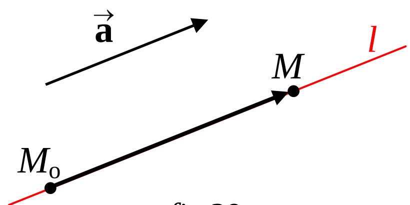


fig.29

Theorem 6. *A straight line l, which passes through the point $M_o(x_o, y_o)$ and has the directing vector $\vec{\mathbf{a}}(a_1, a_2)$, may be defined by the equation*

$$\frac{x-x_o}{a_1}=\frac{y-y_o}{a_2}, \tag{23}$$

which is called the canonical equation, or it may be defined by the parametric equations:

$$\begin{cases} x = x_o + a_1 t, \\ y = y_o + a_2 t,\ t\in\mathbf{R}, \end{cases} \tag{24}$$

Corollary. *A straight line, which cuts the segments $a\neq 0, b\neq 0$ off the coordinate axes may be defined by the equation*

$$\frac{x}{a}+\frac{y}{b}=1, \tag{25}$$

which is called the equation 'in segments'.

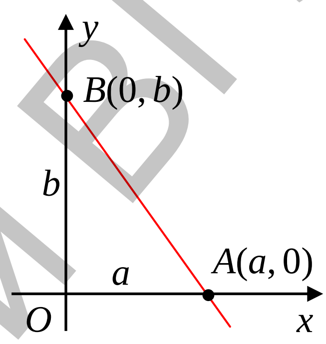


*fig.*30

Proof. Let $M(x, y)$ be an arbitrary point on the straight line l. Then the vector $\overrightarrow{M_oM}(x-x_o, y-y_o)$ is parallel to $\vec{\mathbf{a}}(a_1, a_2)$. According to the second criterion of collinearity of two vectors it is equivalent to (23).

According to the first criterion of collinearity of two vectors $\overrightarrow{M_oM} \,||\, \vec{\mathbf{a}} \Leftrightarrow$ $\Leftrightarrow \exists\, t\in\mathbf{R}$ such that $\overrightarrow{A_oM}=t\vec{\mathbf{a}}$. The latter equality in coordinates looks like

$$x-x_o = ta_1,\ \ y-y_o = ta_2,$$

and it obviously equivalent to (24).

Conversely, let (23) or (24) take place for the coordinates of a point M. Then according to the same criterions $\overrightarrow{M_oM} \,||\, \vec{\mathbf{a}}$, and it means that $M\in l$.

The condition of the corollary means, that the straight line passes through the points $A(a, 0)$ and $B(0, b)$. Then the vector $\overrightarrow{AB}(-a, b)$ is the directing vector of the line. We substitute its coordinates in (22):

$$\frac{x-a}{0-a}=\frac{y-0}{b-0} \ \Leftrightarrow\ \frac{x-a}{-a}=\frac{y}{b} \ \Leftrightarrow\ (25). \ \blacksquare$$

§14. *Transformations of the Cartesian coordinate system*

Let two Cartesian systems Oxy and $O'x'y'$ be given on the plane and suppose, that the directions of corresponding axes coincide, but the initial points O and O' are different. We say, that the *second coordinate system was obtained from the first one by the parallel translation of the coordinate axes* (fig. 31). Often we call the first system 'old' and the second system 'new'.

Suppose that we know coordinates of the point O' in respect to the old coordinate system: $O'(a,b)$. Let M be an arbitrary point on the plane, (x,y) are its old coordinates and (x',y') are its new coordinates. We need to find the relation between these coordinates.

We have learnt, that coordinates of a point coinside with coordinates of its radius-vector. That is why

$$\overrightarrow{OO'}(a,b),\ \overrightarrow{OM}(x,y),\ \overrightarrow{O'M}(x',y').$$

According to the rule of triangle

$$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}.$$

Hence we have the formulas

$$\begin{cases} x = x' + a, \\ y = y' + b. \end{cases} \quad (26) \qquad \begin{cases} x' = x - a, \\ y' = y - b. \end{cases} \quad (26')$$

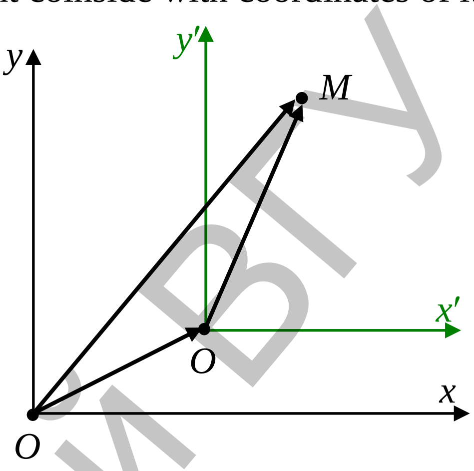


fig.31

Now suppose, that two Cartesian systems with the common initial point are given on the plane: Oxy and $Ox'y'$. Let α be the oriented angle between the positive directions of the axes Ox and Ox' (fig. 32). Then we say that *the second coordinate system was obtained from the first one by the rotation on the angle* α. Let M be an arbitrary point on the plane, (x, y) are its old coordinates and (x',y') are its new coordinates.

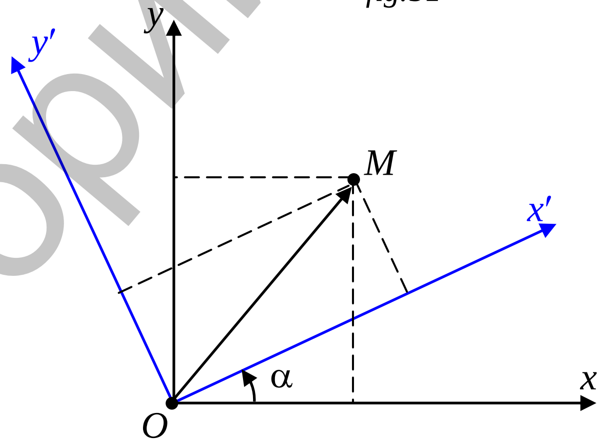


fig.32

We accept without proof, that

$$\begin{cases} x = x'\cdot\mathrm{Cos}\,\alpha - y'\cdot\mathrm{Sin}\,\alpha, \\ y = y'\cdot\mathrm{Sin}\,\alpha + y'\cdot\mathrm{Cos}\,\alpha. \end{cases} \quad (27)$$

$$\begin{cases} x' = \ \ x\cdot\mathrm{Cos}\,\alpha + y\cdot\mathrm{Sin}\,\alpha, \\ y' = -x\cdot\mathrm{Sin}\,\alpha + y\cdot\mathrm{Cos}\,\alpha. \end{cases} \quad (27')$$

And the next case. Suppose, that two arbitrary Cartesian systems Oxy and $O'x'y'$ are given on the plane. Then the second coordinate system can be obtained from the first one as a result of two transformations. First we fulfill a parallel translation of the coordinate axes and we get intermediate coordinate system $O'x''y''$. Then we fulfill a rotation of the coordinate system. We have the following formulas for the first transformation:

$$\begin{cases} x = x'' + a, \\ y = y'' + b. \end{cases} \qquad \begin{cases} x'' = x - a, \\ y'' = y - b. \end{cases}$$

We have the following formulas for the second transformation:

$$\begin{cases} x' = x''\cdot\mathrm{Cos}\,\alpha + y''\cdot\mathrm{Sin}\,\alpha, \\ y' = -x''\cdot\mathrm{Sin}\,\alpha + y''\cdot\mathrm{Cos}\,\alpha. \end{cases}$$

$$\begin{cases} x'' = x'\cdot\mathrm{Cos}\,\alpha - y'\cdot\mathrm{Sin}\,\alpha, \\ y'' = y'\cdot\mathrm{Sin}\,\alpha + y'\cdot\mathrm{Cos}\,\alpha. \end{cases}$$

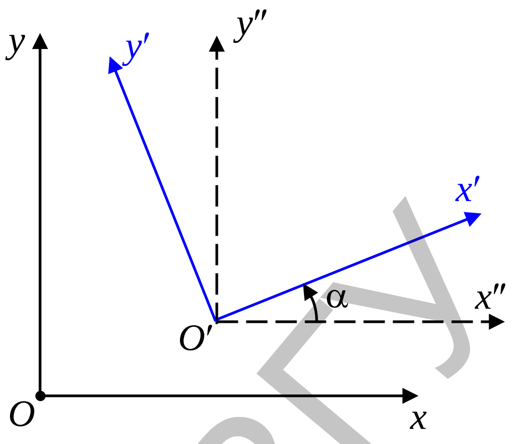


fig.33

We substitute x'' и y'' from the second system to the third one and we get the formulas, which give us an opportunity to calculate the new coordinates (x', y') of a point, if the old one (x, y) are given:

$$\begin{cases} x' = (x-a)\cdot\mathrm{Cos}\,\alpha + (y-b)\cdot\mathrm{Sin}\,\alpha, \\ y' = -(x-a)\cdot\mathrm{Sin}\,\alpha + (y-b)\cdot\mathrm{Cos}\,\alpha. \end{cases} \qquad (28)$$

Exercise. Write the formulas, which give us an opportunity to calculate the old coordinates of a point, if the new one are given (x', y').

§15. *Ellipse*

Definition. A curve, which has a canonical equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \ (a > b > 0) \qquad (29)$$

is called <u>*an ellipse*</u>. Let $c^2 = a^2 - b^2$. Points $F_1(c, 0)$, $F_2(-c, 0)$ are called <u>*focuses*</u>.

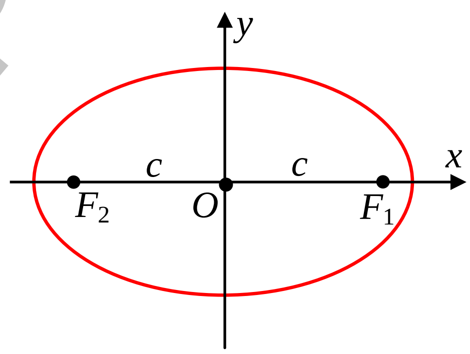


fig.34

<u>*Geometric properties of the ellipse.*</u>

1. Equation (29) implies that $|x| \le a$, $|y| \le b$. It means that the ellipse is located inside the rectangle, which is defined by these inequalities (fig.35).

2. Coordinate axes intersect ellipse at the points $A_1(a, 0)$, $A_2(-a, 0)$, $B_1(0, b)$, $B_2(0, -b)$, which are called its <u>*vertexes*</u>. The segment A_1A_2 and B_1B_2 are called <u>*the big and the small diameters*</u>, and together they are called <u>*the principle diameters*</u>. Numbers a and b are called <u>*the big and the small semiaxes*</u>.

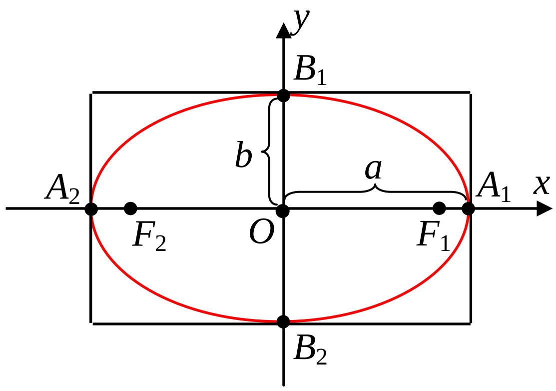


fig.35

3. Coordinate axes are the symmetry axes of the ellipse and the initial point of the coordinate system is its center of symmetry.

In fact, let $M(x, y)$ be an arbitrary point of the ellipse. Then the pair of numbers (x, y) satisfies the equation (29). Then the following pairs $(x, -y)$, $(-x, y)$, $(-x, -y)$ also satisfy the equation (29) and these pairs define the points, which are symmetric to M in respect to axes Ox, Oy and the point O.

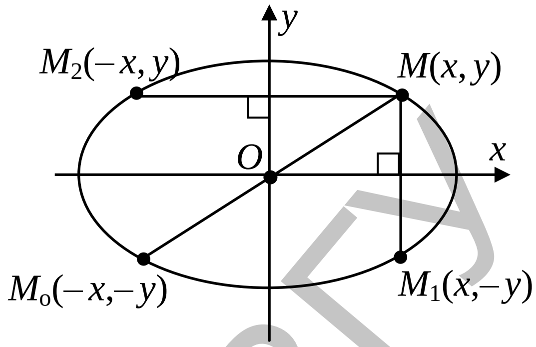


fig.36

4. The focuses have the following property. The sum of distances from an arbitrary point M on the ellipse γ to F_1 and from M to F_2 is the constant value:

$$|MF_1| + |MF_2| = 2a = \text{const}, \qquad (1)$$

i.e. it doesn't depend of the choice of the point точки $M \in \gamma$.

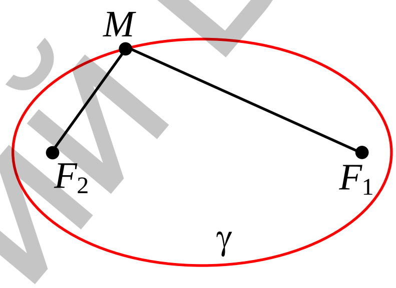


fig.37

In fact, let $M(x, y)$ be an arbitrary point of the ellipse Then

$$|MF_1| = \sqrt{(x-c)^2 + y^2}, \quad |MF_2| = \sqrt{(x+c)^2 + y^2}$$

From (29) we get

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) = (a^2 - c^2)\left(1 - \frac{x^2}{a^2}\right).$$

We substitute this equality in the expression for $|MF_1|$:

$$|MF_1| = \sqrt{(x-c)^2 + y^2} = \sqrt{x^2 - 2xc + c^2 + (a^2 - c^2)(1 - \frac{x^2}{a^2})} =$$

$$= \sqrt{x^2 - 2xc + c^2 + a^2 - c^2 - x^2 + \frac{c^2x^2}{a^2}} =$$

$$= \sqrt{a^2 - 2xc + \frac{c^2x^2}{a^2}} = \sqrt{\left(1 - \frac{x^2}{a^2}\right)^2} = \left| a - \frac{cx}{a} \right|.$$

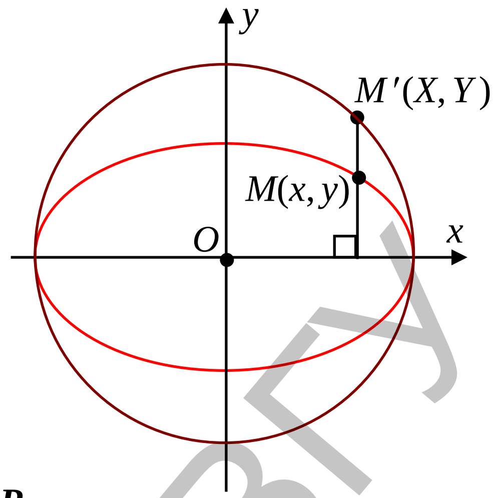
Analogously we get $|MF_2| = \left|a + \frac{cx}{a}\right|$. As we mentioned above, (29) implies $|x| \le a$, and according to the definition $a > c$. Hence both expressions under the modulus sign are nonnegative and we may omit the modulus signs. Therefore

$$|MF_1| + |MF_2| = a - \frac{cx}{a} + a + \frac{cx}{a} = 2a.$$

5. Ellipse may be obtained from the circle

$$\gamma': X^2+Y^2=a^2$$

as a result of proportional contraction along the axis Oy with the coefficient $k=a/b$ (without prove).



6. Convince by yourself that parametric equations of the ellipse have the form:

$$\begin{cases} x=a\cos\alpha, \\ y=b\sin\alpha, \; t\in \boldsymbol{R}. \end{cases}$$

fig.38

§16. *Hyperbola*

Definition. A curve, which has a canonical equation of the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\,. \qquad (30)$$

is called a hiperbola. Let $c^2=a^2-b^2$. Points $F_1(c,0)$, $F_2(-c,0)$ are called focuses.

Geometric properties of the hyperbola.

1. Let $M(x,y)$ be an arbitrary point of a hyperbola. Then (30) implies

$$x^2=a^2\left(1+\frac{y^2}{b^2}\right)$$

Hence

$$|x|\geq a, \text{ and } |x|>\frac{a}{b}\,|y|.$$

It means that the hyperbola is located in the domain, which is defined by these inequalities. This domain is shaded on the drawing (fig. 39).

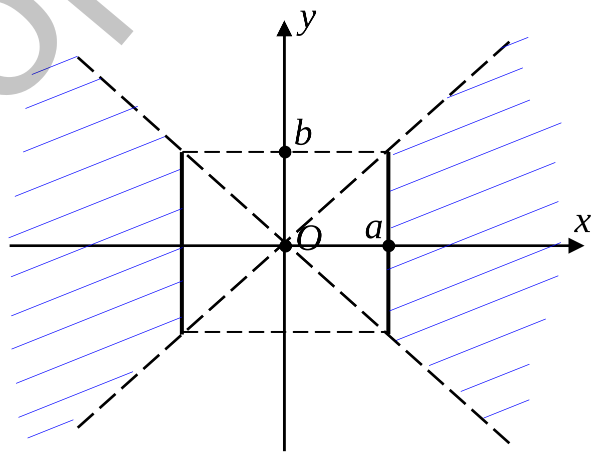


fig.39

2. Axis Ox intersects the hyperbola in the points $A_1(a, 0)$, $A_2(-a, 0)$, which are called the vertexes. Axis Oy doesn't intersect the hyperbola. Numbers a and b are called semi-axes of the hyperbola; a is called real semiaxis and b is called imaginary semiaxis.

3. In the same way as for the ellipse it is easy to demonstrate that coordinate axes are symmetry axes for the hyperbola and the initial point of the coordinate system is the symmetry center.

4. The focuses have the following property. The modulus of difference of distances from an arbitrary point M of the hyperbola γ to F_1 and from M to

F_2 is a constant value:

$$||MF_1|-|MF_2||=2a=\text{const}, \qquad (31)$$

i.e. it doesn't depend of the choice of the point точки $M\in\gamma$ (without proof).

5. The straight lines

$$l_1\colon y=\frac{b}{a}x \text{ и } l_2\colon y=-\frac{b}{a}x$$

are called *asymptotes of the hyperbola*. Hyperbola approaches them on the infinity but doesn't intersect them.

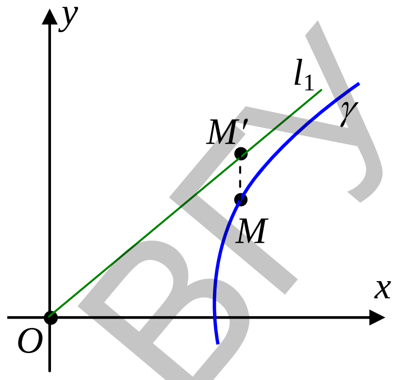


fig.40

In fact, let $M(x, y)$ be a point on a hyperbola and $M'(x, y')$ be a point on a corresponding asymptote. Then

$$|MM'|=|y'|-|y|.$$

$$(y')^2=\frac{b^2}{a^2}x^2, \quad y^2=b^2\left(\frac{x^2}{a^2}-1\right)$$

This equalities implies

$$(y')^2-y^2=b^2 \Leftrightarrow |y'|-|y|=\frac{b^2}{|y'|+|y|}\to 0 \text{ при } |y|\to\infty.$$

Both asymptotes may be defined by the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$$

In order to get this equation it is sufficient to replace 1 in the equation of the hyperbola by 0.

Asymptotes pass through the diagonals of the rectangle, which is defined by inequalities $|x|\le a$, $|y|\le b$. It is called *the fundamental rectangle of the hyperbola*. In order to draw the hyperbola one must draw this rectangle first.

6. If $a=b$ then the hyperbola is called *equilateral*. Its equation is

$$x^2-y^2=a^2, \qquad (32)$$

and asymptotes have the equations

$$l_1\colon y=x, \; l_2\colon y=-x.$$

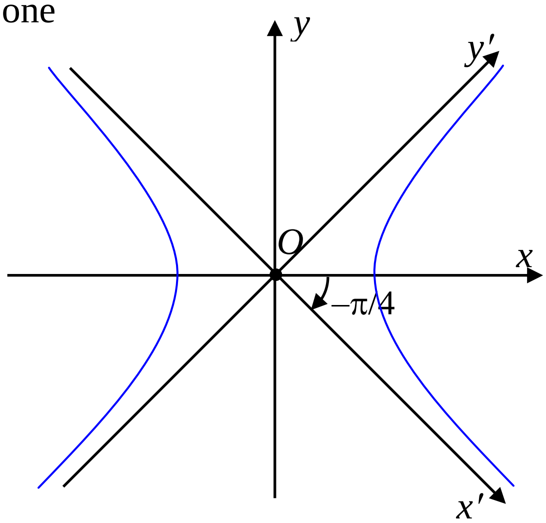


fig.41

It is obvious, that $l_1\perp l_2$, so we may choose these straight lines as the axes of new coordinate system $Ox'y'$, which is obtained from Oxy by the rotation on the angle -45°. Then the formulas of transformation of coordinates look like as follows:

$$\begin{cases} x = \dfrac{\sqrt{2}}{2}(x'+y'), \\ y = \dfrac{\sqrt{2}}{2}(-x'+y'). \end{cases}$$

We substitute them in (32) and we get the equation

$$2x'y' = a^2 \quad \Leftrightarrow \quad y' = \frac{k}{x'} \ ,$$

where $k = a^2/2$. So, the equilateral hyperbola is the graph of inverse proportionality.

7. Parametric equations of the hyperbola are:

$$\begin{cases} x = \pm a \operatorname{ch} t, \\ y = b \operatorname{sh} t, \ t \in \mathbf{R}. \end{cases} \quad \text{or} \quad \begin{cases} x = a(t + 1/t), \\ y = b(t - 1/t), \ t \in (-\infty, 0) \cup (0, +\infty). \end{cases}$$

Here the sign '+' corresponds to one branch of the hyperbola and '–' corresponds to another branch.

Exercise. Check it by yourself.

8. The hyperbola γ', which is defined by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \ ,$$

is called conjugated to the hyperbola γ, which is defined by the equation (30). It has the same fundamental rectangle, the same asymptotes, but it is located into the other pair of vertical angles defined by the asymptotes.

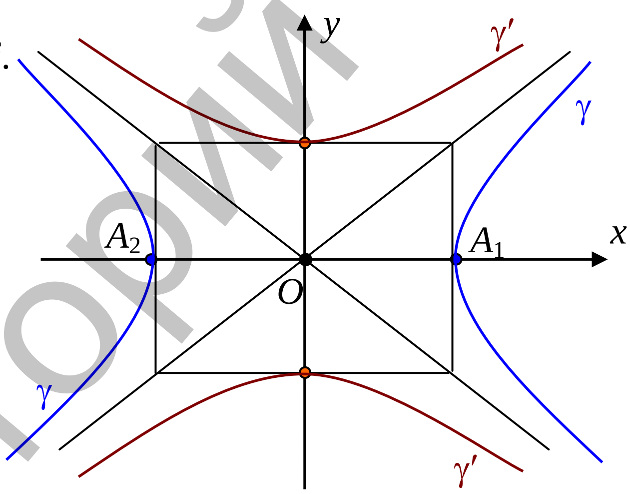


*fig.*42

§17. *Conic sections. Parabola*

Definition. A conic section is a curve, which is obtained by intersection of a conic surface and a plane, which doesn't pass through the vertex of the surface.

In the next chapter we are going to study that a conic surface looks like it is shown on the fig. 43 and we will make certain that conic sections are the ellipse, the hyperbola and the parabola.

We admit the following theorems without a proof.

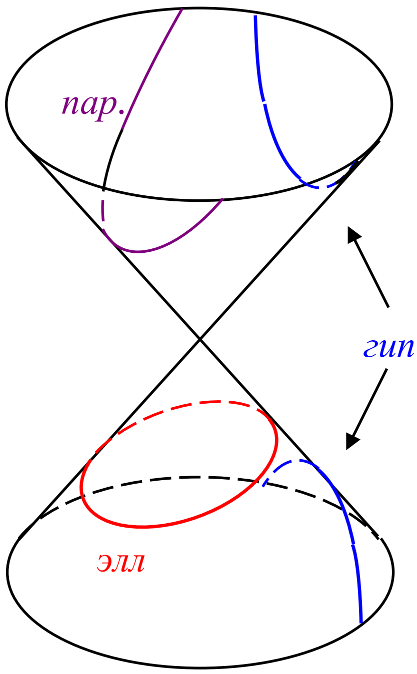


*fig.*43

Theorem 7. *For any conic section* γ, *except the circle there are a point* F *, which is called* [*the focuses*](#), *and a straight line* δ *, which is called* [*the directrix*](#), *such that the ratio of distances from an arbitrary point* $M \in \gamma$ *to* F *and from* M *to* δ *is a constant value* (*i.e. it doesn't depend on the choice of a point* $M \in \gamma$).

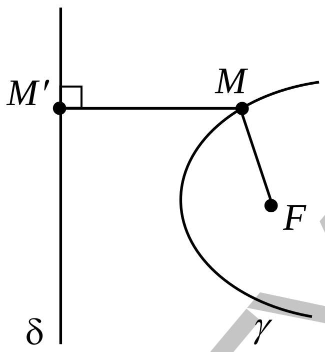


fig.44

This value $\varepsilon = |MF|/|MM'|$ is called *the eccentricity* of the conic section. The lesser is ε, the closer the curve is located to the focus. For $0<\varepsilon<1$ a curve is closed and it is an ellipse. The closer ε to 1, the more ellipse is stretched. If ε becomes equal to 1, an ellipse achieves the infinite length and the ellipse turns into a parabola.

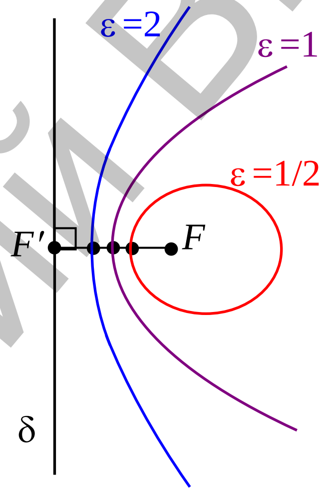


fig.45

The greater is ε, the closer a curve is located to directrix and for $1<\varepsilon<\infty$ we get a hyperbola.

It is obvious that the eccentricity and the distance $|FF'|$ from the focus to the directrix define a conic section uniquely. In fact, if two conic sections have the same distance from the focus to the directrix, then we can match their focuses and their directrixes. If the conic sections in addition have the same ε, then they will mach together. If two conic sections have the same ε, but different distances from F to δ, then these conic sections are similar. So all the parabolas are similar to each other.

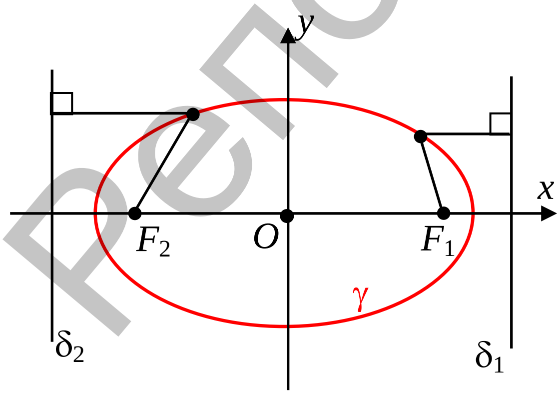


fig.46

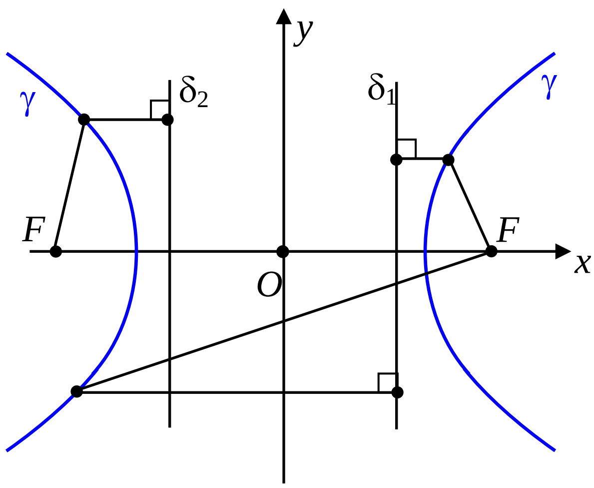


fig.47

***Theorem* 8.** *Let an ellipse and a hyperbola are defined by its canonical equations* (29) *and* (30). *Then the eccentricity of the curve is equal to* c/a, *focuses have the coordinates* $F_1(c, 0)$, $F_2(-c, 0)$, *and directrixes are defined by the equations*

$$\delta_1\colon x = \frac{a^2}{c}, \quad \delta_2\colon x = \frac{a^2}{c}$$

(we remind that $c^2 = a^2 - b^2$ *for an ellipse and* $c^2 = a^2 + b^2$ *for a hyperbola).*

This theorem implies, that focuses defined §15 and in §16, coincide with the focuses defined in this paragraph. Moreover, an ellipse and a hyperbola have two pairs focus-directrix and a curve may be defined by any of the pairs (fig. 46, 47).

Definition. Parabola is a conic section with the eccentricity equal to 1.

Let's make an equation of a parabola. Let $p = |FF'|$ be the distance from the focus to the directrix. We locate the initial point of a coordinate system in the midpoint of the segment FF' and we direct $Ox \uparrow\uparrow \overrightarrow{OF}$. Then the axis Oy is defined uniquely. Coordinates of the focus will be $F(p/2, 0)$, and the directrix will have the equation $x = -p/2$.

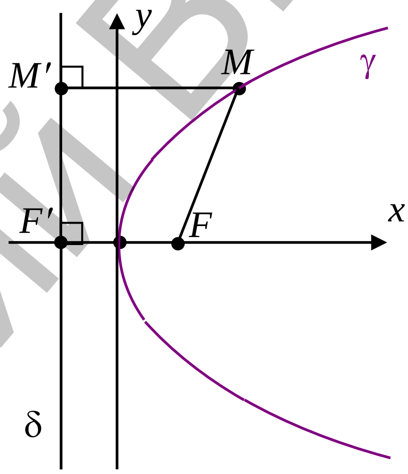


fig.48

Let $M(x, y)$ be an arbitrary point of a parabola. Then

$$|MF| = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2}, \quad |MM'| = x + \frac{p}{2}.$$

According to the definition

$$|MF|^2 = |MM'|^2 \Leftrightarrow \left(x - \frac{p}{2}\right)^2 + y^2 = \left(x + \frac{p}{2}\right)^2 \Leftrightarrow$$

$$y^2 = 2px. \tag{33}$$

Backwards, if coordinates of the point $M(x, y)$ satisfy (33), then

$$|MF|^2 = \left(x - \frac{p}{2}\right)^2 + y^2 = x^2 - px + \frac{p^2}{4} + 2px =$$

$$= x^2 + px + \frac{p^2}{4} = \left(x + \frac{p}{2}\right)^2 = |MM'|^2 \ \blacksquare$$

Equation (33) is called the canonical equation of a parabola.

Geometric properties of a parabola.

1. All the points of a parabola belong to the semi plane $x \geq 0$ (fig. 49).

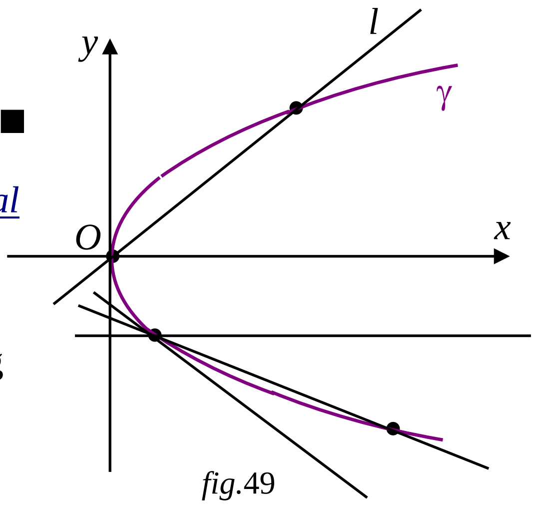


fig.49

2. If $M(x,y)\in\gamma$, i.e. the pair (x,y) satisfies (33), then the pair $(x,-y)$ also satisfies (33). This pair defines the point, which is symmetric to M in respect to Ox. Therefore Ox is the symmetry axis for the parabola. Parabola has no other symmetries.

3. The coordinate axes intersect parabola only in the point O and this point is called *the vertex of the parabola*. Any other straight line, which passes through the vertex intersects parabola in one more point.

In fact, an arbitrary straight line l, which passes through the vertex O, except Oy may be defined by the equation $y=kx$. In order to find the intersection points with the parabola γ we should solve the system

$$\begin{cases} y^2=2px, \\ y=kx. \end{cases} \Leftrightarrow \begin{cases} k^2x^2-2px=0, \\ y=kx. \end{cases} \Leftrightarrow \begin{cases} x(k^2x-2p)=0, \\ y=kx. \end{cases}$$

If $k\neq 0$ we get two solutions $(0,0)$ and $\left(\frac{2p}{k^2},\frac{2p}{k}\right)^2$, and if $k=0$ we get the only one solution. The value $k=0$ corresponds to the axis Ox.

Analogously, we can prove that any straight line parallel to Ox intersects parabola in one point N (fig. 47), and any other straight line, which passes through N, except the tangent line, intersects parabola in one more point.

We note several interesting *optical properties* of the conic sections.

A ray of the light, which comes out a focus of an ellipse, after the reflection from the ellipse passes through the second focus (fig. 50). From the mathematical point of view it means that $\forall M\in\gamma$, the segments MF_1 and MF_2 forms the equal angles with the tangent line in the point M. Let a ray of the light come out a focus of a hyperbola. Then after the reflection from the hyperbola, it seems as if this ray comes out the second focus (fig. 51).

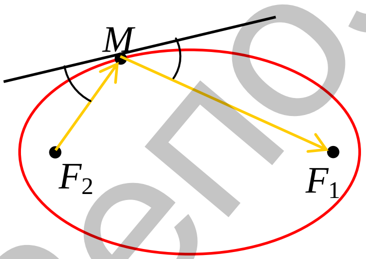


fig.50

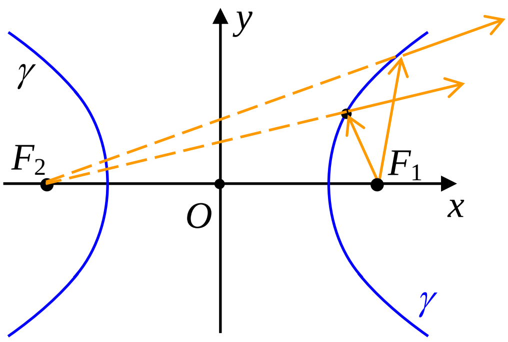


fig.51

A ray of the light, which comes out a focus of a parabola, after the reflection from the parabola, moves parallel to the axis Ox (fig. 52). Vice versa, the rays, which come from the infinity parallel to the axis, concentrate in the focus. The action of parabo-

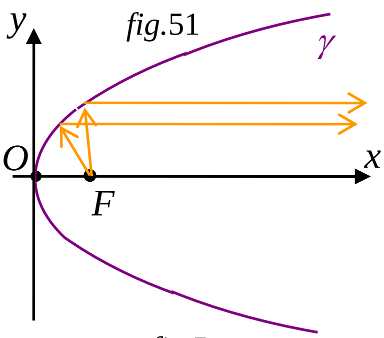


fig.5

lic reflectors aerials and the radars is based on this property of the parabola.

§18. *Examples of solving the problems*

***Problem* 1.** *Coordinates of four vertexes of a quadrangle ABCD are given*: $A(-3,-1)$, $B(7,-3)$, $C(8,2)$, $D(-2,4)$. *Prove that ABCD is a rectangle*.

Proof. We find the coordinates of vectors, which are defined by opposite sides of the quadrangle $ABCD$. In order to find the coordinates of the vector $\overrightarrow{AB}$, we should subtract the coordinates of a point A from the coordinates of a point B:

$$\overrightarrow{AB}(7-(-3),-3-(-1)),\ \overrightarrow{AB}(7+3,-3+1),\ \overrightarrow{AB}(10,-2)$$

And in the same way we get

$$\overrightarrow{DC}(8-(-2),2-4),\ \overrightarrow{DC}(8+2,-2),\ \overrightarrow{DC}(10,-2).$$

We see that $\overrightarrow{AB}=\overrightarrow{DC}$, i.e. the opposite sides of the quadrangle $ABCD$ are parallel and have the equal length. It means, that $ABCD$ is a parallelogram. Now we will prove, that adjacent sides are perpendicular. We know the coordinates of $\overrightarrow{AB}$. Let's find the coordinates of $\overrightarrow{AD}$: $\overrightarrow{AD}(1,5)$. Then we compute the scalar product of $\overrightarrow{AB}$ and $\overrightarrow{AD}$:

$$\overrightarrow{AB}\cdot\overrightarrow{AD}=10\cdot 1+(-2)\cdot 5=10-10=0.$$

Therefore $\overrightarrow{AB}\perp\overrightarrow{AD}$. Hence, $ABCD$ is a rectangle.

***Problem* 2.** *ABCD is a parallelogram and O is the intersection point of the diagonals AC and BD. Coordinates of three vertexes are given*: $A(-5,1)$, $B(1,3)$, $D(-4,5)$.

i) *find coordinates of the vertex C*;

ii) *calculate the square of the parallelogram*;

iii) *find the height h dropped from the vertex D to the side AB*;

iv) *find the coordinates of the point D.*

Solution. i) First we find coordinates of the vector $\overrightarrow{AB}$. We will subtract the coordinates of the point A from the coordinates of the point B:

$$\overrightarrow{AB}(1-(-5),3-1)\ \Leftrightarrow\ \overrightarrow{AB}(6,2).$$

But $\overrightarrow{AB}=\overrightarrow{DC}$ (fig. 53), so $\overrightarrow{DC}(6,2)$. In order to find coordinates of the point C, we add the coordinates of the vector $\overrightarrow{DC}$ to the coordinates of the point D: $C(-4+6,5+2)$; $C(2,7)$.

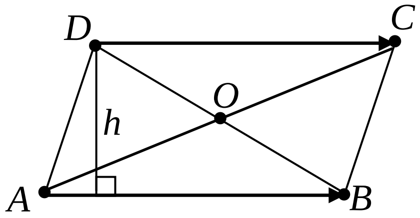


*fig.*53

ii) Then we find the coordinates of the vector $\overrightarrow{AD}$. We will subtract the coordinates of the point A from the coordinates of the point D: $\overrightarrow{AD}(-4-(-5), 5-1) \Leftrightarrow \overrightarrow{AD}(1, 4)$. Now we use formula (16):

$$S_{ABCD} = \text{mod}\begin{vmatrix} 6 & 2 \\ 1 & 4 \end{vmatrix} = |6\cdot 4 - 2\cdot 1| = 22.$$

iii) From the school program we know the formula $S_{ABCD} = |AB|\cdot h$. Consequently $h = \dfrac{S_{ABCD}}{|AB|}$. First we find the length of the side AB:

$$|AB| = \sqrt{6^2 + 2^2} = \sqrt{40} = 2\sqrt{10}.$$

and then

$$h = \frac{22}{2\sqrt{10}} = \frac{11}{\sqrt{10}} = \frac{11\sqrt{10}}{10}.$$

iv) Coordinates of the point O we can calculate as the arithmetic average of the coordinates of two points, B и D:

$$O\left(\frac{1-4}{2}, \frac{3+5}{2}\right) \Leftrightarrow O(-1{,}5; 4).$$

Answer: $C(2, 7)$; $S_{ABCD} = 22$; $h = \dfrac{11\sqrt{10}}{10}$, $O(-1{,}5; 4)$.

***Problem* 3.** *Vertexes of a quadrangle are the points* $A(1, 2)$, $B(7,-6)$, $C(11,-3)$, $D(8, 1)$. *Prove that ABCD is a trapezium. Find the lengths of its bases, its square and* $\cos\angle DAB$.

Solution. First we find coordinates of the vectors $\overrightarrow{AB}(6,-8)$, $\overrightarrow{BC}(4, 3)$, $\overrightarrow{CD}(-3, 4)$, $\overrightarrow{AD}(7,-1)$. Then we check if the vectors defined by the opposite sides are collinear:

$-\dfrac{6}{3} = -\dfrac{8}{4}$ – it's true, so $\overrightarrow{AB}$ is collinear to $\overrightarrow{CD}$.

$\dfrac{4}{7} = \dfrac{3}{-1}$ – it's false, so $\overrightarrow{BC}$ is not collinear to $\overrightarrow{AD}$.

So we have that two opposite sides are collinear and that two opposite sides are not. Hence $ABCD$ is a trapezium, and its bases are AB and CD (fig. 54). Let's find the length of the sides:

$$|\overrightarrow{AB}| = \sqrt{6^2 + 8^2} = 10,$$

and analogously $|\overrightarrow{BC}| = 5$; $|\overrightarrow{CD}| = 5$; $|\overrightarrow{AD}| = 5\sqrt{2}$.

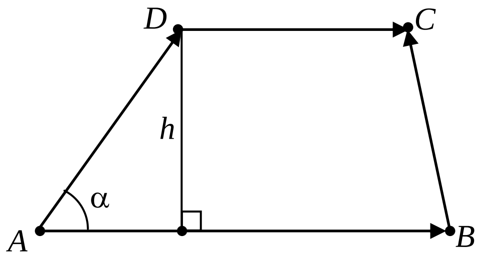


*fig.*54

Denote $\alpha = \angle BAD$. Then

$$\text{Cos}\,\alpha = \frac{\vec{AB}\cdot\vec{AD}}{|\vec{AB}||\vec{AD}|} = \frac{6\cdot 7 + (-8)\cdot(-1)}{10\cdot 5\sqrt{2}} = \frac{1}{\sqrt{2}},$$

hence $\angle BAD = 45^{\circ}$ and $\text{Sin}\,\alpha = \frac{1}{\sqrt{2}}$. Now we can find

$$h = |\vec{AD}|\cdot \text{Sin}\,\alpha = 5.$$

We know two bases and the height. So we can find the square:

$$S = \frac{1}{2}(|AB| + |CD|)\cdot h = \frac{75}{2}.$$

One more method how to find height by means of the equation of the straight line, is given in the next problem.

***Answer*:** $|\vec{AB}| = 10$, $|\vec{BC}| = 5$, $\text{Cos}\,\alpha = \frac{1}{\sqrt{2}}$, $S_{ABCD} = \frac{75}{2}$.

***Problem* 4.** *ABC is a triangle, CD is its height (fig. 55). Coordinates of the vertexes are given*: $A(-1, 3)$, $B(11, 0)$, $C(9, 9)$.

i) *Make an equation of the side AB, an equation of the height CD and find coordinates of the point D.*

ii) *Calculate the height of the triangle by the formula of distance from a point to a line.*

iii) *Calculate the length of the base AB and the square of the triangle by the formula* $S = \frac{1}{2}|AB|\cdot h$.

iv) *Calculate the square of the triangle* ΔABC *by formula* (17). *Compare with the previous result.*

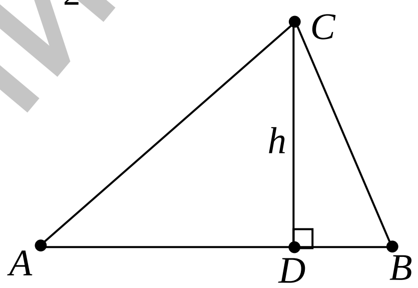


*fig.*55

Solution. i) First we find the angle coefficient of the straight line AB:

$$k_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{0-3}{11-(-1)} = -\frac{1}{4}.$$

Then we make the equation of AB:

$$y - y_A = k_{AB}(x - x_A) \;\Leftrightarrow\; y - 3 = -\frac{1}{4}(x-(-1)) \;\Leftrightarrow\; y = -\frac{1}{4}x + \frac{11}{4}.$$

Straight lines AB and CD are perpendicular. Thus $k_{CD} = -\frac{1}{k_{AB}} = 4$. So we can compose the equation of the line CD:

$$y - y_C = k_{CD}(x - x_C) \Leftrightarrow\; y - 9 = 4(x-9) \;\Leftrightarrow\; y = 4x - 27.$$

The point D is the common point of the lines AB and CD. Thus its coordinates must satisfy both of the equations AB and CD. So in order to find the coordinates of the point D we join the equations of AB and CD in one system and then we solve this system.

$$\begin{cases} y=-\frac{1}{4}x+\frac{11}{4}, \\ y=4x-27. \end{cases} \Leftrightarrow \begin{cases} 4x-27=-\frac{1}{4}x+\frac{11}{4}, \\ y=4x-27. \end{cases} \Leftrightarrow \begin{cases} x=7, \\ y=1. \end{cases} \Rightarrow D(7,1).$$

ii) Let's rewrite the equation of the straight line AB in general form:

$$y=-\frac{1}{4}x+\frac{11}{4} \Leftrightarrow 4y=-x+11 \Leftrightarrow x+4y-11=0.$$

Then we use formula (22) for this equation and for the point C:

$$h=\frac{|9+4\cdot 9-11|}{\sqrt{4^2+1^2}}=\frac{|34|}{\sqrt{17}}=2\sqrt{17}.$$

iii) First we find coordinates of the vectors $\overrightarrow{AB}(12,-3)$, $\overrightarrow{AC}(10,6)$. Then we can find the length of the base:

$$|\overrightarrow{AB}|=\sqrt{12^2+(-3)^2}=3\sqrt{4^2+1^2}=3\sqrt{17}.$$

And finally we find the square of the triangle:

$$S=\frac{1}{2}|AB|\cdot h=\frac{1}{2}\cdot 3\sqrt{17}\cdot 2\sqrt{17}=3\cdot 17=51.$$

iv) We find the square of the triangle in another way:

$$S_{ABC}=\frac{1}{2}\,\mathrm{mod}\begin{vmatrix} 12 & -3 \\ 10 & 6 \end{vmatrix}=\frac{1}{2}|12\cdot 6-10\cdot(-3)|=\frac{1}{2}|72+30|=\frac{1}{2}\cdot 102=51.$$

This result coincides with the previous one.

Answer: $D(7,1)$, $h=2\sqrt{17}$, $|AB|=3\sqrt{17}$, $S=51$.

***Problem* 5.** *ABC is a triangle. Coordinates of the vertexes are given*: $A(1,-6)$, $B(-3,0)$, $C(6,9)$. *Find coordinates of the center of the circumscribed circle and its radius (fig 56).*

Solution. The center of the circumscribed circle is located on the intersection of perpendicular bisectors of the triangle sides. We find coordinates of the middle points $M_1(x_1,y_1)$ and $M_3(x_3,y_3)$ of the sides BC and AB respectively:

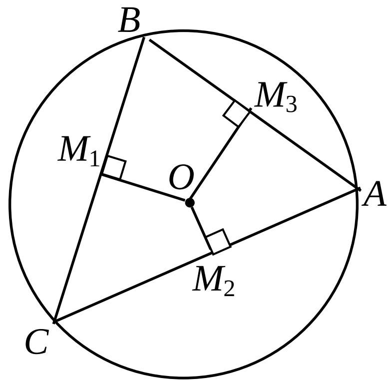


*fig.*56

$$x_1=\frac{x_C+x_B}{2}=\frac{-3+6}{2}=\frac{3}{2},\ y_1=\frac{y_C+y_B}{2}=\frac{0+9}{2}=\frac{9}{2},\ M_1\left(\frac{3}{2},\frac{9}{2}\right).$$

Analogously $M_3(-1,-3)$.

Then we find the angle coefficients of the sides BC and AB:

$$k_{BC}=\frac{y_C-y_B}{x_C-x_B}=\frac{6-(-3)}{9-0}=1,\quad k_{AB}=\frac{y_B-y_A}{x_B-x_A}=\frac{0-(-6)}{-3-1}=\frac{6}{-4}=-\frac{3}{2}.$$

Straight line M_1O is perpendicular to BC and M_3O is perpendicular to AB. So

$$k_{M_1O}=-\frac{1}{k_{BC}}=-1,\quad k_{M_3O}=-\frac{1}{k_{AB}}=\frac{2}{3}.$$

Now we can compose the equations of the straight lines M_1O and M_3O:

$$M_1O:\ y-\frac{3}{2}=-1\left(x-\frac{9}{2}\right),\quad y=-x+6;$$

$$M_3O:\ y-(-3)=\frac{2}{3}(x-(-1)),\ y=\frac{2}{3}x-\frac{7}{3};$$

We have $O=M_1O\cap M_3O$. So, coordinates of the point O must satisfy the system:

$$\begin{cases} y=-x+6, \\ y=\frac{2}{3}x-\frac{7}{3}, \end{cases}$$

We solve this system:

$$\frac{2}{3}x-\frac{7}{3}=-x+6,\quad \frac{2}{3}x+x=6+\frac{7}{3},\quad x=5, y=1,\quad O(5,1).$$

Radius is equal to the distance from the point O to any of the vertexes of the triangle:

$$R=|\overrightarrow{AD}|=\sqrt{(1-5)^2+(-6-1)^2}=\sqrt{65}.$$

***Answer*:** $O(5,1)$, $R=\sqrt{65}$.

***Problem* 6.** *ABC is a triangle and the polar coordinate system is given on the plane. The vertex A is located in the pole, and the vertexes B and C have the following coordinates*: $B(6,\frac{5\pi}{4})$, $C(4,\frac{7\pi}{12})$.

i) *Make the exact drawing of the triangle within the coordinate system*.

ii) *Calculate the square of ΔABC.*

iii) *Calculate the length of the side BC.*

Solution. i) First we draw the polar axis AP. Then we lay aside the angles $\varphi_1=\frac{5\pi}{4}$, $\varphi_2=\frac{7\pi}{12}$ and draw the rays. Finally we draw the points B and C such that $|AB|=6$ and $|AB|=4$ (fig. 57).

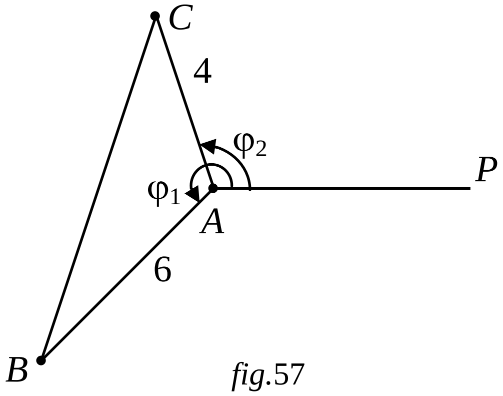


*fig.*57

ii) We see that

$$\angle BAC=|\varphi_2-\varphi_1|=\frac{5\pi}{4}-\frac{7\pi}{12}=\frac{2\pi}{3}.$$

From the school program you must know, that the square of a triangle is equal to the product of two sides and the sinus of the angle between them:

$$S_{\Delta ABC}=\frac{1}{2}AB\cdot AC\cdot \mathrm{Sin}\angle BAC=\frac{1}{2}\cdot 6\cdot 4\cdot \mathrm{Sin}\,\frac{2\pi}{3}=12\cdot\frac{\sqrt{3}}{2}=6\sqrt{3}.$$

iii) According to the cosine theorem

$$BC^2=AB^2+AC^2-2\cdot AB\cdot AC\cdot \mathrm{Cos}\angle BAC=36+16-2\cdot 6\cdot 4\cdot\left(-\frac{1}{2}\right)=76,$$

$$BC=\sqrt{76}=2\sqrt{19}.$$

***Answer*:** $S_{\Delta ABC}=6\sqrt{3}$, $BC=2\sqrt{19}$.

***Problem* 7.** *Let* $|\vec{\mathbf{m}}|=8$, $|\vec{\mathbf{n}}|=3$, $\alpha=\angle(\vec{\mathbf{m}},\vec{\mathbf{n}})=30^\circ$. *Vectors* $\vec{\mathbf{a}}=\vec{\mathbf{m}}-3\vec{\mathbf{n}}$ *and* $\vec{\mathbf{b}}=\vec{\mathbf{m}}+5\vec{\mathbf{n}}$ *are laid aside from one point. A triangle is drawn on this vectors. Find the length of the median, which is drawn from the same point.*

Solution. If $\vec{\mathbf{c}}$ is a vector, which defines the median (fig. 58), then

$$\vec{\mathbf{c}}=\frac{1}{2}(\vec{\mathbf{a}}+\vec{\mathbf{b}})=\frac{1}{2}(\vec{\mathbf{m}}-3\vec{\mathbf{n}}+\vec{\mathbf{m}}+5\vec{\mathbf{n}})=\vec{\mathbf{m}}+\vec{\mathbf{n}}.$$

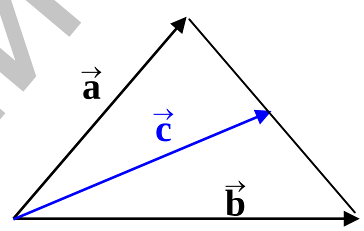


fig.58

We need to find the length of this vector. Let's remind, that the definition of the scalar product immediately implies $\vec{\mathbf{c}}^2=\vec{\mathbf{c}}\cdot\vec{\mathbf{c}}=|\vec{\mathbf{c}}|^2$ (the scalar square of a vector is equal to the square of its length). So we have

$$|\vec{\mathbf{c}}|^2=\vec{\mathbf{c}}\cdot\vec{\mathbf{c}}=(\vec{\mathbf{m}}+\vec{\mathbf{n}})^2=\vec{\mathbf{m}}^2+2\vec{\mathbf{m}}\cdot\vec{\mathbf{n}}+\vec{\mathbf{n}}^2=|\vec{\mathbf{m}}|^2+2|\vec{\mathbf{m}}|\cdot|\vec{\mathbf{n}}|\cdot\cos\alpha+|\vec{\mathbf{n}}|^2=$$

$$=64+2\cdot 8\cdot 3\cdot\frac{\sqrt{3}}{2}+9=73+48\sqrt{3}.$$

Hence, $|\vec{\mathbf{c}}|=\sqrt{73+48\sqrt{3}}$.

***Answer*:** The length of the median is equal $\sqrt{73+48\sqrt{3}}$.

***Problem* 8.** *The new Cartesian coordinate system is obtained from the old one as a result of the parallel translation, such that the new initial point is* $O'(2,-1)$, *and a rotation on the angle* $\alpha=\mathrm{ArcCos}\,\frac{4}{5}$.

i) *Write the formulas which express the new coordinates through the old one. Find the new coordinates of the point* A, *if its old coordinates are given*: $A(6,2)$.

ii) *Write the formulas which express the old coordinates through the new ones. Find the old coordinates of the point* B, *if its new coordinates are given*: $B(5,5)$.

Solution. i) One can find the coordinates by the formulas

$$\begin{cases} x' = (x-a)\cdot\mathrm{Cos}\,\alpha + (y-b)\cdot\mathrm{Sin}\,\alpha, \\ y' = -(x-a)\cdot\mathrm{Sin}\,\alpha + (y-b)\cdot\mathrm{Cos}\,\alpha, \end{cases}$$

where (a, b) are coordinates of the point O', α is the angle of rotation of the coordinate axes. We are given, that $\mathrm{Cos}\,\alpha = \frac{4}{5}$ and $\alpha \in [0, \pi/2]$. We find

$$\mathrm{Sin}\,\alpha = \sqrt{1 - \mathrm{Cos}^2\alpha} = \frac{3}{5}$$

and substitute all the data in the formulas:

$$\begin{cases} x' = \frac{4}{5}(x-2) + \frac{3}{5}(y+1), \\ y' = -\frac{3}{5}(x-2) + \frac{4}{5}(y+1). \end{cases}$$

For the point $A(6, 2)$ we find $x' = 5$, $y' = 0$. So $A(5, 0)_{O'x'y'}$.

ii) The old coordinates may be found by the formulas

$$\begin{cases} x = x'\cdot\cos\alpha - y'\cdot\sin\alpha + a, \\ y = x'\cdot\sin\alpha + y'\cdot\cos\alpha + b. \end{cases}$$

For our case

$$\begin{cases} x = \frac{4}{5}x' - \frac{3}{5}y' + 2, \\ y = \frac{3}{5}x' + \frac{4}{5}y' - 1. \end{cases}$$

We substitute here the coordinates $B(5, 5)_{O'x'y'}$ and we get $B(3, 6)_{Oxy}$.

Answer**:** $A(5, 0)_{O'x'y'}$, $B(3, 6)_{Oxy}$.

***Problem* 9.1.** *The equation of a curve is given*:

$$5x^2 - 9y^2 + 30x + 18y - 9 = 0.$$

Simplify this equation and draw the curve within the coordinate system.

Solution. We single out the full squares by x and by y. In order to get the full square we shall add $b^2/4$ to the expression $x^2 + bx$:

$$5(x^2 + 6x + 9 - 9) - 9(y^2 - 2y + 1 - 1) - 9 = 0.$$

$$5(x^2 + 3) - 45 - 9(y^2 - 1) + 9 - 9 = 0,$$

$$5(x+3)^2 - 9(y-1)^2 = 45.$$

Then we make change of coordinates:

$$\begin{cases} x' = x + 3, \\ y' = y - 1. \end{cases}$$

This change means the translation of the initial point to the $O'(-3,1)$. After the change we get the equation

$$5x'^2 - 9y'^2 = 45.$$

We divide it to 45:

$$\frac{x'^2}{9} - \frac{y'^2}{5} = 1.$$

We get the equation of a hyperbola with semiaxes $a = 3$, $b = \sqrt{5} \approx 2{,}24$. Then we make a drawing.

1. The point $O'(-3,1)$. 2. The coordinate axes $O'x' \| Ox$, $O'y' \| Oy$;
3. The fundamental rectangle with the center O' and the sides 6 и $2\sqrt{5}$;
4. The asymptotes: they comes through the diagonals of the rectangle.
5. The hyperbola: it is tangent to the fundamental rectangle and approaches the asymptotes at the infinity.

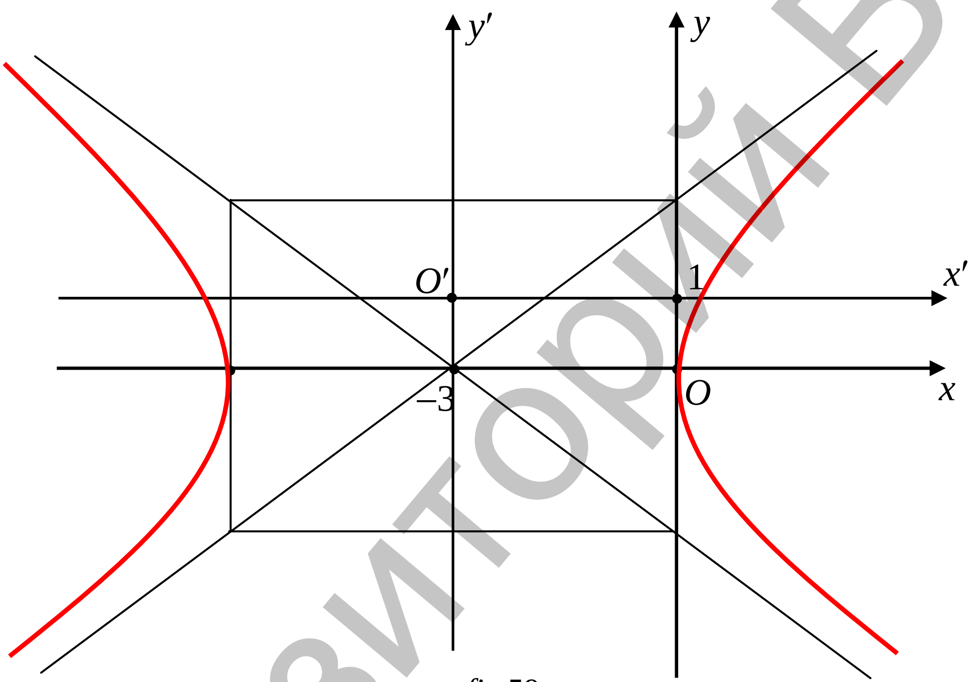


*fig.*59

***Problem* 9.2.** $3y^2 - 2x - 6y + 9 = 0.$

Solution. This equation has no term, which contains x^2, so we single out the full square only by y and then we make a change of coordinates:

$$3(y^2 - 2y + 1 - 1) - 2x + 9 = 0, \quad 3(y-1)^2 - 3 - 2x + 9 = 0.$$

$$3(y-1)^2 - 2(x-3) = 0.$$

$$\begin{cases} x' = x - 3, \\ y' = y - 1. \end{cases}$$

The new initial point is $O'(3,1)$. After the change of coordinates we get the equation of a parabola

$$y'^2 = \frac{2}{3}x'.$$

It has a parameter p=1/3, and its axis is $O'x'$. In order to make the drawing more precise we find an additional point. For example, if $y=0$ then $x=4{,}5$ (fig. 60).

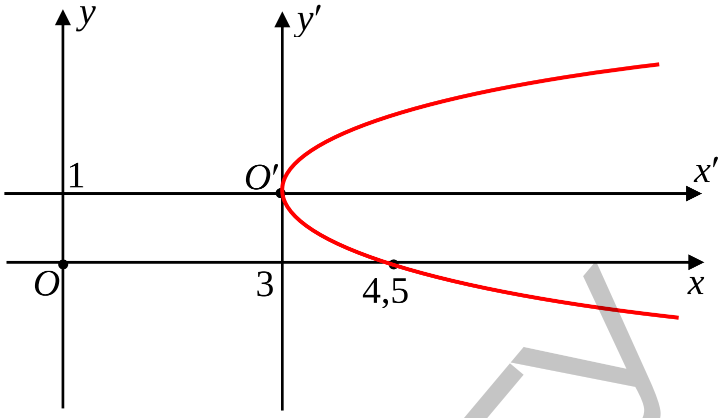


fig.60

Tasks for the independent solving

1. *ABC is a triangle, CD is its height. Coordinates of the vertexes are given*: $A(-1,3)$, $B(11,0)$, $C(9,9)$.

i) *Make an equation of the side AB, an equation of the height CD and find coordinates of the point D.*

ii) *Calculate the height of the triangle by the formula of distance from a point to a line.*

iii) *Calculate the length of the base AB and the square of the triangle by the formula* $S=\frac{1}{2}|AB|\cdot h$.

iv) *Calculate the square of the triangle* ΔABC *by formula* (16). *Compare with the previous result.*

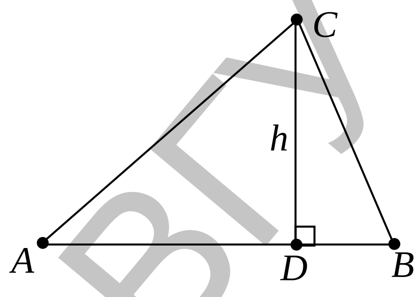


*fig.*36

1. $A(-3,1)$, $B(9,5)$, $C(-2,8)$.
2. $A(-2,-5)$, $B(1,7)$, $C(8,1)$.
3. $A(-5,-4)$, $B(10,-1)$, $C(-1,2)$.
4. $A(0,1)$, $B(12,-3)$, $C(5,6)$.
5. $A(0,2)$, $B(9,-4)$, $C(7,6)$.
6. $A(-1,2)$, $B(5,-2)$, $C(6,6)$.
7. $A(0,4)$, $B(3,-2)$, $C(-4,-3)$.
8. $A(0,-4)$, $B(8,0)$, $C(4,-7)$.
9. $A(0,-3)$, $B(9,0)$, $C(5,-8)$.
10. $A(0,-3)$, $B(-12,0)$, $C(-2,6)$.
11. $A(-6,0)$, $B(6,4)$, $C(-5,7)$.
12. $A(-1,-5)$, $B(2,7)$, $C(8,-3)$.
13. $A(-5,1)$, $B(10,-5)$, $C(2,4)$.
14. $A(-6,1)$, $B(6,-3)$, $C(-1,6)$.
15. $A(-6,2)$, $B(3,-4)$, $C(1,6)$.
16. $A(1,-2)$, $B(7,2)$, $C(8,-6)$.
17. $A(2,5)$, $B(5,-1)$, $C(-2,-2)$.
18. $A(4,0)$, $B(0,-8)$, $C(6,-6)$.
19. $A(0,3)$, $B(9,0)$, $C(5,8)$.
20. $A(-2,0)$, $B(1,-12)$, $C(8,-6)$.
21. $A(-3,2)$, $B(9,-2)$, $C(-2,-5)$.
22. $A(1,-7)$, $B(-2,5)$, $C(-8,-5)$.
23. $A(5,0)$, $B(-10,3)$, $C(1,6)$.
24. $A(0,2)$, $B(9,-1)$, $C(8,6)$.
25. $A(0,-2)$, $B(9,4)$, $C(7,-6)$.
26. $A(4,0)$, $B(0,8)$, $C(-4,1)$.
27. $A(-3,4)$, $B(0,-2)$, $C(-7,-3)$.
28. $A(-6,0)$, $B(4,2)$, $C(7,-2)$.
29. $A(8,0)$, $B(0,12)$, $C(-2,-2)$.
30. $A(0,-4)$, $B(-6,2)$, $C(0,4)$.

2. *ABC is a triangle. Coordinates of the vertexes are given. Find coordinates of the center of the circumscribed circle and its radius (fig 37).*

1. $A(-3,-2)$, $B(4,-3)$, $C(1,6)$.
2. $A(-6,-6)$, $B(1,-7)$, $C(-2,2)$.
3. $A(-10,-3)$, $B(7,-10)$, $C(2,15)$.
4. $A(3,2)$, $B(2,-5)$, $C(-1,4)$.
5. $A(3,0)$, $B(1,-4)$, $C(-2,5)$.
6. $A(9,5)$, $B(2,12)$, $C(-3,13)$.
7. $A(-4,-4)$, $B(8,2)$, $C(0,8)$.
8. $A(-5,-3)$, $B(7,1)$, $C(-1,9)$.
9. $A(-2,-5)$, $B(10,1)$, $C(4,-7)$.
10. $A(-5,-1)$, $B(7,5)$, $C(-1,-3)$.

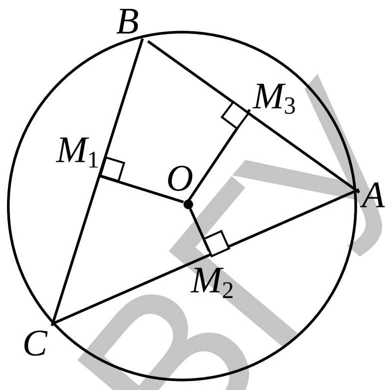


*fig.*37

11. $A(-5,0)$, $B(7,4)$, $C(1,-2)$.
12. $A(-6,-3)$, $B(9,2)$, $C(0,9)$.
13. $A(-5,-2)$, $B(10,3)$, $C(3,-6)$.
14. $A(6,5)$, $B(4,-9)$, $C(-2,9)$.
15. $A(6,1)$, $B(2,-7)$, $C(5,4)$.
16. $A(2,2)$, $B(-6,6)$, $C(-7,-1)$.
17. $A(8,0)$, $B(-4,4)$, $C(2,-8)$.
18. $A(1,-6)$, $B(4,-5)$, $C(-7,-2)$.
19. $A(15,-2)$, $B(-3,10)$, $C(-10,-7)$.
20. $A(4,1)$, $B(2,-3)$, $C(-5,-2)$.
21. $A(4,1)$, $B(2,-3)$, $C(-5,-2)$.
22. $A(5,2)$, $B(0,-3)$, $C(-4,-1)$.
23. $A(13,3)$, $B(5,-9)$, $C(-12,-2)$.
24. $A(8,0)$, $B(-4,4)$, $C(2,-8)$.
25. $A(9,1)$, $B(-3,5)$, $C(1,-7)$.
26. $A(-7,-4)$, $B(-5,2)$, $C(1,-10)$.
27. $A(-3,1)$, $B(-1,5)$, $C(5,-7)$.
28. $A(-2,-1)$, $B(0,5)$, $C(4,-7)$.
29. $A(-6,-3)$, $B(-2,5)$, $C(3,-10)$.
30. $A(9,2)$, $B(5,-6)$, $C(-9,-4)$.

3. *ABC is a triangle and the polar coordinate system is given on the plane. The vertex A is located in the pole, and the vertexes B and C have the following coordinates:* $B(6,\frac{5\pi}{4})$, $C(4,\frac{7\pi}{12})$.

i) *Make the exact drawing of the triangle within the coordinate system.*
ii) *Calculate the square of* ΔABC.
iii) *Calculate the length of the side BC.*

1. $C(2,-\frac{\pi}{3})$, $B(3,-\pi)$.
2. $C(2,-\frac{\pi}{12})$, $B(3,\frac{\pi}{4})$.
3. $C(1,\frac{\pi}{4})$, $B(2,\frac{5\pi}{12})$.
4. $C(3,\frac{\pi}{4})$, $B(2,\frac{7\pi}{12})$.
16. $B(1,-\frac{7\pi}{12})$, $C(2,-\frac{11\pi}{12})$.
17. $B(5,-\frac{\pi}{4})$, $C(3,-\frac{5\pi}{12})$.
18. $B(2,\frac{3\pi}{4})$, $C(5,\frac{11\pi}{12})$.
19. $B(1,\frac{5\pi}{12})$, $C(3,\frac{7\pi}{12})$.

5. $C(2, \frac{\pi}{3})$, $B(5, \frac{\pi}{12})$.

6. $B(4, -\frac{\pi}{9})$, $C(1, -\frac{5\pi}{18})$.

7. $B(2, \frac{7\pi}{12})$, $C(3, \frac{11\pi}{12})$.

8. $B(4, -\frac{\pi}{6})$, $C(7, \frac{\pi}{6})$.

9. $B(3, \pi)$, $C(4, \frac{2\pi}{3})$.

10. $C(5, \frac{7\pi}{12})$, $B(2, \frac{5\pi}{6})$.

11. $B(2, \frac{\pi}{3})$, $C(1, \frac{7\pi}{12})$.

12. $B(3, -\frac{\pi}{2})$, $C(1, \frac{\pi}{4})$.

13. $B(1, \frac{3\pi}{4})$, $C(3, \frac{11\pi}{12})$.

14. $B(5, \frac{3\pi}{4})$, $C(3, \frac{13\pi}{12})$.

15. $B(2, -\frac{\pi}{3})$, $C(3, -\frac{\pi}{6})$.

20. $B(2, \frac{11\pi}{12})$, $C(3, \frac{3\pi}{4})$.

21. $B(3, \frac{3\pi}{4})$, $C(4, \frac{7\pi}{12})$.

22. $B(5, \frac{5\pi}{12})$, $C(3, \frac{3\pi}{4})$.

23. $B(2, \frac{3\pi}{4})$, $C(3, \frac{\pi}{12})$.

24. $B(3, \frac{\pi}{12})$, $C(1, -\frac{\pi}{4})$.

25. $B(3, -\frac{3\pi}{4})$, $C(2, \frac{\pi}{12})$.

26. $B(1, \frac{\pi}{4})$, $C(2, \frac{11\pi}{12})$.

27. $B(3, \frac{11\pi}{12})$, $C(2, \frac{\pi}{4})$.

28. $C(5, \frac{7\pi}{12})$, $B(2, \frac{5\pi}{6})$.

29. $B(1, \frac{5\pi}{12})$, $C(3, \frac{7\pi}{12})$.

30. $C(5, \frac{7\pi}{12})$, $B(2, \frac{5\pi}{6})$.

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Tasks for the practical classes

Practical class 1. *Notion of a vector. Operations over vectors*

1.1. How many different vectors may be defined by the various ordered pairs of points, composed from the vertexes of a parallelogram?

1.2.1 Draw the vector а) $\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}}$; б) $2\vec{\mathbf{b}}$; в) $-2\vec{\mathbf{b}}$ (fig. 40).

1.2.2 Let $\vec{\mathbf{a}} = \overrightarrow{AN}$, $\vec{\mathbf{b}} = \overrightarrow{DQ}$. Tasks are the same.

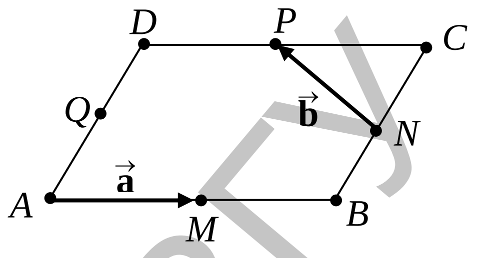

*fig.*40

1.3. Let $ABCD$ be a tetrahedron. Find sums of the vectors:

1. $\overrightarrow{AB}+\overrightarrow{BD}+\overrightarrow{DC}$; 2. $\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{DC}$.

Is it possible to find this sums without a drawing?

1.4. A point O is given inside a triangle ABC. Put the following vectors aside from the point O:

1. $\overrightarrow{OA}-\overrightarrow{OB}$; 2. $-\overrightarrow{OA}-\overrightarrow{OC}$; 3. $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$.

1.5. Let O be the center of regular hexagon $ABCDEF$. Draw the following vector: $2\overrightarrow{FO}-\overrightarrow{OB}$.

1.6. Three medians $\overrightarrow{AD}$, $\overrightarrow{BE}$, $\overrightarrow{CK}$ are drawn in the triangle ABC. Prove that $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CK}=\vec{\mathbf{o}}$.

1.7. Let $ABCD$ be a quadrangle and E and F are the midpoints of the sides AB и CD. Prove that $\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{BC} + \overrightarrow{AD})$.

1.8. Let $ABCD$ be a quadrangle and E and F are the midpoints of the diagonals AC and BD. Prove that $\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{CD}) = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{CB})$.

Home task.

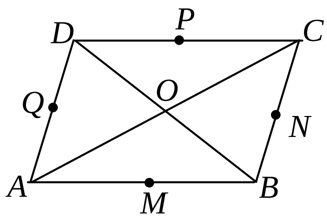

*fig.*41

1.9. Let A, B, C, D be the midpoints of the sequential sides of a quadrangle. Prove, that $\overrightarrow{AB}+\overrightarrow{CD}=\vec{\mathbf{o}}$.

1.10. Draw the vector $2\overrightarrow{NP}+\overrightarrow{AM}$ (fig.41).

Practical class 2.
Coordinates of a vector. Criterion of collinearity of vectors.

2.1. On what number we shall multiply a non-null vector $\vec{\mathbf{a}}$ in order to get a vector $\vec{\mathbf{m}}$, which satisfies the following conditions

1) $\vec{\mathbf{m}}\uparrow\uparrow\vec{\mathbf{a}}$ и $|\vec{\mathbf{m}}|=1$. 2) $\vec{\mathbf{m}}\uparrow\downarrow\vec{\mathbf{a}}$ и $|\vec{\mathbf{m}}|=3$. 3) $\vec{\mathbf{m}}\uparrow\downarrow\vec{\mathbf{a}}$ и $|\vec{\mathbf{m}}|=b$.
4) $\vec{\mathbf{m}}=\vec{\mathbf{o}}$.

2.2. Find a vector, which defines the direction of the bisectrix of the angle between two non-null vectors $\vec{\mathbf{a}}=\overrightarrow{AB}$ и $\vec{\mathbf{b}}=\overrightarrow{AC}$.

2.3. Let ABC be a triangle and AD be a bisectrix. Find decomposition of the vector $\overrightarrow{AD}$ through the vectors $\vec{\mathbf{a}}=\overrightarrow{AB}$ и $\vec{\mathbf{b}}=\overrightarrow{AC}$.

2.4. A vector $\vec{\mathbf{a}}$ forms an angle α with the vector $\vec{\mathbf{i}}$. Find the Cartesian coordinates of the vector $\vec{\mathbf{a}}$, if

а) $\alpha=45°$, $|\vec{\mathbf{a}}|=2$. б) $\alpha=60°$, $|\vec{\mathbf{a}}|=6$. в) $\alpha=-120°$, $|\vec{\mathbf{a}}|=2\sqrt{3}$.

2.5. Let $\vec{\mathbf{a}}(5,-2)$, $\vec{\mathbf{b}}(2,2)$. Find coordinates of the vector $\vec{\mathbf{c}}=\vec{\mathbf{a}}+5\vec{\mathbf{b}}$. Find the length of the vector $\vec{\mathbf{c}}$.

2.6. Let $ABCD$ be a parallelogram and coordinates of three vertexes are given: $A(1,2)$, $B(4,3)$, $D(2,9)$. Find coordinates of the vertex C. Find the square of the parallelogram and its height dropped from the vertex D.

2.7. The following points are the vertexes of a quadrangle: $A(2,2)$, $B(3,4)$, $C(6,1)$, $D(8,-4)$. Demonstrate that $ABCD$ is a trapezium. Find its square and $\cos\angle DAB$.

2.8. Let the vectors $\vec{\mathbf{a}}=\vec{\mathbf{i}}+\vec{\mathbf{j}}$, $\vec{\mathbf{b}}=\vec{\mathbf{i}}-2\vec{\mathbf{j}}$ define the sides of a parallelogram and $\vec{\mathbf{c}}$, $\vec{\mathbf{d}}$ define its diagonals. Find coordinates of the vectors $\vec{\mathbf{c}}$, $\vec{\mathbf{d}}$.

2.9. Select the non-null numbers α, β, γ such that $\alpha\vec{\mathbf{a}}+\beta\vec{\mathbf{b}}+\gamma\vec{\mathbf{c}}=\vec{\mathbf{o}}$, if $\vec{\mathbf{a}}(5,3)$, $\vec{\mathbf{b}}(2,0)$, $\vec{\mathbf{c}}(4,2)$.

2.10. Find a decomposition of the vector $\vec{\mathbf{c}}$ through the vectors $\vec{\mathbf{a}}$ и $\vec{\mathbf{b}}$.

а) $\vec{\mathbf{a}}(4,-2)$, $\vec{\mathbf{b}}(3,5)$, $\vec{\mathbf{c}}(1,-7)$.

Home task.

2.11. Find a decomposition of the vector $\vec{\mathbf{c}}$ through the vectors $\vec{\mathbf{a}}$ и $\vec{\mathbf{b}}$.

б) $\vec{\mathbf{a}}(5,4)$, $\vec{\mathbf{b}}(-3,0)$, $\vec{\mathbf{c}}(19,8)$.

Practical class 3. *Scalar product of the vectors. The length of the vector. The angle between two vectors*

3.1. Calculate the scalar product of the vectors $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$, if

1) $|\vec{\mathbf{a}}|=8$, $|\vec{\mathbf{b}}|=5$, $\angle(\vec{\mathbf{a}},\vec{\mathbf{b}})=120°$.

3.2. Let $|\vec{\mathbf{a}}|=2$, $|\vec{\mathbf{b}}|=5$, $\angle(\vec{\mathbf{a}},\vec{\mathbf{b}})=60°$. Calculate

1) $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$, 2) $\vec{\mathbf{a}}^2$, 3) $\vec{\mathbf{b}}^2$, 4) $(\vec{\mathbf{a}}+\vec{\mathbf{b}})^2$, 5) $(2\vec{\mathbf{a}}+\vec{\mathbf{b}})\cdot(3\vec{\mathbf{a}}+4\vec{\mathbf{b}})$.

3.3. Let vectors $\vec{\mathbf{a}}=2\vec{\mathbf{m}}+\vec{\mathbf{n}}$ and $\vec{\mathbf{b}}=\vec{\mathbf{m}}-2\vec{\mathbf{n}}$ define the sides of a parallelogram, where $\vec{\mathbf{m}}$, $\vec{\mathbf{n}}$ are the unit vectors and the angle between them is equal to $60°$. Calculate the length of the diagonals of this parallelogram.

3.4. Let $\vec{\mathbf{m}}$ and $\vec{\mathbf{n}}$ be the unit vectors and the angle between them is equal to 45°. Vectors $\vec{\mathbf{a}} = \vec{\mathbf{m}} + 3\vec{\mathbf{n}}$ and $\vec{\mathbf{b}} = 2\vec{\mathbf{m}} - \vec{\mathbf{n}}$ are drawn from the same point and they form the sides of a triangle. Calculate the length of the median, which is drawn from the same point.

3.5. Let $\vec{\mathbf{s}}, \vec{\mathbf{t}}$ be the unit vectors and the vectors $\vec{\mathbf{p}} = \vec{\mathbf{s}} + 2\vec{\mathbf{t}}$ и $\vec{\mathbf{q}} = 5\vec{\mathbf{s}} - 4\vec{\mathbf{t}}$ are perpendicular. Find the angle between $\vec{\mathbf{s}}$ and $\vec{\mathbf{t}}$.

3.6. Using the scalar product of vectors prove that:
1) diagonals of a rhombus are perpendicular;
2) diagonals of a rectangle have equal length.

3.7. Find the sum of squares of the medians in the triangle, if the sum of squares of its sides is equal to k.

3.8. Find the angle between two vectors
1) $\vec{\mathbf{a}}(1, 2), \vec{\mathbf{b}}(3, -4)$; 1) $\vec{\mathbf{a}}(2, 1), \vec{\mathbf{b}}(2, 6)$.

3.9. Find a vector, which is collinear to the bisectrix of the $\angle A$ in a triangle ABC, if $\overrightarrow{AB}(4, 3)$, $\overrightarrow{AC}(8, 15)$.

Home task.

3.11. Let vectors $\vec{\mathbf{a}} = 3\vec{\mathbf{m}} + \vec{\mathbf{n}}$ and $\vec{\mathbf{b}} = \vec{\mathbf{m}} - 2\vec{\mathbf{n}}$ define the sides of a parallelogram, where $\vec{\mathbf{m}}, \vec{\mathbf{n}}$ are the unit vectors and the angle between them is equal to 30°. Calculate the length of diagonals of the parallelogram.

Practical class 4.
Distance between two points. Partition of a segment in the prescribed ratio.

4.1. Determine if the following points A, B, C belong to one straight line. If they do, then find the simple ratio of the points A, B, C: $\lambda = (AB, C)$.
1) $A(1, 1), B(3, 4), C(7, 10)$; 2) $A(0, 1), B(1, 3), C(2, 2)$.

4.2. Prove that a triangle with the vertexes $A(3, 0), B(8, -5), C(2, -3)$ is a rectangular. Find 1) lengths of its sides, 2) internal angles, 3) its square, 4) length of the height to the hypotenuse.

4.3. Let $A(3, 0), B(8, -5)$ and the segment AB be divided by the points C, D, E into 4 equal parts. Find coordinates of the points C, D, E.

4.4. Let the points $A(3, 0), B(8, -5), C(2, -3)$ be the vertexes of a triangle and M be the point of intersection of medians of ΔABC. Find the coordinates of the point M.

4.5. Let AD be the bisectrix of the triangle ABC. Coordinates of the vertexes are given: $A(4, 1), B(7, 5), C(-4, 7)$. Find the coordinates of the point D and the length of AD.

Home task.

4.6. Let AD be the bisectrix of the triangle ABC. Coordinates of the vertexes are given: $A(1, 1)$, $B(8, 2)$, $C(5,–3)$. Find the coordinates of the point D and the length of AD.

Practical class 5.

Polar coordinates on the plane. Transformation of Cartesian coordinates

5.1. Find the polar coordinates of the points if its Cartesian coordinates are given: $A(1, 1/2)$, $B(1, 1)$, $C(\sqrt{3}, 1)$, $D(-3, 3)$. ***Home task*:** $E(1,-\sqrt{3})$.

5.2. Find the Cartesian coordinates of the points, if its polar coordinates are given: $A(2, \pi/3)$, $B(\sqrt{2}, 3\pi/4)$, $C(5, \pi/2)$. ***Home task*:** $D(3, \pi/6)$.

5.3. Let $ABCD$ be a parallelogram and O be the intersection point of the diagonals. Let O coincide with the initial point of the polar coordinate system and polar coordinates of two more vertexes are given: $A(2,-4\pi/9)$, $B(\sqrt{2}, 3\pi/14)$. Calculate the polar coordinates of the vertexes C and D.

5.4. Calculate the square of a triangle ABC, if the vertex A is located in the pole and coordinates of two more vertexes are given: $B(5, \pi/4)$, $C(8,-\pi/12)$. Find the length of the side BC. Make the exact drawing of the triangle within the coordinate system.
Home task: $B(5, \pi/9)$, $C(8, 7\pi/9)$.

5.5. Calculate the square of a triangle ABC if its vertexes have the following polar coordinates: $A(3, \pi/8)$, $B(8, 7\pi/24)$, $C(6, 5\pi/8)$.

5.6. The new Cartesian coordinate system is obtained from the old one by a parallel translation and the new initial point is $O'(6,-1)_{Oxy}$. Find the new coordinates of the points, if its old coordinates are given: $A(1, 2)$, $B(-3, 4)$.

5.7. Coordinate system $Ox'y'$ is obtained from the system Oxy by the rotation on the angle $60°$. Make the formulas, which express the new coordinates of an arbitrary point M through its old coordinates and vice versa. The new coordinates of a point A are given: $A(2, \sqrt{3})_{Ox'y'}$. Find its old coordinates. What will be the equation of the line l: $x+y+4=0$ in respect to the new coordinate system?

5.8. The new coordinate system is obtained from the old one by the parallel translation in such a way that new initial point is $O'(-3, 1)$ and by the rotation on the angle α in such a way that $\text{Cos}\,\alpha=\frac{3}{5}$, $\text{Sin}\,\alpha=\frac{4}{5}$. Make the formulas of transformation of coordinates (from old coordinates to the new one and vise versa). A point A has the new coordinates $A(3, 1)$. Find its old coordinates.

Home task.

5.9. Two points are given: $A(-2, 5)$, $B(-6, 1)$. Find their new coordinates if the coordinate axes have been turned on the angle $\alpha=45°$, and the initial point has been translated in the point $O'(-2, 3)$.

Practical classes 6, 7.

Equation of a straight line on the plane. Mutual location of straight lines. Distance from a point to a straight line

6.1. Let ΔABC be a triangle and coordinates of the vertexes are given: $A(1, 2)$, $B(4, 3)$, $C(2, 9)$.

i) Make equations of all the sides.

ii) Find the angles between these straight lines.

6.2. Let ΔABC be a regular triangle and coordinates of two vertexes are given: $A(-3, 2)$, $C(1, 6)$. Compose the equation of the median BM.

6.3. Compose an equation of a straight line, which passes through the point $A(-2, 3)$ and forms the angle $60°$ with the axis Ox.

6.4. Let ΔABC be a triangle and coordinates of the vertexes are given: $A(1, 1)$, $B(-2, 3)$, $C(4, 7)$. Compose the equation of the median AM.

6.5. Let ΔABC be a triangle and let $M_1(1, 1)$, $M_2(-2, 3)$, $M_3(4, 7)$ be the midpoints of the sides. Compose the equations of the sides.

6.6. Let ΔABC be a triangle and coordinates of the vertexes are given: $A(1, 5)$, $B(-4, 3)$, $C(2, 9)$. Compose the equation of the side BC and the equation of the height AD. Find coordinates of the point D.

6.7. Let ABC be a triangle and H be the intersection point of its heights. Coordinates of two vertexes and the point H are given: $A(-6, 2)$, $B(2, -2)$, $H(1, 2)$. Find coordinates of the third vertex C.

6.8. Find coordinates of the center and radius of the circle, circumscribed around the triangle with the vertexes $A(1, 2)$, $B(3, -2)$, $C(5, 6)$. Compose the equation of the circle.

6.9. Compose the canonical and parametric equations of a straight line, which passes through:

i) the point $M_0(1, 2)$ and it is parallel to the vector $\vec{\mathbf{a}}(3, -1)$;

ii) two points $M_1(2, 4)$, $M_2(2, -5)$.

6.10. Which of the following pairs of the straight lines coincide, parallel or intersect:

i) $y=2x+1$ and $y=2x-5$; ii) $x-2y+4=0$ and $-3x+6y-12=0$;

iii) $x-5y=0$ and $2x-10y+7=0$; iv) $2x+3y-8=0$ and $x+y-3=0$.

6.11. Calculate the angle between the straight lines

1) $y=2x+1$ и $y=-x+2$; 2) $y=-\frac{3}{2}x+1$ и $y=\frac{2}{3}x+7$;

6.12. Make equations of the straight lines, which pass through the point $A(3,1)$ and form the angle $45°$ with the straight line $2x+3y-1=0$.

6.13. Let ABC be the equilateral right triangle. Compose the equations of the legs if the equation of hypotenuse AB and coordinates of the vertex C are given: $y=3x+5$, $C(4,-1)$.

6.14. Parametric equations of a straight line are given: $\begin{cases} x=1-4t, \\ y=3+t. \end{cases}$

Find:

i) the directing vector of the straight line;

ii) coordinates of a point for which $t_1=3$;

iii) values of the parameter for the intersection points of the line with the coordinate axes and the coordinates of these points;

iv) among the points $A(-3,4)$, $B(1,1)$ chose the point, which belongs to the straight line and which doesn't belong.

6.15. Rewrite the equation of a straight line in parametric form:

i) $y=2x-3$; ii) $6x+11y+9=0$.

6.16. Calculate the square of a triangle, which is bounded by the coordinate axes and the straight line $2x-3y-18=0$.

6.17. Find the distances from the points $O(0,0)$, $A(1,2)$, $B(-5,7)$, $C(4,-3)$ to the straight line $6x+8y-15=0$. If the segments AB and BC intersect the given straight line?

6.18. Find the distances between the parallel straight lines:

i) $x-2y+3=0$ и $2x-4y+7=0$; ii) $3x-4y+1=0$ и $\begin{cases} x=1+4t, \\ y=3t. \end{cases}$

6.19. Let ABC be a triangle. Coordinates of its vertexes are given: $A(2,5)$, $B(1,3)$, $C(7,0)$. Calculate the length of its heights (one height in the class and the others as a home task).

Home task.

6.20. Individual task, which coincides in the context of the problem 6.6.

6.21. Find coordinates and the vertexes of a triangle, if the equations of its sides are given $3x+y=0$; $y=3$; $-2x+y+3=0$.

6.22. Let $ABCD$ be a square. Compose the equations of the sides, if coordinates of the vertex A and the equation of the diagonal BC are given: $A(-4,5)$, BC: $7x-y+8=0$.

Practical class 8.
The circle and the ellipse

8.1. Find out, which figure is defined by the following equation
i) $x^2+y^2-2x+4y-4=0$; ii) $x^2+y^2-2x+4y+5=0$;
iii) $x^2+y^2-2x+4y+6=0$.

8.2. Compose the equation of a circle, which has the center $S(1,-3)$ and passes through the point $A(5,-3)$.

8.3. Compose the equation of a circle, which passes through the points:
i) $A(-1,5), B(7,1), C(2,6)$; ii) $A(-1,5), B(-2,-2), C(1,19)$.

8.4. For the ellipses $25x^2+9y^2=225$ and $9x^2+25y^2=225$ find
i) semi-axes; ii) focuses; iii) the eccentricity;
iv) equations of the directrixes.
Draw this ellipse within the coordinate system.

8.5. Make the equation of the ellipse, if its focuses belong to the axis Oy, they are symmetric in respect to the initial point and:
i) the distance between focuses $2c=6$ and the big axis is equal to 10.
ii) the big axis is equal to 26 and the eccentricity $\varepsilon=12/13$.

8.6. How are the points located on the plane if their coordinates satisfy the following equation (draw this set):
i) $5x^2+9y^2+30x-18y+9=0$;
ii) $5x^2+9y^2+30x-18y+54=0$;
iii) $5x^2+9y^2+30x-18y+60=0$;
iv) $y=1-\frac{4}{3}\sqrt{-6x-x^2}$.

Home task.

8.6. v) $x=-2+\sqrt{-5-6y-y^2}$.

Practical class 9.
Hyperbola. Parabola

9.1. The hyperbola is given: $\frac{x^2}{25}-\frac{y^2}{144}=1$. Find:
i) semiaxis a and b; ii) focuses; iii) the eccentricity; iv) equations of the asymptotes; v) equations of the directrixes.
Draw this hyperbola within the coordinate system.

9.2. Make the equation of a hyperbola, if its focuses belong to Oy, they are symmetric in respect to the initial point and the distance between focuses is equal $2c=26$, the eccentricity $\varepsilon=6/5$.

9.3. What figures are defined by the following equations:
i) $16x^2-25y^2+32x-100y+316=0$;

ii) $x = 9 - 2\sqrt{y^2 + 4y + 8}$
iii) $xy = 4$.

9.4. Make the equation of a parabola, whose vertex is located at the point $A(1,-2)$, if the parabola is:
i) symmetric in respect to the straight line $x = 1$ and passes through the point $B(2, 0)$;
ii) symmetric in respect to the straight line $y + 2 = 0$ and passes through the point $B(2, 0)$.

9.5. The water stream of the fountain has the form of a parabola and it attains the maximal height 4 miters on the distance 0,5 miters from the vertical line, which passes through the point O outflow of the stream. Find the height of the stream on the distance 0,75 miters from the same vertical line.

9.6. The diameter of a parabolic reflector is equal to 15 sm., and its depth is equal to 10 sm. On what distance from the vertex is the focus of the reflector located?

9.7. A parabola is defined by the equation $x^2 + 14x + 6y + 37 = 0$. Find coordinates of its vertex, the direction of its axis and the value of the parameter. Draw the parabola.

Home task.

9.8. A stone is thrown under the acute angle to the horizon. The stone move by the parabola and fall on the distance 24 miters from its initial position. The maximal height achieved by the stone is equal to 6 m. Find the height of the stone's flight on the distance 20 miters from the initial point.

Additional tasks.

Conic sections. Their equations in polar coordinate system

10.1. Each point of a curve is distant from the straight line $x = 3$ two times farther than from the initial point of the coordinate system. Make the equation of the curve.

10.2. The equations of the circles in polar coordinate system are given. Write their equations in Cartesian coordinate system.
1) $r = 4\cos\varphi$; 2) $r = \cos\varphi + \sin\varphi$.

10.3. Find out what figures are defined by the equations in polar coordinates. Make their equations in Cartesian coordinate system and draw these figures.
1) $r = \dfrac{3}{4 - 4\cos\varphi}$; 2) $r = \dfrac{5}{3 - 4\cos\varphi}$.

Учебное издание

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MATHEMATICAL ANALYSIS

Study guide
for the self-organized work of the students of the specialty
"Applied Informatics"

Методические рекомендации

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