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ДИФФЕРЕНЦИРУЕМЫЕ МНОГООБРАЗИЯ

DIFFERENTIAL MANIFOLDS

Методические рекомендации

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Данное издание подготовлено для студентов второй ступени высшего образования факультета математики и информационных технологий в соответствии с учебной программой по дисциплине «Дополнительные главы топологии» (специальность «Математика и компьютерные науки»). Излагается теоретический материал и представлены упражнения для самостоятельного решения.

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Содержание

Intruduction 4
Chapter 1. Differential Manifold 5
§1. *Differentiability…* 5
§2. *Separability axioms* 9
§3. *Notion of a manifold. Examples* 10
§4. *Mappings of manifolds* 15
§5. *Tangent vectors* 18
§6. *Fiber bundle* 21
§7. *Vector fields. Integral curves* 23
§8. *Tangent mapping* 26
§9. *Submanifold* 27
§10. *Tensors* 30
§11. *Connectivity on manifold or covariant derivation* 31
§12. *Geodesic lines. Exponential mapping* 34
Chapter 2. Riemannian manifold 38
§1. *Definition of Riemannian manifold* 38
§2. *Riemannian connectivity* 39
§3. *Curvature tensor* 41
§4. *Function of distance of Riemannian manifold. Extremal property of geodesic lines* 44
§5. *Compaction* 48
§6. *Complete Riemannian manifold* 50
§7. *Comparison theorems. Connection with curvature and topological structure* 51

Intruduction

This textbook is intended for the organization of independent work of students of the second stage of higher education, studying in the specialty "Mathematics and Computer Science". The theory of differentiable manifolds and Riemannian manifolds is presented. We avoid detailed proves. To get a positive mark on the exam, students should be sure to compile a glossary of terms for the entire subject. It is recommended to use an electronic resource https://www.multitran.com/. The presentation of the theoretical material is accompanied by simple exercises that must also be performed.

Chapter 1. Differential Manifold

§1. *Differentiability*

Recall that $\boldsymbol{R}^n$ is a vector space, but we will also consider its elements as points. Let U and V be two domains, $U\subset\boldsymbol{R}^n$, $V\subset\boldsymbol{R}^m$, and let $f\colon U\to V$ be some function. Since $U\subset\boldsymbol{R}^n$, we can consider f as a function of n variables $(x_1, x_2, \ldots, x_n)$.

Let $x\in\boldsymbol{R}^n$. Then $y=f(x)\in\boldsymbol{R}^m$. Hence y can be represented in coordinates as $(y_1, y_2, \ldots, y_m)$. Accordingly we can write $f(x)=(f_1(x), f_2(x), \ldots, f_m(x))$, where $f_1(x), f_2(x), \ldots, f_m(x)$ are common functions of n variables.

Recall that $\boldsymbol{R}^n$ are $\boldsymbol{R}^m$ metric spaces. Denote their metrics as ρ_1 and ρ_2.

***Definition* 1.** We write $x\to x_o$ if $\rho_1(x,x_o)\to 0$. We write

$$y_o=\lim_{x\to x_o} f(x)$$

if

$$\lim_{x\to x_o} \rho_2(f(x), y_o)=0\,.$$

This is equivalent to $\lim\limits_{x\to x_o} f_i(x)=y_o^i$, for all $i=1,2,\ldots,m$.

***Definition* 2.** Function $f\colon U\to V$ is said to be continuous at a point $x_o\in U$, if

$$\lim_{x\to x_o} f(x)=f(x_o)\,.$$

Function $f\colon U\to V$ is said to be continuous in the domain if it is continuous at every point $x_o\in U$.

It is easy to prove, that $f\colon U\to V$ is continuous if and only if all the functions $f_1(x)$, $i=1,2,\ldots,m$ are continuous.

Since $\boldsymbol{R}^n$ are $\boldsymbol{R}^m$ vector spaces, there are operations of the sum and the difference of two points.

***Definition* 3.** $f'_{x_i}(x_o)=\lim\limits_{x\to x_o}\dfrac{f(x)-f(x_o)}{x_i-x_i^o}$, $i=1,2,\ldots,m,$ if this limit exists and is finite.

It is easy to prove, that $f'_{x_i}(x_o)=(f'_{1x_i}(x), f'_{2x_i}(x), \ldots, f'_{mx_i}(x))$, $i=1,2,\ldots,m$, i.e. we can find partial derivatives coordinatewise.

Suppose that $f'_{x_i}(x_o)$ exists at every point $x_o\in U$. Than it is also a function. Therefore we can calculate its partial derivatives, which are also functions and so on.

***Definition* 4.** We say, that $f(x)$ belongs to the class $C^n(U)$ if it has partial and mixed derivatives of the orders $1, 2, \ldots, n$ and they are continuous functions. We write that $f(x)\in C^\infty(U)$ if it has partial and mixed derivatives of any order.

Each coordinate function can be expanded into a Taylor series at each point and in this way the entire function $f(x)$ can be also expanded into a Taylor series at each point.

***Definition* 5.** We say that $f(x)$ is analytic function and write $f(x)\in C^{\omega}(U)$ if its Taylor series at each point converges to the function $f(x)$ itself.

All elementary functions of one variable of the class $C^{\infty}(U)$ are analytic in their domains of definition.

***Example* 1.** The following function is not analytic:

$$f(x)=\begin{cases} e^{\frac{1}{x^2}}, x\neq 0 \\ 0, x=0 \end{cases}$$

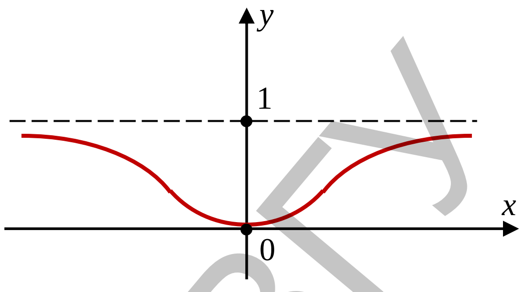

fig. 1

All the derivatives and the function itself are equal zero at point 0. Therefore its Taylor series converges to the null function, and not to the function $f(x)$. You can see its graph on figure 1.

***Definition* 6.** Consider numbers $f_{ij}(x_o) = (f_i)'_{x_j}(x_o)$. The matrix composed of this numbers is called the Jacobi matrix of the mapping f at the point x_o. We denote it as $J(x_o)$.

***Definition* 7.** Mapping $f\colon U\to V$ is called diffeomorphism, if

1) $f\in C^1(U)$;
2) f is homeomorphism.

Of course, f can be homeomorphism only if $m=n$, i.e. we can consider, that U and V are in the same space. In this case $J(x)$ is square matrix and we can calculate its determinant.

***Definition* 8.** The determinant $|J(x_o)|$ is called the Jacobian of the mapping $f\colon U\to V$ at the point x_o.

It turns out, that f is a diffeomorphism if and only if $f\in C^1(U)$ and $|J(x)|\neq 0 \quad \forall x\in U$.

Let $f\colon U\to V$, $g\colon V\to W$ be two mappings, $U,V,W\in \boldsymbol{R}^n$, $x_o\in U$, $f(x_o)=y_o\in V$, $g(y_o)=z_o\in W$. Consider the composition of the mappings

$$h=g\circ f\colon U\to V$$

(figure 2).

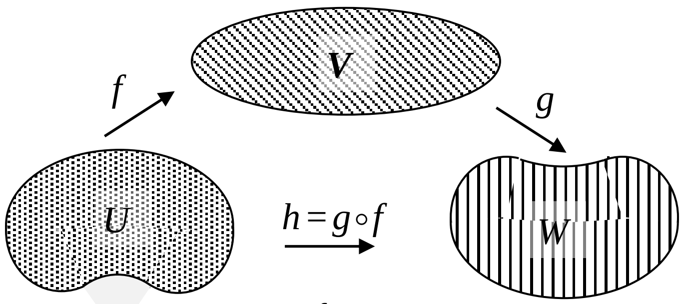

fig. 2

Let $J_1(x_o)$ be Jacobi matrix for the mapping f at the point x_o, $J_2(y_o)$ be Jacobi matrix for the mapping g at point y_o, $J(x_o)$ be Jacobi matrix for the mapping h at the point x_o. Then it turns out that

$$J(x_o) = J_1(y_o)\circ J_2\,(x_o) \tag{1}$$

and therefore $|J(x_o)| = |J_1(y_o)| \circ |J_2(x_o)|$. Consequently, h is a diffeomorphism if and only if f and g are diffeomorphisms.

***Example* 2.** Let coordinates (x^1,x^2), (y^1,y^2), (z^1,z^2), be defined respectively on the domains U, V, W. Consider mappings f and g, which are defined by the formulas

$$f: \begin{cases} y_1 = (x^2)^3, \\ y_2 = x^1, \end{cases} \quad g: \begin{cases} z^1 = y^1, \\ z^2 = y^1 y^2. \end{cases}$$

Then

$$h: \begin{cases} z_1 = (x^2)^3, \\ y_2 = x^1 (x^2)^3. \end{cases}$$

Jacobi matrixes are

$$J = \begin{pmatrix} 0 & 3(x^2)^2 \\ 1 & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 1 & 0 \\ y^2 & y^1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 3(x^2)^2 \\ (x^2)^3 & 3x^1(x^2)^2 \end{pmatrix}.$$

***Exercise* 1.** Check, that formula (1) is true in this case.

We see, that $|J| = 0$ at all points where $x^2 = 0$, and $|J_1| = 0$ at all points where $y^1 = 0$. Therefore if we consider the following domains U and V (figure 3), then restrictions of mappings $f|_U$ and $g|_V$ are diffeomorphisms.

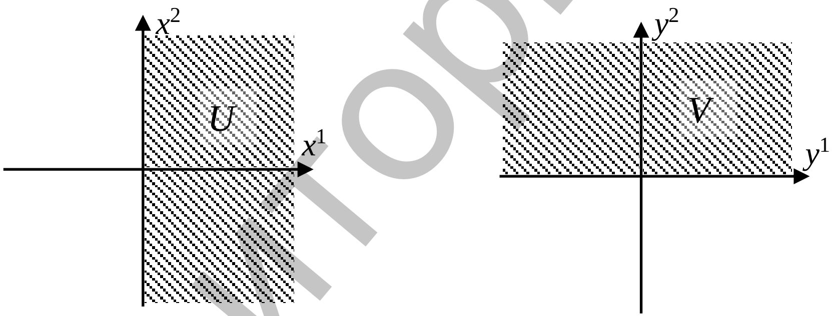

fig. 3

But f just maps U on V. That is why restriction $h|_U$ is diffeomorphism and it is easy to find out, that $W = h(U) = U$.

Let $c: I \to V$ be differentiable path, $\gamma = c(I)$ be its image – a curve in $\boldsymbol{R}^n$, $p \in \gamma$ – be a point on the curve and let $p = c(t_o)$ (figure 4). Suppose, that a function $f: \boldsymbol{R}^n \to \boldsymbol{R}$ is defined in some neighborhood of point p. If we substitute $c(t)$ in the function f,

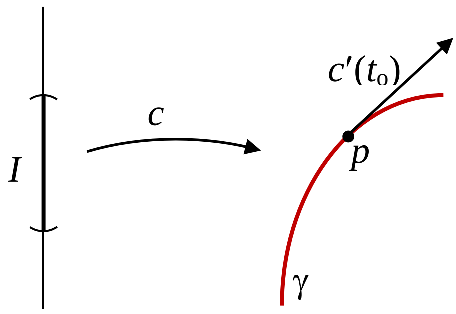

fig. 4

we get a function of one variable $f(c(t))$. We can calculate its derivative $\frac{d}{dt}(f(c(t)))$, and its value at the point p: $\frac{d}{dt}(f(c(t_o)))$.

***Definition* 9.** *The derivative of the function* f *along the path* c *at point* p is the value $\frac{d}{dt}(f(c(t_0)))$.

Suppose that $(x^1, x^2, \dots, x^n)$ are coordinates in $\boldsymbol{R}^n$. Then equations of the path are:

$$\begin{cases} x^1 = c^1(t), \\ x^2 = c^2(t), \\ \dots \\ x^n = c^n(t), \end{cases} \tag{2}$$

According to the rules of derivation

$$\frac{d}{dt}(f(c(t))) = \frac{\partial f}{\partial x^1}(c^1(t))' + \frac{\partial f}{\partial x^2}(c^2(t))' + \dots \frac{\partial f}{\partial x^n}(c^n(t))'. \tag{3}$$

Vector with coordinates $\left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \dots, \frac{\partial f}{\partial x^n}\right)$ is called the gradient of the function f, and we denote it $\mathbf{grad} f$. Vector with coordinates $\left((c^1(t))', (c^2(t))', \dots, (c^n(t))'\right)$ is tangent vector to the path c. It is the vector of the first derivative: $c'(t)$. Therefore formula (3) can be written in the vector form as follows:

$$\frac{d}{dt}(f(c(t))) = (\mathbf{grad} f) \cdot c'(t). \tag{3′}$$

The principal result is following. The derivative of function f along the path c doesn't depend on path c itself: it depends on the vector $c'(t)$ only. Therefore for all the paths, that have the same tangent vector of the first derivative at point p (figure 5), have the same derivatives along the path for any function.

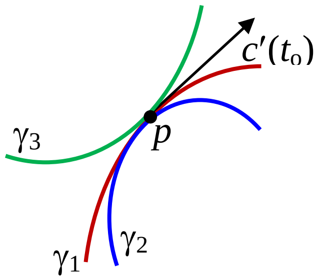

fig. 5

***Definition* 10.** We will say that such paths *are equivalent* at point p.

This result gives us opportunity to define the directional derivative of the given function.

***Definition* 11.** *The derivative of the function* f *in the direction of vector* $Y(y^1, y^2, \dots, y^n)$ at point p is the value

$$Y_p f = \mathbf{grad}_p f \cdot Y = \frac{\partial f}{\partial x^1} y^1 + \frac{\partial f}{\partial x^2} y^2 + \dots \frac{\partial f}{\partial x^n} y^n, \tag{4}$$

where all partial derivatives should be calculated at the point p.

We see that each vector at the point p defines an operator acting on functions. We will say that each vector acts on functions. It is easy to prove the following properties, using formula (4).

1) $(X_p + Y_p) f = X_p f + Y_p f$;
2) $\alpha X_p f = \alpha (X_p f)$;
3) if $Y_p = c'(t_0)$, then $Y_p f = (f(c(t_0)))'$.

Let $(E_1, E_2, \ldots, E_n)$ be a basis in $\boldsymbol{R}^n$. According to the definition 10

$$(E_1)_p f = \mathbf{grad}_p f \cdot E_1 = \frac{\partial f}{\partial x^1} \cdot 1 + \frac{\partial f}{\partial x^2} \cdot 0 + \ldots \ \frac{\partial f}{\partial x^n} \cdot 0 = \frac{\partial f}{\partial x^1}.$$

Analogously, $(E_i)_p f = \frac{\partial f}{\partial x^i}$, $i = 1, 2, \ldots, n$. We will call the derivative in the direction of a basis vector a basis derivative. Finally we get the following formula:

$$Y_p f = y^1 (E_1)_p f + y^2 (E_2)_p f + \ldots \ y^n (E_n)_p f.$$

It means the following. Each vector Y can be decomposed in a linear combination of the basis vectors with definite coefficients. Then $Y_p f$ is the linear combination of the basis derivatives with the same coefficients.

It is important to emphasize, that value $X_p f$ depends only on the value of the function f at some neighborhood of the point p, i.e. if two functions coincide in little domain around p, then they have equal directional derivatives.

§2. *Separability axioms*

The concept of a topological space is too general. It acquires its geometric content only after the introduction of additional axioms. An arbitrary topological space can be very different from a metric one. Let (M, τ) be a topological space.

Zero axiom of separability (Axiom of Kolmogorov). *Among two points $x, y \in M$ at least one of them has a neighborhood, that doesn't contain the second point.*

We will call topological spaces satisfying this axiom T_0-spaces.

The first axiom of separability. *For any two different points $x, y \in M$ there is a neighborhood $U(x)$ of point x, that doesn't contain y and there is a neighborhood $V(y)$ of point y, that doesn't contain x.*

We will call topological spaces satisfying this axiom T_1-spaces.

***Exercise* 2.** This axiom is equivalent to the requirement that any point be a closed set. Prove this statement.

The second axiom of separability (Hausdorff axiom). *For any two different points $x, y \in M$ there are a neighborhood $U(x)$ of point x, and a neighborhood $V(y)$ of point y, which doesn't intersect* (*figure* 6).

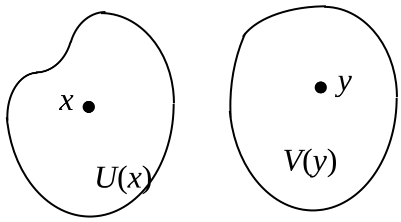

fig. 6

We will call topological spaces satisfying this axiom T_2-spaces or the Hausdorff spaces.

***Definition* 12.** Let s and S be two collections of sets (families of sets). If each set from the collection S is union of sets from the collection s, we say that collection s additively generates S or it is the additive base of S. If each set from collection S is intersection of sets from collection s, we say that collection s multiplicatively generates S or that s is the multiplicative base of S.

***Example* 3.** Consider a collection s of all infinite intervals $(-\infty, a)$ and $(b, +\infty)$ and consider a collection S of all intervals on the numerical line $\boldsymbol{R}$. Each finite interval (a, b) from S is intersection of two intervals $(-\infty, a)$ and $(b, +\infty)$. Each infinite interval from S is intersection of two equal intervals from s. Thus s multiplicatively generates S. Any open set in $\boldsymbol{R}$ is union of several intervals (may be of infinite number of intervals). A collection of all open sets is just the topology τ of $\boldsymbol{R}$. Therefore s additively generates τ.

***Definition* 13.** Additive base s of the topology τ is often called the *net* of topological space (M, τ). If collections of sets s_o multiplicatively generates the net of topology, then it is called subbase of topology.

It turns out that collection of sets s is a net in (M, τ) if and only if for any point x and its neighborhood $U(x)$ there is $V \in s$, such that $x \in V \subseteq U(x)$.

***Example* 4.** The collection of all open squares is a net of metric topology in the plane $\boldsymbol{R}^2$. We can inscribe a square V in any neighborhood $U(x)$ of any point x in $\boldsymbol{R}^2$ so that $x \in V \subseteq U(x)$ (figure 7).

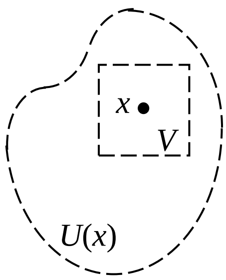

fig. 7

***Definition* 14.** We say that topological space (M, τ) *has numerable weight* if it has a numerable net and we say that this space satisfies *the second axiom of countability*.

It is proved, that any space $\boldsymbol{R}^n$ (for finite n) satisfies the second axiom of countability.

§3. *Notion of a manifold. Examples*

We have defined the concept of a two-dimensional surface in three-dimensional space. However, in various branches of mathematics, surfaces of a higher dimension are often used, which are located either in some kind of space or inside some other surface. In addition, these surfaces are often viewed on their own with no enclosing space.

The simplest example of the need for such an approach. It has already been proven that the space we are in is curved. This is not Euclidean space, but a three-dimensional surface. It can be viewed as embedded in four-dimensional space-time (Minkowski space). But if we are not talking about the theory of relativity, but only about the geometric shape of space, this surface should be considered "by itself."

magine that some two-dimensional creature lives in a two-dimensional world, where the sum of the angles of any triangle is equal to two straight lines and the Pythagorean theorem is fulfilled, as in the plane. This creature can measure distances between points. But his world is large enough, and it is still impossible to measure it entirely. (i.e. our being does not go beyond some small area). Can this creature determine if its world is a plane, a cylinder, or a cone? No, he can not. From the point of view of internal geometry, small parts of a cylinder or cone are arranged in the same way as a piece of a plane.

A man is in a similar situation. We cannot imagine the geometry of the Universe "as a whole". We can only find out that the nearest part of the Universe is arranged topologically, like Euclidean space; but this part is not isometric to the part of the Euclidean space (i.e. the lengths of the curves do not coincide with the lengths of the curves in the Euclidean space).

So how can we define the concept of a multidimensional surface that is not located in its enclosing space?

***Definition* 15.** Hausdorff topological space (M,τ) with countable base is called *m-dimensional topological manifold*, if it is locally homeomorphic to an open subset of the Euclidean space $\boldsymbol{R}^m$. This means that for each point $x \in M$ there is its neighborhood W_i in M and a homeomorphism $\varphi_i : W_i \to U_i$, where U_i is a domain in the Euclidean space. Pair (W_i, φ_i) is called *a map*, and the set of all such maps is called *atlas of maps* of M. Additionally it is assumed that the manifold must have an atlas consisting of a finite or countable number of maps, which cover the entire manifold.

Suppose, that two maps (W_i, φ_i) and (W_j, φ_j) of the manifold M intersects:

$$W_{ij} = W_i \cap W_j \neq \varnothing.$$

Then the set W_{ij} turns out to be depicted on two maps at once (figure 8):

$$U_{ij} = \varphi_i(W_{ij}),\ U_{ji} = \varphi_j(W_{ij})$$

(figure 7). There we have a mapping

$$\varphi_{ij} = \varphi_i^{-1} \circ \varphi_j : U_{ij} \longrightarrow U_{ji}.$$

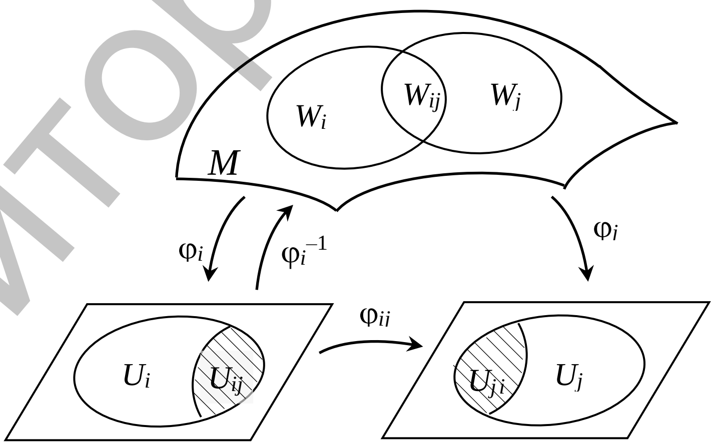

fig. 8

Since both mappings φ_i are φ_j are homeomorphisms, then φ_{ij} is also a homeomorphism.

***Definition* 16.** Homeomorphism $\varphi_{ij} : U_{ij} \longrightarrow U_{ji}$ is called *a transition function* from the map (W_i, φ_i) to the map (W_j, φ_j). If this functions for all maps are differentiable of the class C^k, than manifold M is called *a differentiable manifold of the class* C^k $(k = 1, 2, \ldots, \infty)$. If this functions for all the maps are differentiable of the class C^ω, than manifold M is called *an analytic manifold*. Manifold of the class C^0 is called topological manifold.

Thereafter (henceforth, from henceforth) it is always assumed by default that the manifold is differentiable manifold of the class C^∞.

***Definition* 17.** We say that a map (V,φ) *is consistent with the atlas* $\mathcal{A}$, if $\mathcal{A}' = \mathcal{A} \bigcup (V,\varphi)$ is an atlas of the same class of differentiability. An atlas is called full if it contains all maps consistent with the atlas $\mathcal{A}$.

It is obvious, that full atlas is not countable.

***Examples.* 5.** A circle (figure 9) is a one-dimensional manifold. At large, it is not homeomorphic to an open interval of numerical line, but a neighborhood of any it's point is a simple arc, i.e it is homeomorphic to an interval.

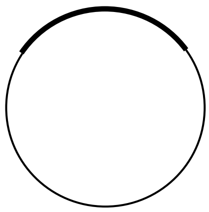

fig. 9

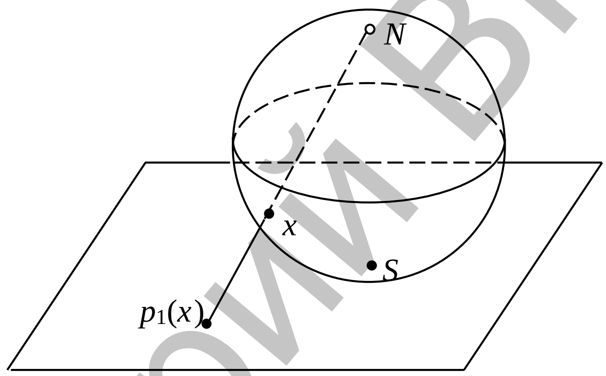

fig .10

6. Any simple surface is two-dimensional manifold. For instance, for the sphere S^2 (figure 10) an atlas can consist of two maps: $(S^2\backslash\{N\}, p_1)$ и $(S^2\backslash\{S\}, p_2)$, where p_1 is stereographic projection from the north pole N on the plane, and p_2 is stereographic projection from the south pole S.

We will demonstrate later that mapping $p_2^{-1}p_1$ is a differentiable of the class C^∞. Therefore, the sphere is a differentiable manifold of the class C^∞.

***Exercise* 3.** Show, that a set $M = \{p \in \boldsymbol{R}^2 \,|\, p_2 \geq 0,\ p_1(p_1^2 - p_2^2) = 0\}$ in $\boldsymbol{R}^2$ is not locally Eucledean, but the sets $M \backslash \{0\}$ and $\{p \in M \,|\, p_1 \geq 0\}$ are locally Eucledean.

Let $(V,\varphi) \in \mathcal{A}$ be a map of a manifold M. Then the mapping $\varphi : V \to U$ gives us opportunity to introduce coordinates on the domain V. We say that the point p has coordinates $(x^1, x^2, \ldots, x^m)$ if its image $q = \varphi(p)$ in the domain U has the same coordinates.

Let $(W,\psi) \in \mathcal{A}$ be another map, $\psi : W \to U_1$, and $q_1(y^1, y^2, \ldots, y^m) = \psi(p) \in U_1$. Then p acquires another coordinates $(y^1, y^2, \ldots, y^m)$. The transition function $\theta = \psi \circ \varphi^{-1}$ is coordinate replacement function in domain $S = V \bigcap W$ (figure 11) and it is diffeomorphism of the class C^∞.

We shell note, that θ is not defined on S itself, nevertheless we call it coordinate replacement function just in S. On the other hand, if there are given a map $(V,\varphi) \in \mathcal{A}$ and a coordinate replacement function θ, which is diffeomorphism of the class C^∞ and is defined in some part U' of the set U, then we get a new map (S,ψ), $\psi=\theta\circ\varphi$, $S=\varphi^{-1}(U')$ (figure 11).

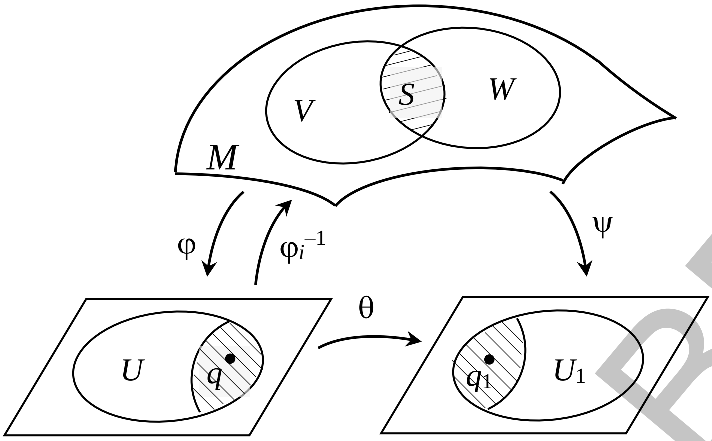

fig. 11

Example **7.** Let $M=R$, $V=R$, and let $\varphi:\boldsymbol{R}\to\boldsymbol{R}$ act by the formula $\varphi(t)=t$. Then (V,φ) is a map of the manifold and its atlas $\mathcal{A}$ may consist of one map. Because there are no transition functions, this manifold of class C^ω.

Consider one more map of the same manifold (W,ψ), $W=M$, $\psi(t)=t^3$. (figure 12). The transition function $\theta=\psi\circ\varphi^{-1}$ acts by formula $\theta(t)=t^3$. But θ is not diffeomorphism, because its Jacobian $|\mathrm{J}|$ equal 0 at the point $t=0$. It means that the new map doesn't consistent with the atlas $\mathcal{A}$. However, if we consider an atlas $\mathcal{A}'$ that consists of one new map, the manifold atlas $\mathcal{A}'$ is also of the class C^ω. But the manifold with atlas $\{(V,\varphi),(W,\psi)\}$ is only topological manifold (of the class C^0).

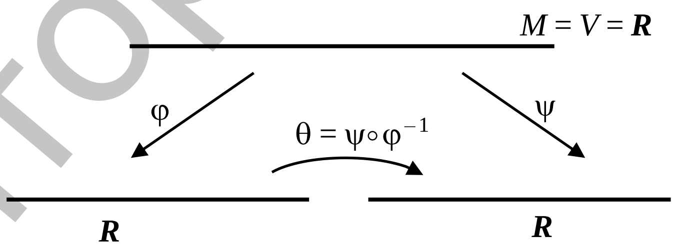

*fig.*12

Let's get back to Example 2 on a new level.

Example **8.** Consider the n-dimensional sphere

$$M=S^n=\{a\in\boldsymbol{R}^{n+1}|\rho(0,a)=1\}=\{a\in\boldsymbol{R}^{n+1}|(a^1)^2+(a^2)^2,\ldots,(a^{n+1})^2=1\}.$$

(here 0 is a point "zero vector"). Let $p(0,0,..,1)$ be the north pole and $q(0,0,..,-1)$ be the south pole. Consider two domains that cover the whole sphere: $V=S^n\backslash p$ and $W=S^n\backslash q$. We are going to do projections on the n-dimensional plane, which contains 0 and is perpendicular to the line pq.

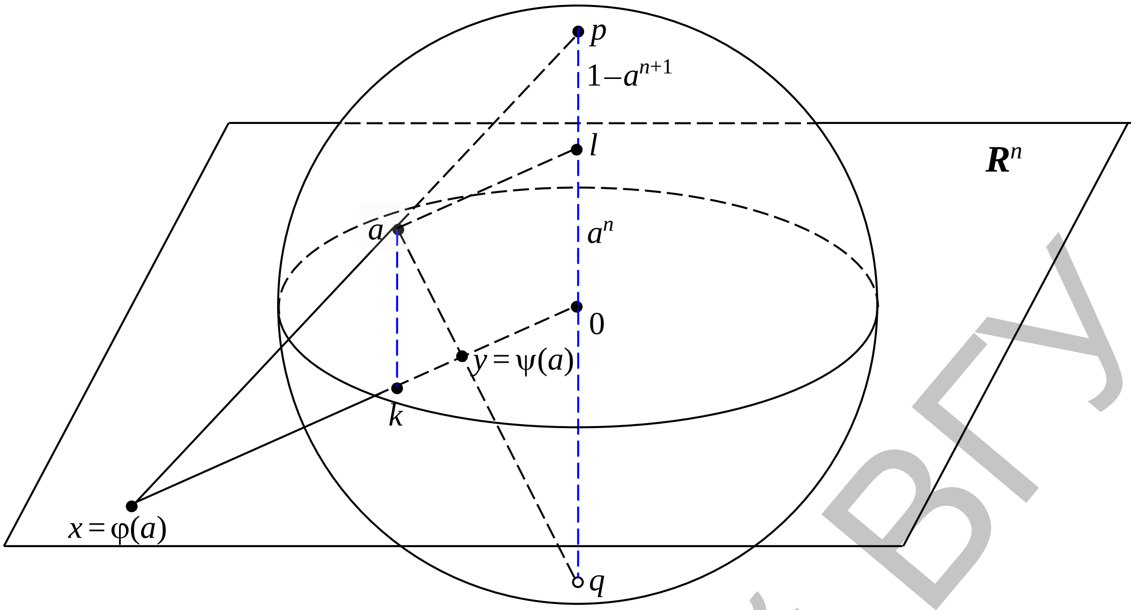

*fig.*13

Let φ be the projection from the north pole and ψ be the projection from the south pole (figure 13). If $a(y^1,y^2,\ldots,y^{n+1})$, then $\varphi(a)=x(x^1,x^2,\ldots,x^n)$, where

$$x^1=\frac{a^1}{1-a^{n+1}},\ldots,x^n=\frac{a^n}{1-a^{n+1}}. \tag{4}$$

Analogously, $\psi(a)=y(y^1,y^2,\ldots,y^n)$, where

$$y^1=\frac{a^1}{1+a^{n+1}},\ldots,y^n=\frac{a^n}{1+a^{n+1}}. \tag{5}$$

We are going to find formulas of transformation of the coordinates. From formulas (4) and (5) we get

$$y^1=x^1\cdot\frac{1-a^{n+1}}{1+a^{n+1}},\ldots,y^n=x^n\cdot\frac{1-a^{n+1}}{1+a^{n+1}}. \tag{6}$$

Further on, from (4) we get

$$(x^1)^2+(x^2)^2+\ldots+(x^n)^2=\frac{(a^1)^2+(a^2)^2+\ldots+(a^n)^2}{(1-a^{n+1})^2}=\frac{1-(a^n)^2}{(1-a^{n+1})^2}=\frac{1+a^{n+1}}{1-a^{n+1}}.$$

Thus

$$y^1=\frac{x^1}{(x^1)^2+(x^2)^2+\ldots+(x^n)^2},\ldots,y^n=\frac{x^n}{(x^1)^2+(x^2)^2+\ldots+(x^n)^2}.$$

These formulas are defined on $\boldsymbol{R}^n\backslash 0$, and the function $\theta:\boldsymbol{R}^n\backslash 0\to\boldsymbol{R}^n\backslash 0$, which is defined by these formulas, is of the class C^ω.

We should underline, that in the definition of a manifold we does not mention the ambient space. We consider a manifold "by itself". In order to study the geometric properties of a manifold, it is necessary to define the notion of a curve on a manifold, and the more general notion of a submanifold (manifold, that contains in the given manifold), as well as the notions of a tangent vector and a vector field on a manifold.

Remark. The definition of a manifold does not say that it must be connected. If a manifold is not connected, it can be represented as a union of some number (finite or infinite) of connected manifolds. Therefore from henceforth it is always assumed by default that the manifold is connected.

§4. *Mappings of manifolds*

***Definition* 18.** Let M and N be differential manifolds of the dimension m and n respectively. Let $f: M \to N$ be a mapping defined in some neighborhood of a point $p \in M$. Let (V, φ) and (W, ψ) be maps of the manifolds defined in a neighborhoods of points p and $f(p)$ respectively. Let $U_1 = \varphi(V)$, $U_2 = \psi(W)$ (figure 14). Let $(x^1, x^2, \dots, x^m)$ and $(y^1, y^2, \dots, y^n)$ be coordinates, that are defined by the maps on V and W.

***Definition* 19.** Mapping $\tilde{f} = \psi \circ f \circ \varphi^{-1} : \tilde{V} \to \tilde{W}$ is called coordinate representation (координатной записью) of mapping f in the maps (V, φ) and (W, ψ) or coordinate representation of mapping f in coordinates $(x^1, x^2, \dots, x^m)$ and $(y^1, y^2, \dots, y^n)$.

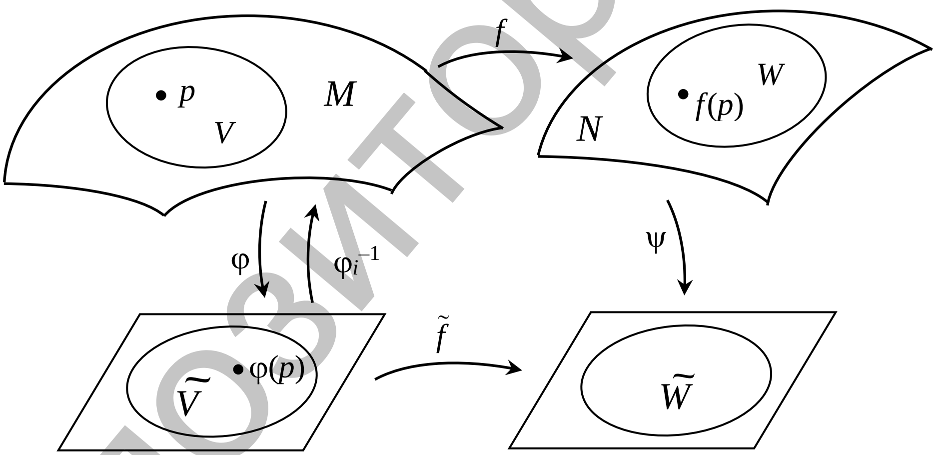

fig. 14

***Definition* 20.** Mapping f is called differentiable of the class C^k $(k=0,1,2,\dots,\infty)$ at the point $p \in V$ if its coordinate representation $\tilde{f}$ is differentiable of the class C^k at the point $\varphi(p) \in U_1$. We can't speak about analytical mappings because the manifold is supposed to be only of the class C^∞.

Suppose that maps (V, φ) and (W, ψ) belong to atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ of manifolds M and N. Points p and $f(p)$ may belong to several maps from these atlases. It is easy to prove, that in this case, the coordinate representation $\tilde{f}$ has the same class of differentiability in all these maps.

A function on a manifold is usually called a mapping $f:M\to\boldsymbol{R}$. We may consider $\boldsymbol{R}$ as manifold also with atlas, consisting of one identical map. Therefore the definition 18 is valid for such kind of mapping also. As a result we get the following definition.

***Definition* 21.** Let M be a manifold, $\mathcal{A}$ be its atlas, $p\in M$, and f be a function which is defined in some neighborhood of point p. Let $(V,\varphi)\in\mathcal{A}$ be a map, which defines coordinates $(x^1,x^2,\ldots,x^m)$ and $p\in V$. Then the function

$$\tilde{f}=f\circ\varphi^{-1}:\tilde{V}\to\boldsymbol{R}$$

(figure 15) is called *the coordinate representation of mapping f in the map* (V,φ) *or in coordinates* $(x^1,x^2,\ldots,x^m)$.

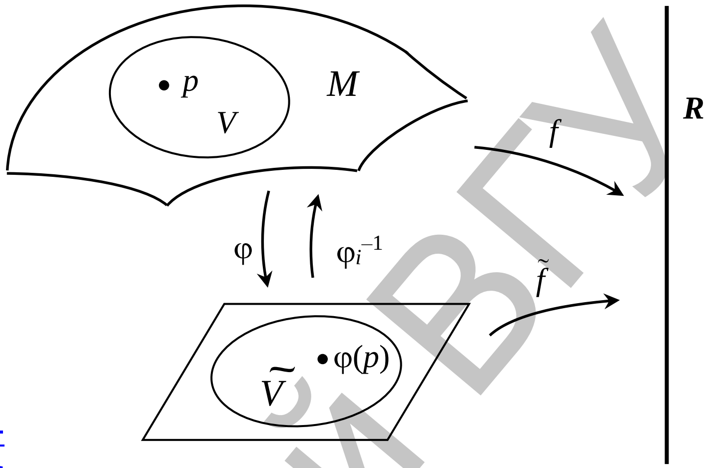

fig. 15

***Definition* 22.** A function f is called *differentiable of the class* C^k in some neighborhood of point p, if its coordinate representation is differentiable of the same class in some map from the atlas.

***Definition* 23.** Let M and N be differentiable manifolds. A mapping $f:M\to N$ is called the diffeomorphism if it is a homeomorphism and it is differentiable of the class C^∞. If there exists a diffeomorphism $f:M\to N$, then the manifolds M and N are called *diffeomorphic*.

If two manifolds are diffeomorphic, then they are equally arranged in terms of their differential structure. Using the diffeomorphism we can transfer the atlas from M to N and its maps will be consistent with the atlas of N. Any differential function on M generates differential function on N.

More exactly, if (V,φ) is a map on M, then $(f(V),\varphi\circ f^{-1})$ is a map on N (figure 16).

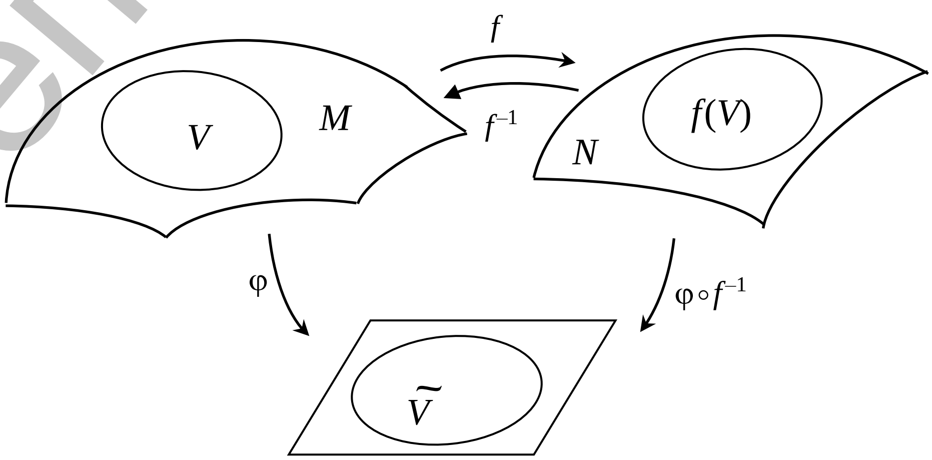

fig. 16

If $g: N \to P$ is a differentiable mapping of manifold N in some manifold P, then $g \circ f: M \to P$ is also differentiable mapping (figure 17). If $g: N \to \boldsymbol{R}$ is differentiable function, then $g \circ f: M \to \boldsymbol{R}$ is also differentiable function of the same class.

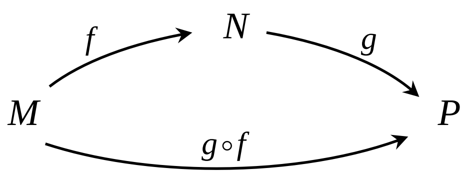

fig. 17

***Example* 9.** Consider $M = \boldsymbol{R}$ with usual structure and $N = (-1, 1)$. Then

$$f: N \to M, \quad f(t) = \frac{t}{1 - t^2}$$

is diffeomorphism.

***Exercise* 4.** Let $M = \boldsymbol{R}$ with usual structure and $N = \boldsymbol{R}$ with atlas, that consists of one map $(\boldsymbol{R}, \psi)$, $\psi(t) = t^3$. Consider mapping $f: N \to M$, that acts by formula $f(t) = t^3$. Find its coordinate representation (figure 18).

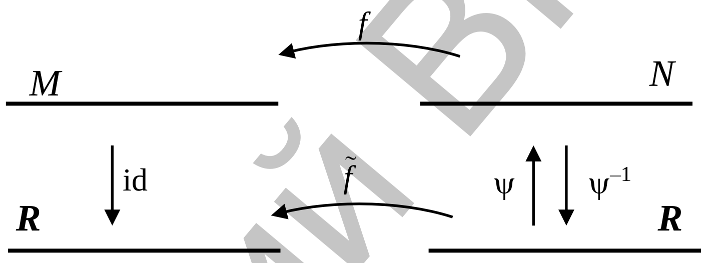

fig. 18

$$\tilde{f}(t) = (\mathrm{id} \circ f \circ \psi^{-1})(t)$$

and prove that f is diffeomorphism. So, although M and N have different differentiable structures, these manifolds are diffeomorphic.

***Exercise* 5.** Suppose now, that M and N are the same as above with the same atlases. Consider mapping $f: N \to M$, that acts by formula $f(t) = t$. Find its coordinate representation. Is this mapping it is differentiable on the entire manifold?

***Example* 10.** Consider the sphere S^2 with the atlas like in example 4. Consider a function $f: S^2 \to \boldsymbol{R}$, that acts by formula $f(a) = a^3$. We have formulas for mapping φ:

$$x^1 = \frac{a^1}{1 - a^3}\,;\; x^2 = \frac{a^2}{1 - a^3}\,.$$

From this formulas we can derive

$$a^3 = \frac{1 - (x^1)^2 - (x^2)^2}{1 + (x^1)^2 + (x^2)^2}\,.$$

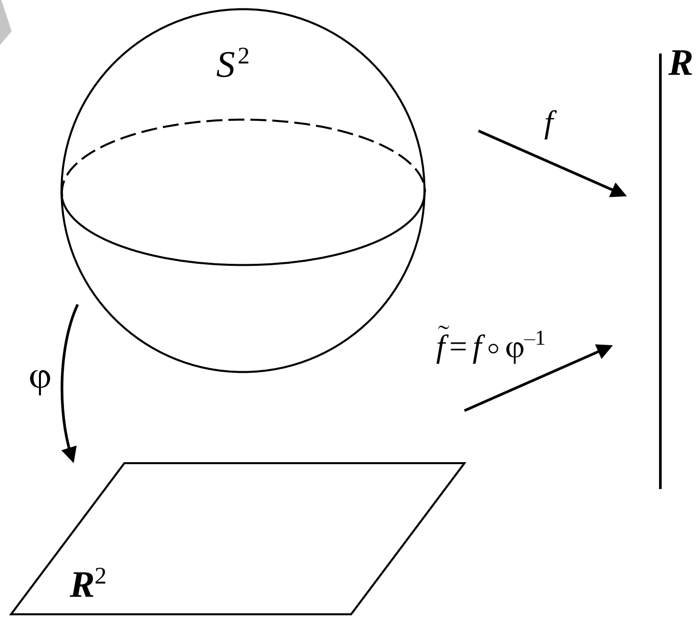

*fig.*19

Thus $\tilde{f}: \boldsymbol{R}^2 \to \boldsymbol{R}$ (figure 19) acts by the formula

$$\tilde f(x) = \frac{1-(x^1)^2-(x^2)^2}{1+(x^1)^2+(x^2)^2}.$$

We have infinitely differentiable function. Therefore f is also infinitely differentiable.

***Exercise* 6.** Find coordinate representation of the function $f(a) = a^3$. Is this function differentiable?

One more special case of a mapping of a manifold is a curve on a manifold.

***Definition* 24.** A curve on differentiable manifold M is such set $\gamma \subset M$, that for any map (V, φ) the set $\tilde\gamma = \varphi(\gamma \cap V)$ is a curve in the domain $\tilde V = \varphi(V) \subset \boldsymbol{R}^n$.

***Definition* 25.** A path on manifold M is continuous mapping $c: I \to M$, where I is an interval on numerical line $\boldsymbol{R}$.

According to the definition c is a path if and only if its coordinate representation

$$\tilde c = \varphi \circ c : I \to \boldsymbol{R}^n$$

is a path (figure 20); c is differentiable if and only if $\tilde c$ is differentiable.

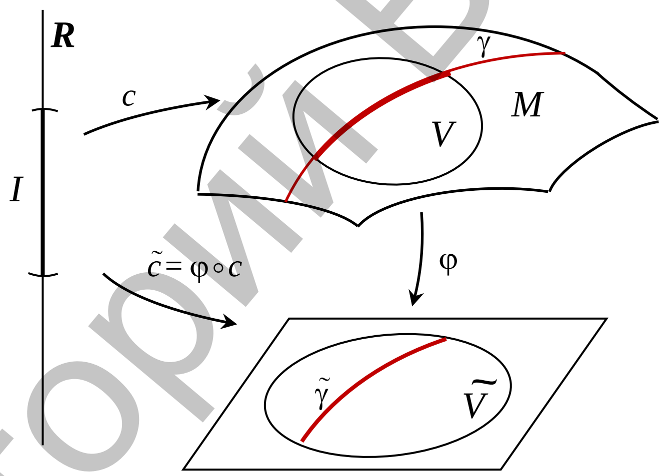

fig. 20

If $(x^1, x^2, \ldots, x^n)$ are coordinates defined by some map, than equations of a curve γ in such coordinates coincide with equations of $\tilde\gamma$ in $\boldsymbol{R}^n$.

Remind that a path $c: I \to \boldsymbol{R}^n$ is called regular, if $c'(t) \neq \vec 0$ for all $t \in I$. If the path is regular and differentiable of the class C^k $(k = 1, 2, \ldots, \infty)$, we say that it is smooth of the class C^k.

§5. *Tangent vectors*

***Definition* 26.** Let M be a differential manifold, $p \in M$ and (V, φ) be a map defined in the neighborhood V of the point p. Suppose that function $f: V \to \boldsymbol{R}$ is defined in the domain V and curve γ goes through point p. Let $c(t)$ be a parametrization of the curve and $p = c(t_0)$. Then the derivative of function f along the path $c(t)$ is the derivative of function $\tilde f$ along the path $\tilde c(t)$, i.e.

$$[f(c(t))]'|_{t=t_0} = [\tilde f(\tilde c(t))]'|_{t=t_0}.$$

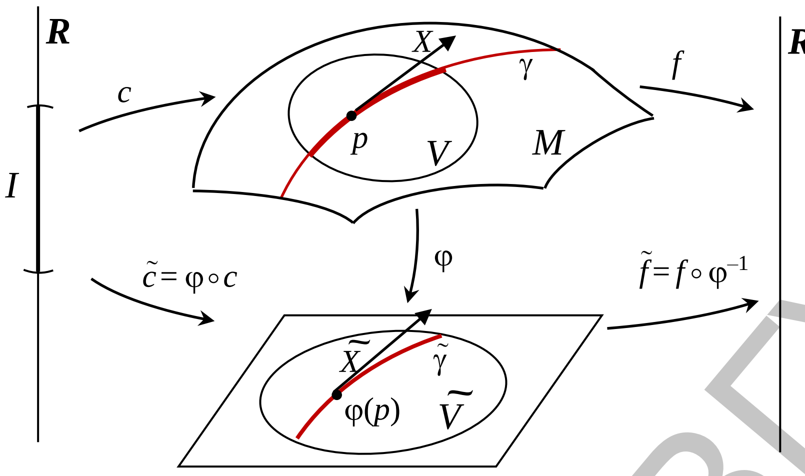

fig. 21

Let $(x^1, x^2, \ldots, x^n)$ be coordinates, that are defined by the map on V. We consider that

$$\frac{\partial f}{\partial x^i}(x^1, x^2, \ldots, x^n)_p = \frac{\partial \tilde{f}}{\partial x^i}(x^1, x^2, \ldots, x^n)_{\varphi(p)}.$$

(we denote coordinates on V and coordinates on $\widetilde{V}$ in the same way).

Let γ, $\tilde{\gamma}$, $c(t)$ and $\tilde{c}(t)$ be the same as in the previous paragraph. Let $\widetilde{X} = \tilde{c}'(t)$ be the tangent vector to path $\tilde{c}(t)$ at point $\varphi(p)$. We know, that the derivative of the function $\tilde{f}$ in the direction of vector $\widetilde{X}$ is equal

$$\widetilde{X}\tilde{f} = [\tilde{f}(\tilde{c}(t))]'|_{t=t_0}. \quad (7)$$

If M were a surface in the Euclidean space, then we could consider tangent vector X to path $c(t)$ at point p, and could define the derivative in the direction of vector X. But our manifold M is not embedded anywhere. Therefore we can define vector X abstractly through its property to differentiate functions.

Remind, that two functions are called equivalent at point p if they coincide in some neighborhood of point p.

***Definition* 27.** The set of all functions which are equivalent to function f are called *the germ (росток) of function* f *at point* p.

Define $\mathfrak{F}_p M$ the set of all germs of differentiable functions at point p. In other words, $\mathfrak{F}_p M$ consists of all functions, that are defined in some neighborhood of point p, taking into account, that we identify equivalent functions.

By the way, if we consider analytic functions only, then the germ of function f consists only of one function f.

***Definition* 28.** Tangent vector X to manifold M at point p is a mapping $X : \mathfrak{F}_p M \to \boldsymbol{R}$ that has the following properties:

1. $X(af + bg) = aXf + bXg$ (linearity);

2. $X(fg)=Xf\cdot g+f\cdot Xg$
$\forall a,b\in R,\ \forall f,g\in\mathfrak{F}_pM$.

In other words, a tangent vector is a mapping that behaves like differentiation. We can say, that tangent vector X acts on functions, but we must add the property, that vector acts in the same way on equivalent functions.

***Definition* 29.** We say that vector X *is tangent to the path* $c(t)$ *at point* $p=c(t_0)$ if $Xf=[f(c(t))]'|_{t=t_0}$ for any $f\in\mathfrak{F}_pM$. We write, that $X=c'(t)$.

***Definition* 30.** The set of all vectors tangent to all the paths, that goes through point $p\in M$ is called *the tangent space* to the manifold M at the point p. We denote it T_pM.

We can define linear operations on tangent vectors from one tangent space as follows. We write $Z=X+Y$ if $Zf=Xf+Yf\ \ \forall f\in\mathfrak{F}_pM$, and we write $Z=\alpha X$ if $Zf=\alpha(Xf)\ \ \forall f\in\mathfrak{F}_pM$.

According to definition 26 $[f(c(t))]'|_{t=t_0}=[\tilde f(\tilde c(t))]'|_{t=t_0}$ and according to formula (7) $\tilde f(\tilde c(t))'|_{t=t_0}=\tilde X\tilde f|_{\varphi(p)}$, where $\tilde X=\tilde c'(t)$. Finally we get

$$Xf=[f(c(t))]'|_{t=t_0}=[\tilde f(\tilde c(t))]'|_{t=t_0}=\tilde X\tilde f. \qquad (8)$$

for any $f\in\mathfrak{F}_pM$. It means, that there is correspondence between tangent vector X to path $c(t)$ and tangent vector $\tilde X$ to path $\tilde c(t)$ at point $\varphi(p)$. Taking into account properties 1 and 2 after definition 11, we can say that this correspondence is linear.

***Definition* 31.** Two paths $c(t)$ and $d(\tau)$ that goes through point $p\in M$ are called *eqivalent at point* p, if paths $\tilde c(t)=\varphi(c(t))$ and $\tilde d(\tau)=\varphi(d(\tau))$ are equivalent at point $\varphi(p)$.

Suppose that $c(t)$ and $d(\tau)$ are equivalent at point p, $p=c(t_0)=d(\tau_0)$, $X=c'(t_0)=d'(\tau_0)$). Then

$$f(c(t))'|_{t=t_0}=[\tilde f(\tilde c(t))]'|_{t=t_0}=\tilde X\tilde f=[\tilde f(\tilde d(\tau))]'|_{\tau=\tau_0}=f(d(\tau))'|_{\tau=\tau_0}.$$

We see that equivalent paths on the manifold define the same operator of differentiation, i.e. they define the same tangent vector.

Let's assume that point p has coordinates $(x_0^1,x_0^2,\ldots,x_0^m)$ relative to map (V,φ). It means, that the same coordinates has point $\varphi(p)$. Consider a straight line l_i in $\boldsymbol{R}^n$, which is defined by parametric equations

$$\{x^i=x_0^i;\ x^j=0,\ j\neq i.$$

It is called coordinate line x^i, which goes through point $\varphi(p)$. Then $\varphi^{-1}(l_i)$ is called coordinate line x^i, which goes through point p. We say, that the set of all coordinate lines in domain V forms the coordinate net in V (figure 22).

It is easy to prove, that the derivative of arbitrary function along the coordinate line x^i is equal to $\partial f/\partial x^i$. That is why the tangent vector to coordinate line x^i at point p we denote as $\partial/\partial x^i$ or as X_i. So,

$$X_i f = \left(\frac{\partial}{\partial x^i}\right) f = \frac{\partial f}{\partial x^i}. \qquad (9)$$

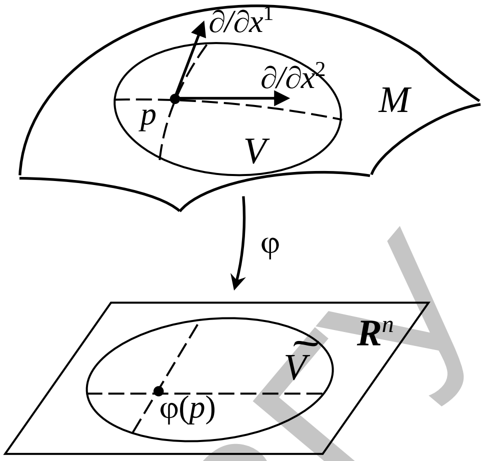

fig. 22

***Definition* 32.** We say that vectors $(\partial/\partial x^1, \partial/\partial x^1, \dots, \partial/\partial x^m)$ *form the coordinate basis or basis of coordinate vectors in the tangent space* T_pM.

Let's prove, that

$$(\partial/\partial x^1, \partial/\partial x^2, \dots, \partial/\partial x^m) \qquad (10)$$

is really a basis. This will prove, that the dimension of T_pM is equal to the dimension of the manifold itself. We must prove, that any vector $X \in T_pM$ can be expanded in system (10) and this system is linearly independent.

According to formula (8) $Xf = \tilde{X}\tilde{f}$. Suppose, that $(\alpha^1, \alpha^2, \dots, \alpha^m)$ are coordinates of vector $\tilde{X}$. It means, that

$$\tilde{X} = \alpha^1 e_1 + \alpha^2 e_2 + \dots + \alpha^m e_m,$$

where $(e_1, e_2, \dots, e_m)$ is a basis in $\boldsymbol{R}^m$. According to definition 11 and formula (9)

$$\tilde{X}\tilde{f} = \frac{\partial f}{\partial x^1}\alpha^1 + \frac{\partial f}{\partial x^2}\alpha^2 + \dots \frac{\partial f}{\partial x^m}\alpha^m = \alpha^1\frac{\partial}{\partial x^1}f + \alpha^2\frac{\partial}{\partial x^2}f + \dots \alpha^m\frac{\partial}{\partial x^m}f.$$

Thus

$$Xf = \tilde{X}\tilde{f} = \alpha^1\frac{\partial}{\partial x^1}f + \alpha^2\frac{\partial}{\partial x^2} + \dots + \alpha^m\frac{\partial}{\partial x^m} = \left(\alpha^1\frac{\partial}{\partial x^1} + \alpha^2\frac{\partial}{\partial x^2} + \dots \alpha^m\frac{\partial}{\partial x^m}\right)f.$$

It means that

$$X = \alpha^1\frac{\partial}{\partial x^1} + \alpha^2\frac{\partial}{\partial x^2} + \dots + \alpha^m\frac{\partial}{\partial x^m}.$$

Why system (10) is linearly independent? Simply because $\frac{\partial}{\partial x^i}x^j = \delta_i^j$ (Kronecker delta), $i,j = 1,2,\dots,m$.

***Exercise* 7.** Give the detailed proof that system (10) is linearly independent.

§6. *Fiber bundle*

Remind, that direct or Cartesian product of two sets X and Y consists of all ordered pairs of elements: $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$.

Suppose now, that (X,τ_1) and (Y,τ_2) are topological spaces. We can introduce topology τ in $X\times Y$ in the following way. Subbase of τ consists of sets $U\times V$ such that $U\in\tau_1$, $V\in\tau_2$. We will submit examples below.

***Definition* 33.** Topological space (X,τ) is called *a fiber bundle* (or *a fibration* (расслоение)) over base M with fiber F, if M and F are topological spaces and each point $p\in X$ has neighborhood $\widetilde{U}$, which is homeomorphic to $U\times F$, where U is a domain in M. Such neighborhood $\widetilde{U}$ is called *tubular*.

***Definition* 34.** A fiber bundle is called *trivial*, if X is homeomorphic to $M\times F$.

***Examples* 11.** The cylinder (infinite one) is trivial fibration $S^1\times\boldsymbol{R}$ (figure 23).

12. Mobius band is fibration over base S^1 with fiber I (or $\boldsymbol{R}$ if it is infinite).This fibration is not trivial (figure 24).

13. Torus is trivial fibration $S^1\times S^1$ (figure 25).

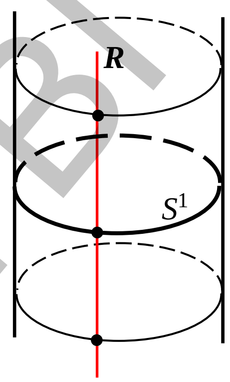

fig.23

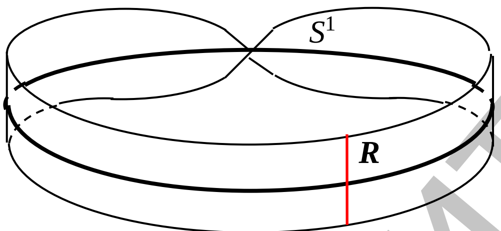

fig.24

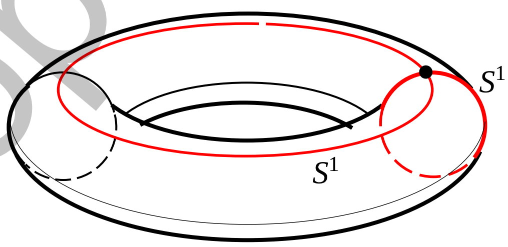

fig.25

***Definition* 35.** *The projection* of fiber bundle X on base M is called mapping $\pi: X\to M$, which acts by the following rule. For any neighborhood $\widetilde{U}=U\times F$ holds $\pi(\widetilde{U})=U$ and $\pi(p,q)=p$ (figure 26).

***Definition* 36.** *Bundle cut* of fiber bundle X is such mapping of base M into X $\sigma: M\to X$, that for any $p\in M$ holds $\sigma(\pi(p))=p$ (figure 27).

Bundle cut is kind of immersion or embedding of the base into the fibration ()figure 27. The exact definition of such notions we will study later.

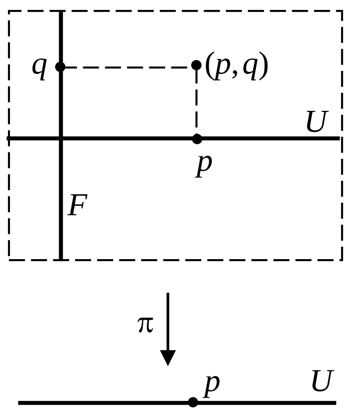

fig.26

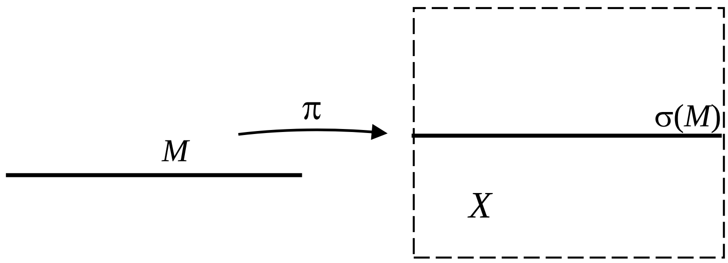

*fig.*27

***Examples* 14.** Consider trivial fibration $\boldsymbol{R}\times\boldsymbol{R}=\{(x,y)|\ x\in\boldsymbol{R},\ y\in\boldsymbol{R}\}$. Then the mapping $\pi(x,y)=x$ is the projection, $\sigma(x)=(x,x^2)$ is a bundle cut, because $\pi(\sigma(x))=\pi(x,y)=x$. Mapping $\sigma(x)=(2x,x^2)$ is not a bundle cut.

15. In figures 23, 24, 25 we can see the result of action of a bundle cut – the base embedded in fiber bungle.

***Exercise* 8.** Make the drawing for $\sigma(x)$ in Example 9.

§7. *Vector fields. Integral curves*

***Definition* 37.** A set $TM=\bigcup\limits_{p\in M} T_pM$ is called *a tangent bundle* (касательное расслоение) of the manifold M.

We can introduce a structure of differential manifold on T_pM. Let $\mathcal{A}$ be an atlas on M. Then atlas $\overline{\mathcal{A}}$ consists of such maps $(\overline{V},\overline{\varphi})$ that

1. $(V,\varphi)\in\mathcal{A}$, $\varphi(V)=\widetilde{V}$;

2. $\overline{V}=TV=\bigcup\limits_{p\in V} T_pM$ (tubular neighborhood, each element of it is a pair (p,Y), $Y\in T_pM$);

3. $\overline{\varphi}:\overline{V}\to\widetilde{V}\times\boldsymbol{R}^n\subset\boldsymbol{R}^{2n}$ according to the following rule. Let point p have coordinates $(x^1,x^2,\ldots,x^n)$, and let $Y=\alpha^1\dfrac{\partial}{\partial x^1}+\alpha^2\dfrac{\partial}{\partial x^2}+\ldots+\alpha^m\dfrac{\partial}{\partial x^m}$ be a tangent vector at point p. Then $\overline{\varphi}(p,Y)=(x^1,x^2,\ldots,x^m,\alpha^1,\alpha^2,\ldots,\alpha^m)$.

It is important to emphasize here that tangent planes to a surface in 3-dimensional space are located in the same space and can intersect. In the theory of manifolds M is not embedded anywhere. Each tangent space is considered separately and they do not intersect. We should note also, that tangent bundle of a manifold is not always trivial, i.e. it is not always diffeomorphic to $M\times\boldsymbol{R}^n$. With our definition of the atlas, the projection will be differentiable mapping because in each map it will be defined by the formula

$$\pi(x^1,x^2,\ldots,x^m,\alpha^1,\alpha^2,\ldots,\alpha^m)=(x^1,x^2,\ldots,x^m).$$

***Definition* 38.** *Vector field* X on manifold M is a cut $X:M\to TM$, of tangent bundle TM. Vector field X is called differentiable if this cut is differentiable mapping.

In other words, we can say that a vector field is a mapping that assigns to each point $p \in M$ a vector $X_p \in TM$, tangent to the manifold (figure 28).

At each point p, we can expand the vector X_p in the basis $((\partial/\partial x^1)_p, (\partial/\partial x^2)_p, \ldots, (\partial/\partial x^m)_p)$:

$$X_p = \alpha^1(\partial/\partial x^1)_p + \alpha^2(\partial/\partial x^2)_p + \ldots + \alpha^n(\partial/\partial x^m)_p.$$

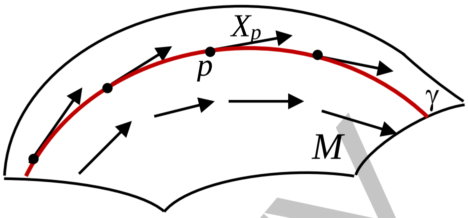

fig. 28

Therefore, in whole, on the manifold we can expand the vector field in basis vector fields:

$$X = \alpha^1 \frac{\partial}{\partial x^1} + \alpha^2 \frac{\partial}{\partial x^2} + \ldots + \alpha^m \frac{\partial}{\partial x^m} = \sum_{i=1}^{m} \alpha^i \frac{\partial}{\partial x^i}.$$

And here $(\alpha^1, \alpha^2, \ldots, \alpha^n)$ are functions.

***Proposition* 1.** A vector field X is differentiable of the class C^k if and only if all its coordinate functions are differentiable of the class C^k.

Proof. A vector field X is defined by the formulas

$$X(p) = X(x^1, x^2, \ldots, x^m) = (x^1, x^2, \ldots, x^m, \alpha^1, \alpha^2, \ldots, \alpha^m).$$

Therefore this mapping is differentiable of the class C^k if and only if $\alpha^1, \alpha^2, \ldots, \alpha^m$ differentiable of the same class.

A curve $\gamma \subset M$ is called the integral curve of a vector field X, if for any point $p \in \gamma$ the vector X_p is tangent vector to γ. Integral path of the vector field X is such path $c: I \to M$, that any t_o the vector $X_{c(t_o)} = c'(t_o)$.

***Theorem* 1.** Suppose that a vector field X is defined in some domain U on a manifold M and X is continuous. Then for any point $p \in U$ there is the unique path $c: I \to U$, which is the integral path of field X and $c(0) = p$.

Proof. Let (V, φ) be a map in some neighborhood of point p, which defines coordinates $(x^1, x^2, \ldots, x^m)$, $\widetilde{V} = \varphi(V)$. Let $X = \sum_{i=1}^{m} \alpha^i \frac{\partial}{\partial x^i}$. Consider vector field $\widetilde{X} = \sum_{i=1}^{m} \alpha^i e_i$ on $\widetilde{V}$. Let $\tilde{c}: I \to \widetilde{V}$ be an integral path of vector field $\widetilde{X}$ in domain $\widetilde{V}$. Then $\widetilde{X}_{c(t_o)} = c'(t_o)$ for any $t_o \in I$. The last equation can be rewritten in coordinates:

$$\begin{cases} a^1 = (c^1)', \\ \ldots\ldots \\ a^m = (c^m)'. \end{cases}$$

According to the theorem of existence and uniqueness of the solution of system of differential equations this system has a unique solution, if the initial data $\tilde{c}(0) = (x_o^1, x_o^2, \ldots, x_o^m)$ are given. They are coordinates of the initial point $\varphi(p) \in \widetilde{V}$.

Consider a path $c = \varphi^{-1} \circ \tilde{c} : I \to V$ on a manifold (figure 21). Then c is determined by the same equations in the internal coordinates and X is determined by the same coordinate functions as $\widetilde{X}$. Thus c is the integral path for X.

At last, if domain U does not fit in one coordinate neighborhood, we can prolong path c from one neighborhood to another and so on by the chain of neighborhoods. ■

Knowing what is the derivative of a function f in the direction of a vector X_p is, we can determine the derivative of the function f in the direction of a vector field X: $(Xf)_p = X_p f$ for any point p from the domain, where f and X are both defined.

Apart of this there is one more operation for vector fields: Lie bracket or commutator

$$[X, Y] = XY - YX.$$

I.e. we write $Z = [X, Y]$, if for any function f holds $Zf = X(Yf) - Y(Xf)$. Here Xf and Yf are again functions and we can consider their derivatives by another vector field.

Properties of this operation.

1. $[X + Y, Z] = [X, Z] + [Y, Z]$;

2. $[\alpha X, Y] = \alpha[X, Y]$;

3. $[Y, X] = -[X, Y]$;

4. $[X,[Y, Z]] + [Z,[X, Y]] + [Y,[Z, X]] = 0$ (Jacobi identity, here 0 – zero vector field).

Denote $\mathcal{B}M$ the set of all differentiable vector fields on a manifold M.

***Definition* 39.** A manifold M of the dimension m is called parallelizable if there exist m vector fields $X_1, X_2, \dots, X_m$ on M, which are linearly independent at each point, i.e. they form a basis in tangent space at each point of the manifold.

***Examples* 16.** The cylinder $S^1 \times \boldsymbol{R}^1$ is a parallelizable manifold. We can choose X_1 along parallels and X_2 along meridians (figure 29). Moebius band is not parallelizable.

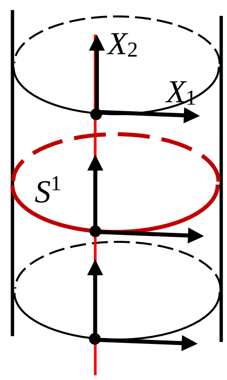

fig. 29

17. S^2 and S^{2k} $(k = 1, 2, \dots)$ are not parallelizable. It is impossible to introduce even one differentiable vector field on even-dimensional sphere, which has no singular points. S^{2k-1} $(k = 1, 2, \dots)$ are parallelizable.

Let $\partial/\partial x^i$ and $\partial/\partial x^j$ $i, j = 1, 2, \dots, m$ be coordinate vector fields on coordinate neighborhood V and f be arbitrary function of the class C^2. Then

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}.$$

Thus

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] f = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} = 0.$$

Since f is arbitrary function we get

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0 \text{ (zero vector field).}$$

The question naturally arises: whether this implies $[X, Y] = 0$ for any vector fields $X = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i}$ and $Y = \sum_{j=1}^{n} \beta^j \frac{\partial}{\partial x^j}$? Let's check.

$$[X, Y] f = \left[\sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i}, \sum_{j=1}^{n} \beta^j \frac{\partial}{\partial x^j}\right] f = \left(\sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i}\right)\left(\sum_{j=1}^{n} \beta^j \frac{\partial}{\partial x^j}\right) f - \left(\sum_{j=1}^{n} \beta^j \frac{\partial}{\partial x^j}\right)\left(\sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i}\right) f =$$

$$= \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i}\left(\sum_{j=1}^{n} \beta^j \frac{\partial f}{\partial x^j}\right) - \sum_{j=1}^{n} \beta^j \frac{\partial}{\partial x^j}\left(\sum_{i=1}^{n} \alpha^i \frac{\partial f}{\partial x^i}\right) =$$

$$= \sum_{i=1}^{n} \alpha^i \sum_{j=1}^{n} \frac{\partial \beta^j}{\partial x^i} \frac{\partial f}{\partial x^j} + \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha^i \beta^j \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{j=1}^{n} \beta^j \sum_{i=1}^{n} \frac{\partial \alpha^i}{\partial x^j} \frac{\partial f}{\partial x^i} - \sum_{j=1}^{n}\sum_{i=1}^{n} \beta^j \alpha^i \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} =$$

$$= \sum_{i,j=1}^{n} \left(\alpha^i \frac{\partial \beta^j}{\partial x^i} \frac{\partial}{\partial x^j} - \beta^j \frac{\partial \alpha^i}{\partial x^j} \frac{\partial}{\partial x^i}\right) f.$$

Thus

$$[X, Y] = \left(\alpha^i \frac{\partial \beta^j}{\partial x^i} \frac{\partial}{\partial x^j} - \beta^j \frac{\partial \alpha^i}{\partial x^j} \frac{\partial}{\partial x^i}\right).$$

***Definition* 40.** If $[X, Y] = 0$, then we say, that vector fields X and Y *commute*.

If both X and Y have constant coordinates, then $[X, Y] = 0$. Constant coordinates is sufficient, but not necessary condition for commutation of two vector fields.

§8. *Tangent mapping*

***Definition* 41.** Let M and N be differential manifolds of dimensions m and n. Let f be differentiable mapping of class C^1, which is defined in the neighborhood of point $p \in M$. Let $q = f(p)$. Let $X \in T_pM$ and let $c(t)$ be a path, that goes through point p, $p = c(0)$, such that $X = c'(t_0)$. Let $d(t) = f(c(t))$ be a path in manifold N and $Y = d'(t_0)$ (figure 30). Consider a mapping $(f_*)_p : T_pM \to T_qN$, which acts by the rule: $(f_*)_p X = Y$, i.e. if $X = c'(t_0)$, then $(f_*)_p X = d'(t_0)$. This mapping is called *tangent mapping to* f *at point* $p \in M$ or *differential of mapping* f *at point* p. If we remove connection to a point, we get mapping of tangent bundles: $f_* : TM \to TN$, which is called *tangent mapping to* f.

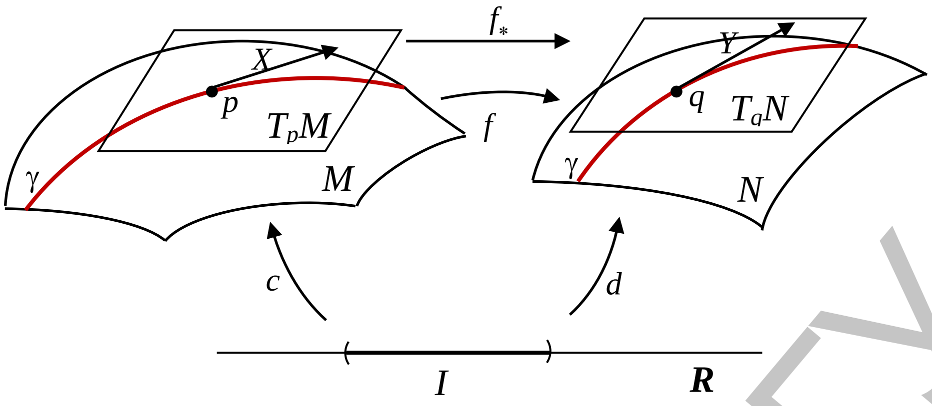

fig. 30

Suppose now, that coordinates $(x^1, x^2, \ldots, x^m)$ are introduced in some neighborhood of point p and $((\partial/\partial x^1)_p, (\partial/\partial x^2)_p, \ldots, (\partial/\partial x^m)_p)$ be the basis in T_pM, which determines coordinates $(\alpha^1, \alpha^2, \ldots, \alpha^m)$. Analogously, let $(y^1, y^2, \ldots, y^n)$ be coordinates in a neighborhood of point q and $(\beta^1, \beta^2, \ldots, \beta^n)$ be coordinates in T_qN determined by basis $((\partial/\partial y^1)_p, (\partial/\partial y^2)_p, \ldots, (\partial/\partial y^n)_p)$.

As we already know, mapping f is determined in coordinates x^i and y^j by the same formulas as its coordinate representation $\tilde{f} : \mathbf{R}^m \to \mathbf{R}^n$. Let J Jacobi matrix of mapping $\tilde{f}$. In terms of coordinates $(\alpha^1, \alpha^2, \ldots, \alpha^m)$ and $(\beta^1, \beta^2, \ldots, \beta^n)$ $\tilde{f}$ is linear mapping and determined by matrix J.

In details, if

$$\begin{cases} y^1 = f_1(x^1, x^2, \ldots, x^m), \\ \quad \cdots\cdots\cdots \\ y^n = f_1(x^1, x^2, \ldots, x^m), \end{cases}$$

are equations of the mapping f, then

$$\begin{cases} (f_*)_p \dfrac{\partial}{\partial x^1} = \dfrac{\partial f^1}{\partial x^1}\dfrac{\partial}{\partial y^1} + \dfrac{\partial f^2}{\partial x^1}\dfrac{\partial}{\partial y^2} + \ldots + \dfrac{\partial f^n}{\partial x^1}\dfrac{\partial}{\partial y^n}, \\ \quad \cdots\cdots\cdots\cdots\cdots \\ (f_*)_p \dfrac{\partial}{\partial x^m} = \dfrac{\partial f^1}{\partial x^m}\dfrac{\partial}{\partial y^1} + \dfrac{\partial f^2}{\partial x^m}\dfrac{\partial}{\partial y^2} + \ldots + \dfrac{\partial f^n}{\partial x^m}\dfrac{\partial}{\partial y^n}. \end{cases}$$

In brief,

$$(f_*)_p \left(\frac{\partial}{\partial x^i}\right)_p = \sum_{j=1}^{n} \frac{\partial f^j}{\partial x^i}\frac{\partial}{\partial y^j}, \; i = 1, 2, \ldots, m.$$

§9. *Submanifold*

***Definition* 42.** Let M and N be differential manifolds of dimensions m and n. Differentiable mapping $f : M \to N$ is called *immersion* (погружение) if for all $p \in M$ differential of this mapping $(f_*)_p : T_pM \to T_qN$ is injective mapping. Injectiveness of $(f_*)_p$ means that matrix of this mapping has rank equal m. Of course it is possible only if $m \le n$.

***Definition* 43.** An immersion $f: M \to N$ is called is called *an embedding* (вложение), if f homeomorphically maps M on its image $f(M) \subset N$.

***Example* 18.** Let F be elementary surface and $r: U \to F \subset \boldsymbol{R}^3$ is its parametrization. Then according to the definition r homeomorphically maps U on F. But it is not sufficient to call r embedding.

Let $p \in U$, $q = f(p) \in F$. As usual we denote coordinates on U as (u,v) and coordinates in $\boldsymbol{R}^3$ as (x, y, z). In coordinates we have equation of the surface $x = r_1(u,v)$, $y = r_2(u,v)$, $z = r_3(u,v)$.

Matrix of mapping $(r_*)_p : T_pU \to T_q\boldsymbol{R}^3$ (figure 31) is

$$J = \begin{pmatrix} (r_1)'_u & (r_2)'_u & (r_3)'_u \\ (r_1)'_v & (r_2)'_v & (r_3)'_v \end{pmatrix}.$$

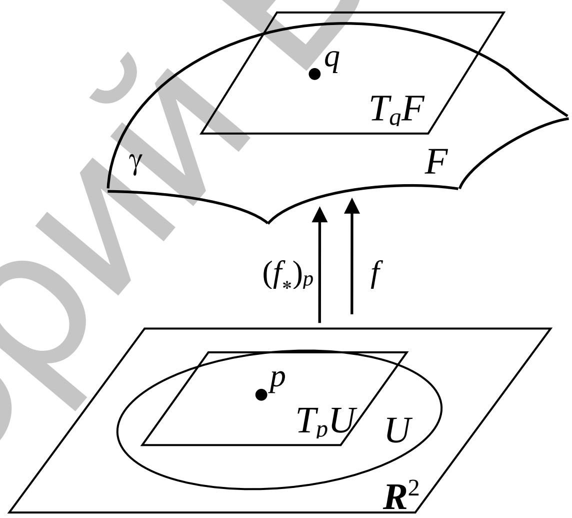

fig. 31

Condition $\operatorname{rank} J = 2$ is equivalent to $\vec{r}_u' \times \vec{r}_v' \ne \vec{0}$. Parameterized surface $r: U \to F \subset \boldsymbol{R}^3$ is called regular just if it holds $\vec{r}_u' \times \vec{r}_v' \ne \vec{0}$ for all $p(u,v) \in U$. It means that f is immersion if and only if it is regular parameterized surface. In this case it will be even embedding. I.e. r as if embeds domain U into $\boldsymbol{R}^3$ and bends it at the same time.

***Example* 19.** Let $M = \boldsymbol{R}$ and $M = \boldsymbol{R}^2$. Consider mapping $f(t) = (\cos t, \sin t)$ (figure 32). This mapping is immersion since its differential has matrix

$$\mathbf{J} = (-\sin t, \cos t).$$

$\operatorname{Rank} J = 1$ for any $t \in \boldsymbol{R}$. This mapping is not embedding, because $f(M) = S^1$ is not homeomorphic to M.

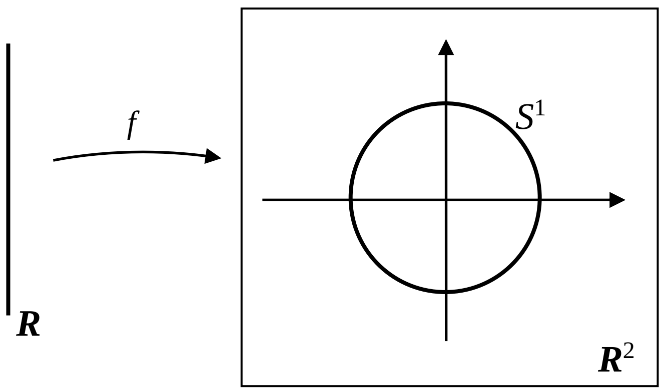

fig.32

For the same reason mapping of the plane in $\boldsymbol{R}^3$

$$f(u,v) = (\cos u, \sin u, v) \quad (9)$$

(figure 33) is immersion, but not embedding.

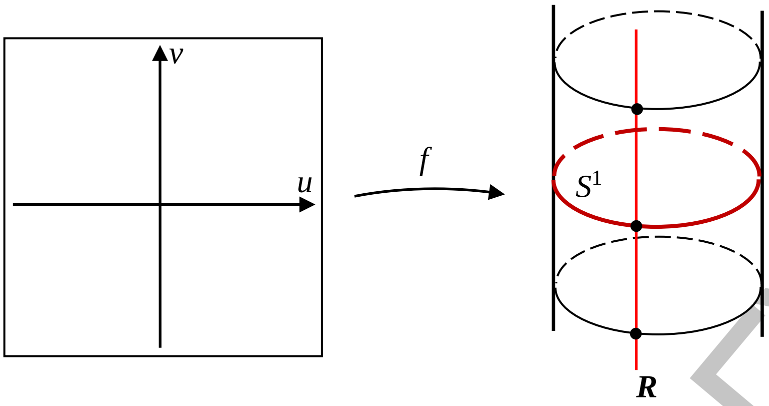

fig. 33

Exersises **6.** Find Jacobi matrix of mapping (9) and prove, that its rank is equal 2 for all $(u,v)\in \boldsymbol{R}^2$.

7. Prove, that the mapping $c:\boldsymbol{R}\to\gamma\subset\boldsymbol{R}^3$ acting by the formula $f(t)=(a\cdot\cos u,\ a\cdot\sin u, bv)$, $ab\neq 0$, is embedding. Remember the name of the curve that is obtained as a result of this embedding.

Example **20.** Mapping $f:\boldsymbol{R}\to\boldsymbol{R}^2$ $f(t)=(t^2,t^3)$ (figure 34) has image "semiqubical parabola". It is homeomorphism, however its Jacobi matrix $\mathbf{J}=(2t,3t^2)$ has rank equal zero at point $t=0$. Thus this mapping is not immersion.

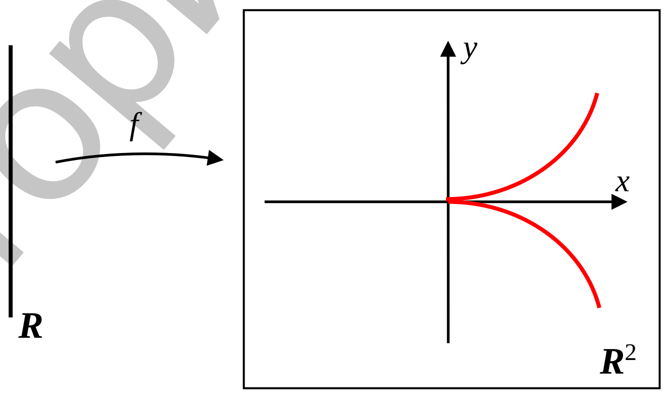

*fig.*34

For the same reason mapping of the plane in $\boldsymbol{R}^3$

$$f(u,v)=(2u,3u^2,v)$$

(figure 35) is not immersion.

Exersise **8.** Find its Jacobi matrix and prove, that its rank is equal 2 for all $(u,v)\in\boldsymbol{R}^2$.

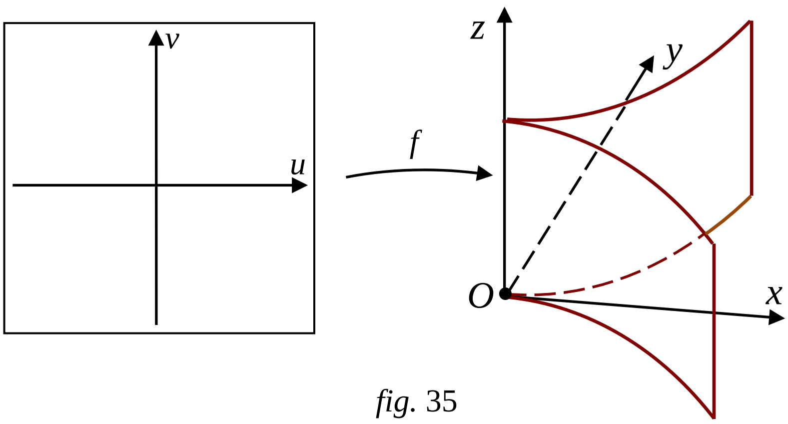

fig. 35

If manifold M is contained in manifold N we can consider M in two ways: as itself and as a part of N (figure 36).

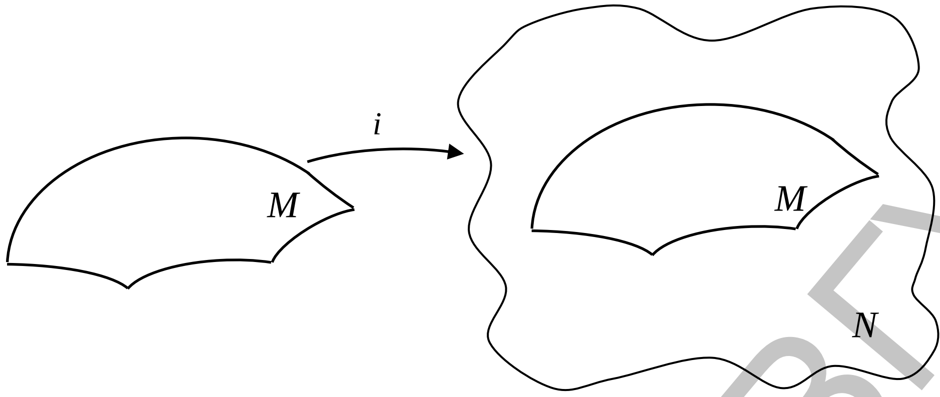

fig. 36

Therefore the following definition makes sense.

***Definition* 44.** Let M and N be differential manifolds. If M is part of N and mapping of inclusion $i:M\to N$ is embedding, then is called differential submanifold in N.

This definition allows us to discard such cases as semiqubical parabola and the cylinder on it. This manifolds are not differential submanifolds in $\boldsymbol{R}^2$ and in $\boldsymbol{R}^3$ respectively. But if we consider semicubial parabola "by itself", it is arranged as the line $\boldsymbol{R}$, and the cylinder on semicubial parabola "by itself" is arranged as the plane $\boldsymbol{R}^2$. For a submanifold to be differentiable, it is not enough that it is differentiable "by itself". We need to embed it smoothly in the enveloping (объемлющее) manifold.

§10. *Tensors*

***Definition* 45.** *Tensor of type (r,s)* $(r,s=0,1,2\dots)$ in space $\boldsymbol{R}^m$ is polylinear mapping

$$T:\underbrace{\boldsymbol{R}^m\times\boldsymbol{R}^m\times\dots\times\boldsymbol{R}^m}_{r\text{ times}}\to\underbrace{\boldsymbol{R}^m\times\boldsymbol{R}^m\times\dots\times\boldsymbol{R}}_{s\text{ times}}$$

(i.e. T is function of r arguments, each of them is a m-vector, and function value is ordered set of s m-vectors) which is linear in each argument:

$$T(X_1,\dots,X_i+Y_i,\dots,X_r)=T(X_1,\dots,X_i,\dots,X_r)+T(X_1,\dots,Y_i,\dots,X_r) \qquad (9)$$

$$T(X_1,\dots,\alpha X_i,\dots,X_r)=\alpha T(X_1,\dots,X_i,\dots,X_r) \qquad (10)$$

In tensor algebra, it is customary to dispense with the summation sign by using indices of two levels: upper and lower ones. For instance, the following notation a^ie_i means $\sum_{i=1}^m a^ie_i$, and notation $g_{ij}a^ib^j$ means $\sum_{i,j=1}^m g_{ij}a^ib^j$. The value, up to which the summation is carried out, is assumed to be known.

In the future we will come across tensors with $r \le 4,\ s \le 1$ only. Therefore all the definitions below will be given for tensors of these types only. The set $\boldsymbol{R}^m \times \boldsymbol{R}^m \times \ldots \times \boldsymbol{R}^m$ zero times is a scalar (a number).

***Definition* 46.** Suppose, that a basis $(e_1, e_2, \ldots, e_n)$ is given in $\boldsymbol{R}^m$. Let T be a tensor of type $(3,1)$. Then we can write $T(e_i, e_j, e_k) = T^l_{ijk} e_l$. The values T^l_{ijk} are called *components of the tensor* T *in the basis* $(e_1, e_2, \ldots, e_n)$. If R is a tensor of type $(4,0)$, then $R(e_i, e_j, e_k, e_l) = R_{ijkl}$.

Components of the tensor gives us possibility to calculate tensor value on any vectors. For instance,

$$T(X, Y, Z) = T(x^i e_i, y^j e_j, z^k e_k) = x^i y^j z^k T(e_i, e_j, e_k) = x^i y^j z^k T^l_{ijk} e_l.$$

Analogeously,

$$R(X, Y, Z, U) = x^i y^j z^k u^l R(e_i, e_j, e_k, e_l) = x^i y^j z^k u^l R_{ijkl}.$$

Let $\mathbf{A} = (a_i^{i'})$ be the matrix of a linear transformation $A: \boldsymbol{R}^m \to \boldsymbol{R}^m$, i.e.

$$Ae_i = a_i^j e_j.$$

and let $\mathbf{B} = (b_l^{l'}) = \mathbf{A}^{-1}$. Consider be the new basis, which is made of vectors $Ae_1,\ Ae_2, \ldots, Ae_n$. Then the transformation rules of tensor components ar

$$T^{l'}_{i'j'k'} = a_{i'}^i a_{j'}^j a_{k'}^k T^l_{ijk} b_l^{l'} \quad a_{i'}^i a_{j'}^j a_{k'}^k$$

$$R_{i'j'k'l'} = a_{i'}^i a_{j'}^j a_{k'}^k a_{l'}^l R_{ijkl}.$$

***Definition* 47.** We say, that T is a tensor field on a manifold M if at each point $p \in M$ T_p is a tensor determined in T_pM and (9) and (10) are true for any vector fields and any function α on M.

§11. *Connectivity on manifold or covariant derivation*

***Definition* 48.** Let X be a vector field on manifold M and $p \in M$. Let $c(t)$ be integral path of the vector field X, which goes through p and $p = c(0)$. Let $q = c(t_o)$. Consider transformation $\Phi_{t_o}: M \to M$, which maps each point p to the point q as described above. We say, that Φ_{t_o} is *the local flew of the vector field* X.

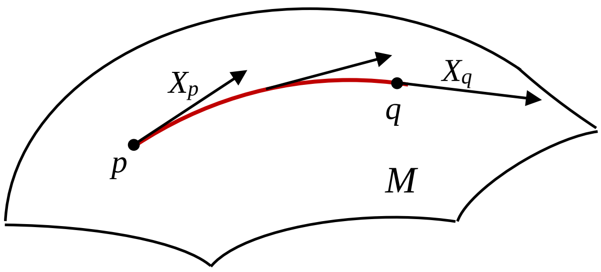

fig.37

This transformation has the following property: $(\Phi_{t_o *})_p X_p = X_q$, i.e. this transformation maps vector field X in itself (figure 37).

In Euclidean space we can easily compare two vectors, which have different initial points. We can translate one of them at initial point of the other. We can't do in the same way on a manifold. If vectors are tangent to the manifold at different points, then they are located in different spaces. However it gives us an opportunity to perform something like translation and map one tangent space on the other.

Let $p \in M$, $r = \Phi_{-t_0}(p) = -c(t_0)$. Then $\Phi_{t_0}(r) = p$. Let $Y_r \in T_r M$ and $Y_p = (\Phi_{t_0 *})_r Y_r$. Then we say that Y_p is obtained from Y_r by parallel translation along vector field X or along the path $c(t)$.

This operation gives us opportunity to define the derivative of vector field Y along vector field X:

$$L_X Y|_p = \lim_{t \to 0} \frac{Y_p - (\Phi_{t*})_r Y_r}{t},$$

where $r = \Phi_{-t}(p)$. This derivative is called Lie derivative of the vector field Y along the vector field X. We accept without proof, that

$$L_X Y = [X, Y].$$

But translation defined in this way is not very well. It turns out that the result of the translation and Lie derivative depends not only on the value of vector field X along path $c(t)$, but on its value in the nearest neighbourhood of the path. We can define the translation more correctly, but this method is less convenient, then than formal description.

***Definition* 49.** Let M be differential manifold. Linear connectivity or covariant derivative on M is mapping $\nabla : \mathfrak{B}M \times \mathfrak{B}M \to \mathfrak{B}M$, written as $Z = \nabla_X Y$ (covariant derivative of vector field Y along vector field X), which has the following properties:

1. $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$;
2. $\nabla_X (fY) = Xf + f \cdot \nabla_X Y$;
3. $\nabla_{X_1 + X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$;
4. $\nabla_{fX} Y = f \cdot \nabla_X Y$.

We shell notice, that $L_X Y$ has all the properties, except 4, therefore it does not fit this definition.

Let $X_i = \partial / \partial x^i$ be coordinate vector fields in some map. Let

$$\nabla_{X_i} X_j = \Gamma_{ij}^k X_k .$$

Then functions Γ_{ij}^k are called *the components of the connectivity in the map*.

***Definition* 50.** Tensor

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

is called *the torsion tensor*. We say, that *a connectivity has no torsion*, if $T(X, Y) \equiv 0$.

Equality $T(X, Y) \equiv 0$ means that $[X, Y] \equiv \nabla_X Y - \nabla_Y X$.

Suppose, that a connectivity has no torsion. Then for any coordinate vector fields holds

$$0 = T(X_i,X_j) = \nabla_{X_i}X_j - \nabla_{X_j}X_i - [X_i,X_j] = \Gamma^k_{ij}X_k - \Gamma^k_{ji}X_k \iff$$
$$\iff \Gamma^k_{ij}X_k = \Gamma^k_{ji}X_k \quad \forall\, i,j = 1,2,\ldots,m.$$

Since coordinate vector fields are linearly independent at each point, then

$$\Gamma^k_{ij} = \Gamma^k_{ji} \quad \forall\, i,j,k = 1,2,\ldots,m. \tag{11}$$

And vice versa condition (11) implies $T(X_i,X_j) \equiv 0 \Rightarrow$ all the components of this tensor are equal zero $\Rightarrow T(X,Y) \equiv 0$.

Let $T(X_i,X_j) = T^k_{ij}X_k$. Then

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}. \tag{12}$$

***Definition* 51.** Let M be a differential manifold, $c: I \to M$ be a differentiable path. Let $X = c'(t)$ be a vector field along c. Then vector field Y is called parallel along c, if

$$\nabla_X Y \equiv 0. \tag{13}$$

***Theorem* 2.** *Let $p = c(t_o)$ and $Y_o \in T_pM$. Then there exists one and only one vector field Y along c, which is parallel along c and $Y_o = Y_p$.*

In the proof of this theorem, equation 2 is reduced to differential equation and theorems on the existence and uniqueness of the solution are used.

This theorem means, that the parallel translation of a vector along the path is uniquely defined by the connectivity. Specifically, if $Y_o \in T_pM$, $p = c(t_o)$ and we want to translate Y_o to the point $q = c(t_1)$, then we construct vector field Y, which is parallel along $c(t)$ with initial condition $Y_p = Y_o$ and then Y_q is the desired result (figure 38).

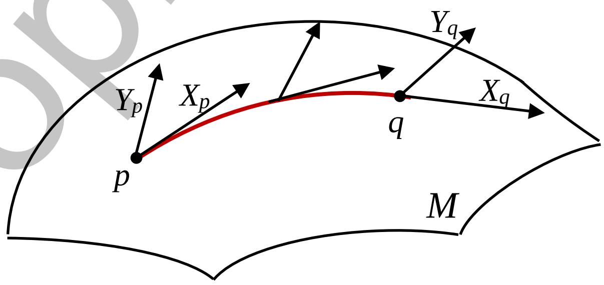

fig.38

Let Y be arbitrary vector field along c, $Y_o = Y_p$, $p = c(t_o)$. Denote $Y_o^{(t)}$ vector obtained by parallel translation of vector $Y_{c(t)}$ to point p. Then the following theorem is true.

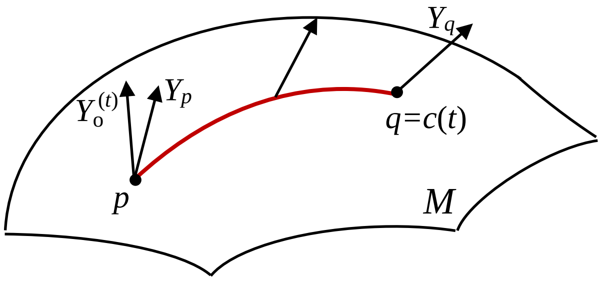

fig. 39

***Theorem* 3.** $\nabla_X Y|_p \equiv \lim_{t\to 0} \dfrac{Y_o^{(t)} - Y_o}{t - t_o}$.

So, the covariant derivative is defined by a completely natural formula, similar to the definition of the ordinary derivative of a function (figure 39).

Let $c(t)$ be a piecewise smooth path. Then parallel translation is defined as sequential translation along each of the smooth pieces (figure 40).

***Definition* 52.** Let B be a tensor of the type (r,s) on a differential manifold M, $s=0,1$. Then covariant derivative of the tensor B along a vector field X is defined as a tensor of the type (r,s) by the following formula:

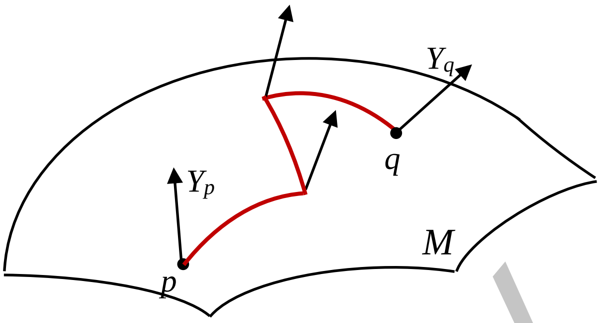

fig. 40

$$(\nabla_X B)(Y_1,\dots,Y_n)=\nabla_X(B(Y_1,\dots,Y_n))+\sum_{i=1}^{r}B(Y_1,\dots,\nabla_X Y_i,\dots,Y_n) \qquad (14)$$

If $s=1$, then $B(Y_1,\dots,Y_n)$ is a vector field, but if $s=1$, then it will be a function. Therefore we must make a reservation (сделать оговорку) that in formula (14) $\nabla_X f$ means Xf.

***Definition* 53.** The covariant derivative of tensor B of type (r,s) is called a tensor ∇B of the type $(r+1,s)$, which is defined by formula

$$\nabla B(X,Y_1,\dots,Y_n)=(\nabla_X B)(Y_1,\dots,Y_n).$$

***Definition* 54.** A tensor field B is said to be parallel on M if $\nabla B\equiv 0$.

If B has the type $(0,0)$, then B is a function f and by ∇B we mean df – differential of the function. According to this definition df is a tensor of the type $(1,0)$ and

$$df(X)=\nabla_X f=Xf.$$

I.e. the differential of a function assigns to each vector X the derivative of the function in the direction of the vector X. Condition $df=0$ means that for any vector field X holds $Xf=0$ and it means $f=\text{const}$.

If B has the type $(0,1)$, then it is a vector field Y and the condition $\nabla Y=0$ means that for any vector field X holds $\nabla_X Y=0$, i.e. Y is parallel along each vector field. In general, there are no such fields on an arbitrary manifold, but on the flat manifold $\boldsymbol{R}^m$ such vector fields are constant fields.

§12. *Geodesic lines. Exponential mapping*

***Definition* 55.** A differentiable path c on a manifold M is called *geodesic*, if

$$\nabla_{c'}c'=0. \qquad (15)$$

i.e. if its tangent vector field $X=c'$ is parallel along c. A curve γ in the manifold M is called *geodesic*, if it is the image of geodesic path. A regular path c is called *pregeodesic*, if its image is a geodesic curve.

Remind, that two paths $c:I\to M$ and $d:I_1\to M$ are called equivalent, if there is a regular change of the parameter $\varphi:I_1\to I$, such that $d=c\circ\varphi$ $(t=\varphi(\tau)$,

$d(\tau)=c(\varphi(\tau))$. Equivalent paths have the same trajectory, but have different parameters.

Let (V,φ) be a map on M, $X_i=\partial/\partial x^i$ be the coordinate vector fields and Γ^k_{ij} be components of the connectivity on M. Denote $\Gamma^k_{ij}(t)=\Gamma^k_{ij}|_{c(t)}$. Let

$$x_1=c^1(t)\,,\,\dots\,,x_m=c^m(t)$$

be coordinate equations of the path c. Then

$$c'(t)=(c^1(t))'\frac{\partial}{\partial x^1}+\dots+(c^m(t))'\frac{\partial}{\partial x^m}.$$

Further on we will denote differentiation by dot. In classical differential geometry the dot usually means differentiation by natural parameter. We have

$$\nabla_{\dot c}\,\dot c=\nabla_{\dot c^i X_i}\dot c^j X_j=\dot c^i\,\nabla_{X_i}\dot c^j X_j=\dot c^i\,(\dot c^j\,\nabla_{X_i}X_j+(X_i\dot c^j)X_j)=\dot c^i\,\dot c^j\,\Gamma^k_{ij}X_k+((\dot c^i X_i)\dot c^j)X_j$$

Let's replace in the second term blind index j on k and equate to zero:

$$(\dot c^i\,\dot c^j\,\Gamma^k_{ij}+\dot c\dot c^k)X_k=0.$$

Since vectors are linearly independent, it holds for any k: $\dot c^i\,\dot c^j\,\Gamma^k_{ij}+(\dot c\dot c^k)=0$.

According to the definition, $\dot c f=\frac{d}{dt}f(c(t))\ \Rightarrow\ \dot c\dot c^k=\ddot c^k$. Thus, finally we get the equation

$$\ddot c^k+\dot c^i\,\dot c^j\,\Gamma^k_{ij}=0,\ k=1,2\,,\dots,m. \qquad (16)$$

These are *equations of geodesic* path in coordinates. This is a system of differential equations of the second order. According to the theory it has unique solution, if the initial data are given: $c^k(0)=x^k_o$, $\dot c^k(0)=v^k$. It means, that there is one and only one geodesic path, which satisfies initial data $c(0)=p$, $\dot c(0)=V\in T_pM$. Therefore the following theorem is true.

Theorem 4. *For any point $p\in M$ and for any vector $V\in T_pM$ there is one and only one geodesic path $c:I\to M$ outgoing from point p in the direction of vector V.*

The previous reasoning proves the uniqueness of the geodesic line within one coordinate neighborhood U. If we want to prolong this path further on in coordinate neighborhood U_1, we shell take a point $q=c(t_1)\in U\cap U_1$ and the vector $Y=\dot c(t_1)$; they uniquely determine the continuation of the path in the next coordinate neighborhood and so on (figure 41).

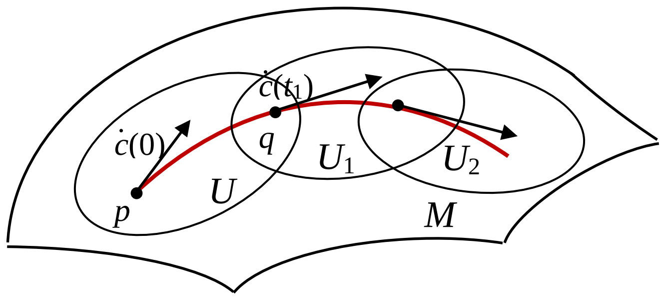

fig. 41

But this reasoning doesn't mean, that finally we will get the path which domain of definition is the whole numerical line. For instance, semiplane is also a manifold. Geodesic lines on it are straight lines. But there are lines, which we can't be prolonged in such way, that their domain of definition will be R (figure 42).

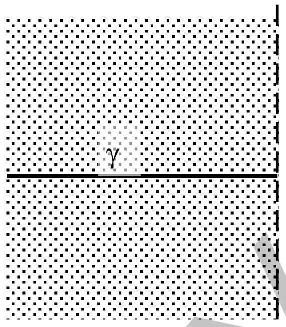

fig. 42

***Definition* 56.** Linear connectivity on a manifold M is called *full*, if each geodesic path can be prolonged to the geodesic path, determined on the whole $\boldsymbol{R}$. A manifold M with full connectivity is called *a full manifold*.

Let M be a differential manifold, $p \in M$, $X \in T_pM$. Let $c: I \to M$ be a geodesic path such that $c(0) = p$, and let $q = c(1)$. Consider the mapping $\exp_p : T_pM \to M$ which assigns to each vector $X \in T_pM$ a point $q \in M$, according to the rule described above.

***Definition* 57.** Such mapping is called *exponential mapping of the manifold* M at the point p.

If the manifold is full, then $\exp_p$ is defined on the whole T_pM. If the manifold is not full, then it is possible, that $\exp_p(X)$ is not defined for some vector X. This mapping may be not injective (i.e. it is possible, that $\exp_p(X) = \exp_p(Y)$ while $X \neq Y$) and may be not surjective (i.e. it is possible, that image of this mapping is not whole M). We will see some examples later. But if we consider neighborhood V of zero vector in T_pM, which is small enough, then the reduction of $\exp_p$ on this neighborhood will be a diffeomorphism. In particular it means, that there is a neighborhood W of a point p, such that any point $q \in W$ can be connected with p by the unique geodesic line. And here is an explanation why.

For the system of differential equations (16), we can set the boundary value problem: $c(0) = p$, $c(1) = q$, i.e. $c^k(0) = x_0^k$, $c^k(1) = x_1^k$, $k = 1, 2, \dots, m$. According to the theory of differential equations this problem also has unique solution, if p and q are in sufficiently small neighborhood. However, on the manifold as a whole, geodesic line connecting two points may not exist or may not be unique.

***Example* 21.** On the sphere S^2 geodesic lines are big circles. Two arbitrary points (figure 43) can be connected by two geodesic lines, and diametrically opposite points can be connected by infinite number of geodesic lines.

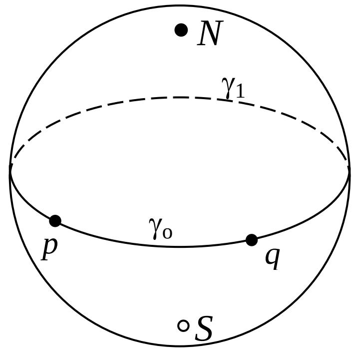

fig. 43

***Example* 22.** On the cylinder $S^1 \times \boldsymbol{R}$ geodesic lines are those lines, that are depicted on the surface development as straight lines. These are circles, ruling lines and helix lines. Two arbitrary point on one circle can be connected by two geodesic lines and two arbitrary points, which are not located on one circle can be connected by infinite number of geodesic lines. We have drawn three of them (figure 44).

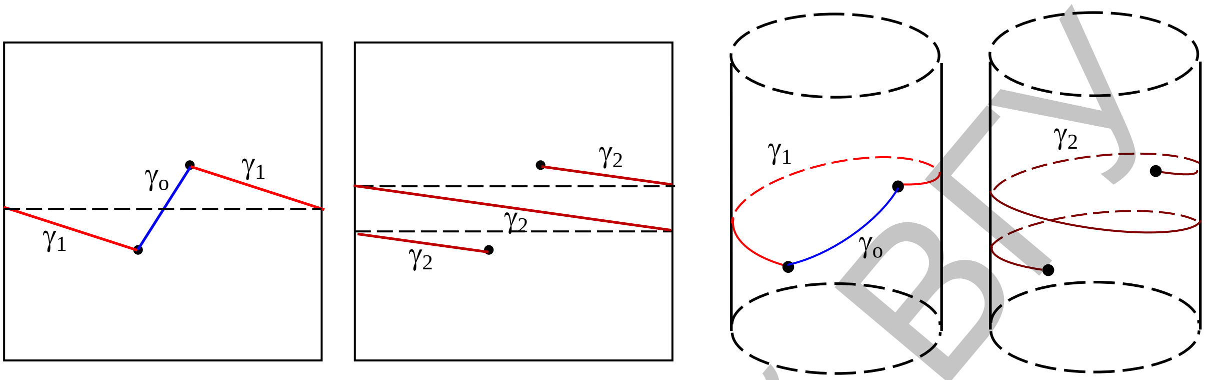

fig. 44

Chapter 2. Riemannian manifold

§1. *Definition of Riemannian manifold*

Let M be a differential manifold. Let's determine in each tangent space $T_pM,\ p\in M$ a scalar product of vectors $X{\cdot}Y$. Let $X=\alpha^i\frac{\partial}{\partial x^i}$, $Y=\beta^j\frac{\partial}{\partial x^j}$. Then

$$X{\cdot}Y=\alpha^i\beta^j\frac{\partial}{\partial x^i}\cdot\frac{\partial}{\partial x^j}=g_{ij}\alpha^i\beta^j.$$

Here we denoted

$$g_{ij}=\frac{\partial}{\partial x^i}\cdot\frac{\partial}{\partial x^j},\ i,j=1,2,\dots,m.$$

This scalar product is usually denoted in the following way:

$$g_p(X,Y)\quad\text{or}\quad\langle X,Y\rangle_p.$$

The function $g_p: T_pM\times T_pM\to R$ is a tensor of the type $(2,0)$, defined in vector space T_pM and g_{ij} are its components.

***Definition* 58.** This tensor is called *the metric tensor of the manifold* M *at the point* p. If we avoid connection to the point, we will get so-called *the metric tensor field* g on M and its components g_{ij} will be a functions. Nevertheless it is common practice to call this tensor field *the metric tensor* on M.

***Definition* 59.** A manifold M with a metric tensor g is called *the Riemannian manifold*. We denote it (M,g).

For brevity, the metric tensor is simply called *the metric*.

The metric tensor $g(X,Y)$ satisfies the following conditions.

1. $g(X,Y)=g(Y,X)$;
2. $g(X+Y,Z)=g(X,Z)+g(Y,Z)$;
3. $g(\alpha X,Y)=\alpha g(X,Y)$;
4. $g(X,X)\geq 0$ and $g(X,X)=0 \Leftrightarrow X=0$

$\forall\, X,Y,Z\in\mathfrak{B}M$ and $\forall\,\alpha\in\mathfrak{F}M$.

Conditions 1 and 2 are included in the definition of a tensor. Condition 1 is symmetric property and 4 means that the tensor is positively definite.

The metric tensor allows us to define the length of a vector:

$$\|X_p\|=\sqrt{\langle X_p,X_p\rangle_p}$$

and the angle between two vectors:

$$\cos\angle(X_p,Y_p)=\frac{\langle X_p,Y_p\rangle_p}{\|X_p\|\,\|Y_p\|}.$$

The angle between two curves is the angle between their tangent

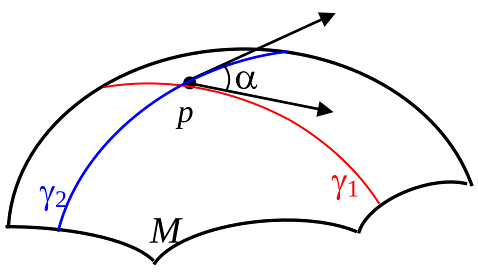

fig. 45

vectors at the point of intersection (figure 45).

Let $c(t)$ be a differential path on the manifold M, $c: I \to M$ and $t_o, t_1 \in I$. The length of the path c from $t = t_o$ to $t = t_1$ is called the value

$$L(t_o, t_1) = \int_{t_o}^{t_1} \|\dot{c}(t)\| \, dt. \qquad (17)$$

If $\|\dot{c}(t)\| \equiv 1$ the path is called *normal*. In this case formula (17) implies

$$L(t_o, t_1) = |t_1 - t_o|.$$

***Theorem* 5.** On each differentiable manifold, we can define the metric tensor, i.e. any differentiable manifold can be turned into a Riemannian one.

***Definition* 60.** Let M and N be two Riemannian manifolds and $f: M \to N$ be differential mapping, $p \in M$, $q = f(p) \in N$. Mapping f is called *isometric at point* p if

$$\langle X, Y \rangle_p = \langle f_*(X), f_*(Y) \rangle_q \quad \forall \; X, Y \in T_pM$$

(figure 46).Mapping f is called *isometric* (or *isometry*) if it is isometric at each point $p \in M$.

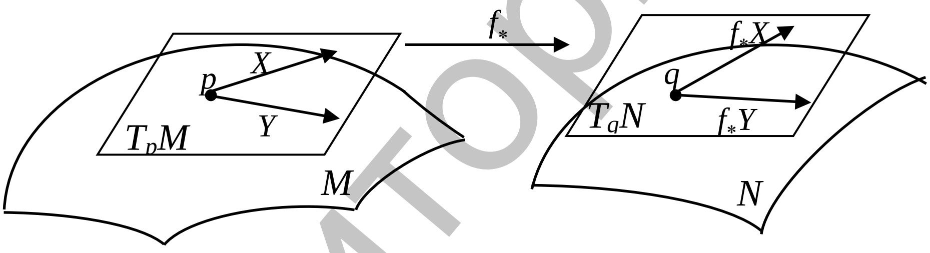

fig. 46

It is obvious that an isometric mapping preserve the length of a vector, the angle between vectors, the length of a curve, i.e. it preserves everything, what can be calculated with the help of the metric tensor. In other words, isometric mapping *preserves the internal geometry of the manifold*.

§2. *Riemannian connectivity*

We have defined the connectivity, defined the metric but haven't defined connection between them. Naturally only those connections are of the interest that agree with the metric.

***Definition* 61.** Linear connectivity ∇ on a differential manifold M is called *the Riemannian connectivity*, if for any differentiable path $c: I \to M$ and for any vector fields X, Y, which are parallel along c, function $\langle X, Y \rangle$ is constant along c.

If a linear connectivity is *Riemannian*, then parallel translation of vectors from T_pM to T_qN $(p = c(t_o), q = c(t_1))$ is isometric mapping of tangent spaces. In particular, function $\|X\|$ is constant for parallel vector field X.

If $c(t)$ is a geodesic path, then $\dot{c}$ is parallel along c, and it means, that $\|\dot{c}\|$ is constant. If X is also parallel along $c(t)$, then both $\|X\|$ and $\langle \dot{c}, X\rangle$ are constant $\Rightarrow$ $\cos\angle(\dot{c}, X)$ is also constant. Therefore a vector field, which is parallel along a geodesic curve γ forms constant angle with tangent vectors to γ and has constant length. This reasoning shows that our definition of Riemannian connectivity is natural.

***Theorem* 6.** Let M be Riemannian manifold and ∇ be a linear connectivity on M. Then ∇ is Riemannian connectivity, if for any vector fields X, Y, Z on M holds

$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle . \tag{18}$$

It is full coincidence with the rule of derivation (without proof).

***Definition* 62.** Among all Riemannian connections on a differentiable manifold, stands out one that is torsion-free, and it is called *the Levi-Civitta connectivity*.

According to the definition the Levi-Civitta connection must satisfy (18) and $T(X,Y)=0$ $\Leftrightarrow$

$$\nabla_X Y = \nabla_Y X + [X,Y] \tag{19}$$

From formulas (18) and (19) we can derive

$$\langle \nabla_Z X, Y\rangle = \frac{1}{2}\{X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle + \langle Z,[X,Y]\rangle + \langle Y,[Z,X]\rangle - \langle X,[Y,Z]\rangle\}. \tag{20}$$

If we take coordinate vector fields X_i, X_j, X_k instead of X, Y, Z, then taking into account $[X_i, X_j]=0$, we get

$$\langle \nabla_{X_i} X_j, X_k\rangle = \frac{1}{2}\{X_i\langle X_j,X_k\rangle + X_j\langle X_k,X_i\rangle - X_k\langle X_i,X_j\rangle\}. \tag{20}$$

Remind that $\nabla_{X_i} X_j = \Gamma^l_{ij} X_l$, $\langle X_i, X_j\rangle = g_{ij}$, and $X_i f = \frac{\partial f}{\partial x^i}$. Thus (20) is equivalent

$$g_{ik}\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right). \tag{20$'$}$$

The following notation is common used: $\frac{\partial g_{ij}}{\partial x^k} = g_{ij,k}$. Using it we obtain

$$g_{kl}\Gamma^l_{ij} = \frac{1}{2}(g_{jk,i} + g_{ki,j} - g_{ij,k}). \tag{20$''$}$$

We know, that $\det(g_{ij})>0$, that is why there is the inverse matrix $(g_{ij})^{-1}$. We denote elements of this matrix as g^{ij}. Remind, that $(g_{ij})(g_{ij})^{-1} = \mathrm{E}$ and elements of unity matrix are Kronecker symbols. Thus $g^{nk}g_{kl} = \delta^n_l$. Because δ^n_l is equal to 0 in all cases, except $n=l$, when it is equal to 1, then, for instance, in the sum $\delta^n_l T_{nv}$ all terms are equal zero, except T_{lv}.

Let's multiply (20″) by inverse matrix (g^{nk}).

$$\delta_l^n \Gamma_{ij}^l = \frac{1}{2} g^{nk}(g_{jk,i} + g_{ki,j} - g_{ij,k}).$$

$$\Gamma_{ij}^n = \frac{1}{2} g^{nk}(g_{jk,i} + g_{ki,j} - g_{ij,k}). \tag{21}$$

This formula shows, that the Levi-Civitta connectivity is totally determined by the metric.

Let M be a Riemannian manifold with the Levi-Civitta connectivity, $p \in M$, $X \in T_pM$. Let $\|X\| = a$ and let $c:[0,1] \to M$ be a geodesic path such that $c(0) = p$, $\dot{c}(0) = X$. Let $q = c(1) = \exp_p X$ (figure 47). And at last, let $X_t = \dot{c}(t)$ be the tangent vector field along c. As we have noted above, $\|X_t\| = a = \text{const}$. Therefore

$$L(0,1) = \int_0^1 \|X_t\|\, dt = \int_0^1 a\, dt = a = \|X\|.$$

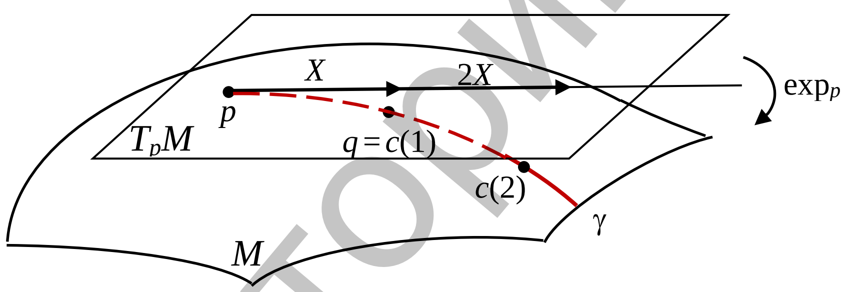

fig. 47

So, the length of geodesic path, that connects p and $q = \exp_p X$ is equal $\|X\|$.

Geodesic path with initial data $c(0) = p$, $\dot{c}(0) = X \in T_pM$ can be defined as $c(t) = \exp_p tX$. For instance, $c(2) = \exp_p 2X$. We get mapping of the ray $tX \subset T_pM$ on the geodesic line γ, which is image of the geodesic path $c(t)$. We proved, that the length of vector tX (it is the same as length of the part of the ray) is equal to the length of geodesic segment from $c(0)$ to $c(t)$. Therefore, the exponential map is said to be *radially isometric*, i.e. it isometrically maps ray tX on the geodesic line. However, it should be noted that it is true only within some domain $V \subset T_pM$, such that reduction $\exp_p|_V$ is one-to-one mapping. Outside this area it may turn out that $\exp_p 0X = p = \exp_p t_o X$ for some $t_o > 0$, i.e. geodesic line can come back to point p (like on the cylinder, for instance).

§3. *Curvature tensor*

***Definition* 63.** Denote

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

This value is a tensor (without proof) of the type (3,1) and it is called *the curvature tensor* of the connectivity ∇ or *the curvature tensor* of the manifold M with the connectivity ∇.

When we consider a surface in Euclidean space we can estimate its curvature looking on its form, or we can estimate how quickly the direction of normal vector changes while moving along the surface. Our manifold is not embedded anywhere. Therefore we need to find a way to measure curvature without going beyond the manifold. Imagine the task: to measure the curvature of the space in which we live.

It turns out, that curvature tensor is a measure of the dependence of a parallel translation on the path along which this translation is carried out.

Let $p \in M$, $X, Y, Z \in T_pM$ and let (V, φ) be a map in the neighborhood of point p, $\widetilde{V} = \varphi(V)$. Consider a square $\widetilde{Q}$ with a vertex $\varphi(p)$ and with the side equal t, which fit in the neighborhood $\widetilde{V}$, such that vectors X and Y are tangent to the sides of curvilinear rectangle $Q = \varphi^{-1}(\widetilde{Q}) \subset V$ (figure 48).

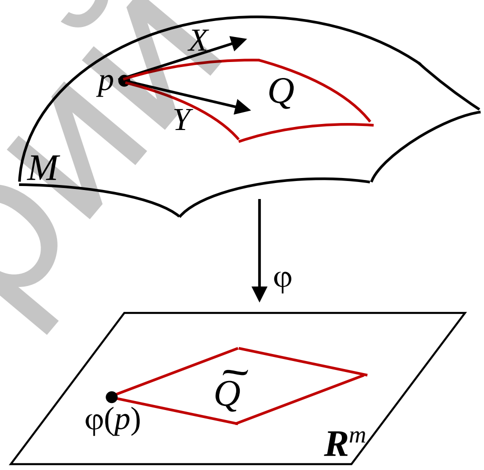

fig. 48

Let's perform consequently the parallel translation of vector Z along the sides of Q. When we come back to p, we will get another vector Z^t. It turns out, that

$$R_p(X, Y)Z = \lim_{t \to 0} \frac{Z^t - Z}{t^2} \qquad (21)$$

(without proof).

If M is the Riemannian manifold, than we can consider the Riemannian curvature tensor of the type (4,0):

$$R(X, Y, Z, U) = \langle R(X, Y)Z, U \rangle. \qquad (22)$$

***Theorem* 7.** Properties of the curvature tensor and the Riemannian curvature tensor. For any vector fields X, Y, Z, U on M holds

1. $R(X, Y)Z = -R(Y, X)Z$;
2. $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$;
3. $R(X, Y, Z, U) = -R(X, Y, U, Z)$;
4. $R(X, Y, Z, U) = R(Z, U, X, Y)$.

Without proof.

We can define components of this tensors.

$$R(X_i, X_j)X_k = R^k_{i\,jl}X_l;$$

$$R(X_i, X_j, X_k, X_l) = R_{ijkl}.$$

Consider a value $k_1(X,Y) = \|X\|\,\|Y\| - \langle X,Y\rangle^2$. If X, Y are vectors in $\boldsymbol{R}^m$, then it is equal $|X \times Y|$, i.e. it is equal to the area of the parallelogram constructed on vectors X and Y. Since T_pM is also $\boldsymbol{R}^m$, then it is true in T_pM also.

***Definition* 64.** Let $X, Y \in T_pM$ be linear independent. The value

$$K_p(X,Y) = \frac{R(X,Y,Y,X)}{k_1(X,Y)}$$

is called *the sectional curvature of the manifold* M *at the point* p *in the direction of two-dimensional area element spanned on vectors* X *and* Y.

Let σ be two-dimensional space, which is defined by X and Y. If we take some other linear independent vectors $Z, U \in \sigma$ then $K_p(X,Y) = K_p(Z,U)$, i.e. the sectional curvature depends only on the plane σ, where vectors are located, but doesn't depend on vectors themselves. Therefore the notation $K_\sigma = K_p(X,Y)$ is often used.

If M is a surface in three-dimensional space, then its tangent plane contains only one two-dimensional direction and the sectional curvature in this direction coincide with the Gaussian curvature of the surface.

If $\|X\| = \|Y\|$ and $\langle X,Y\rangle = 0$, then $k_1(X,Y) = 1$ and

$$K_p(X,Y) = R(X,Y,Y,X) \qquad (22)$$

***Definition* 65.** A Riemannian manifold is called a constant curvature space if $K_\sigma = \kappa = \text{const}$ for all two-dimensional spaces $\sigma \subset T_pM$. A manifold M is called a hyperbolic space, if $\kappa < 0$, elliptic space, if $\kappa > 0$ and it is called flat, if $\kappa \equiv 0$.

It turns out, that if M is flat, then it is locally Euclidean, i.e. each point p has a neighborhood, which is isometric to a flat domain $\widetilde{V} \subset \boldsymbol{R}^m$. An example of elliptic space is the sphere S^2 and an example of elliptic space is the hyperbolic paraboloid.

Remind, that the trace of a matrix is the sum of all its diagonal elements:

$$\operatorname{tr}\mathbf{A} = a_1^1 + a_2^2 + \ldots + a_m^m = \sum_{i=1}^{m} a_i^i .$$

A linear operator has different matrixes in different bases. But the trace of the matrix is invariant. For a tensor of the kind $(p,1)$, we can define operation of contraction of the tensor and get a tensor of the type $(p-1,0)$:

$$T_{i_2 \ldots i_p} = \sum_{k=1}^{m} T^k_{k i_2 \ldots i_p} .$$

If $\mathbf{A}$ is the matrix of a linear operator $\mathcal{A}$ in an orthonormal basis $(e_1, e_2, \ldots, e_m)$, then $a_i^j = (\mathcal{A}e_i) \cdot e_j$. Therefore

$$\operatorname{tr}\mathcal{A} = \sum_{i=1}^{m} (\mathcal{A}e_i) \cdot e_i = \sum_{i=1}^{m} \Big(\sum_{k=1}^{m} a_i^k e_k\Big) \cdot e_i = \sum_{i=1}^{m} a_i^k g_{ki} .$$

Analogously, the trace of a tensor of the type $(p,1)$ is tensor of the type $(p-1,0)$: $T^{k}_{ji_2\dots i_p}\, g_{kj}$.

***Definition* 66.** *Ricci tensor* is a tensor

$$R(X,Y) = \mathrm{trace}\{U \to R(U,X)Y)\}.$$

If basis $(E_1, E_2, \dots, E_m)$ is orthonormal, then

$$R(X,Y) = \sum_{i=1}^{m} R(E_i, X, Y, E_i) = \sum_{i=1}^{m} \langle R(E_i, X)Y, E_i\rangle. \qquad (23)$$

or

$$R_{ij} = R^{k}_{ijk}$$

If the basis is not orthonormal, then

$$R_{ij} = \sum_{k=1}^{m} R^{l}_{ijk}\, g_{kl}.$$

According to the properties of the curvature tensor

$$R(E_i, X, Y, E_i) = R(Y, E_i, E_i, X_i) = -(-R(E_i, Y, X, E_i)) \Rightarrow$$

$$\Rightarrow \quad R(X,Y) = R(Y,X),$$

i.e. the Ricci tensor is symmetric.

***Definition* 67.** Value

$$r_p(X) = \frac{R_p(X,X)}{\|X\|^2}.$$

is called *the Ricci curvature of manifold* M *at the point* p *in the direction of the vector* X.

Let $\|X\| = 1$. Then from formulas (22) and (23) we get

$$r_p(X) = R_p(X,X) = \sum_{i=1}^{m} R_p(E_i, X, X, E_i) = \sum_{i=1}^{m} R_p(X, E_i) = \sum_{i=1}^{m} K_p(X, E_i). \qquad (24)$$

This formula allows us to calculate the Ricci curvature, if we know all the sectional curvatures $K_p(X, E_i)$, $i = 1, 2, \dots, m$.

***Definition* 68.** Trace of the Ricci tensor at a point $p \in M$ is called scalar curvature of manifold M at point p, and we denote it ρ_p.

$$\rho_p = \sum_{i=1}^{m} r_p(E_i) = \sum_{i=1}^{m}\sum_{j=1}^{m} K_p(E_j, E_i) = \sum_{i,j=1}^{m} R_{ijji}.$$

§4. *Function of distance of Riemannian manifold. Extremal property of geodesic lines*

Let M be a connected Riemannian manifold. Denote Ω_{pq} the set of all piecewise smooth paths $[0,1] \to M$ with the beginning at p and end at q. Since M is connected, $\Omega_{pq} \neq \varnothing$. Sometimes we will identify a path and its image – a curve on the manifold.

***Theorem* 8.** (Extremal property of geodesic lines I) *Let* M *be Riemannian manifold,* γ *be a geodesic line in* M *and* $p\in\gamma$. *Then there is such neighborhood* V *of the point* p *that for any point* $q\in V\cap\gamma$ *the segment of geodesic line* γ *connecting* p *and* q *is the shortest line among all the other lines from* Ω_{pq}, *which are included in* V (figure 49).

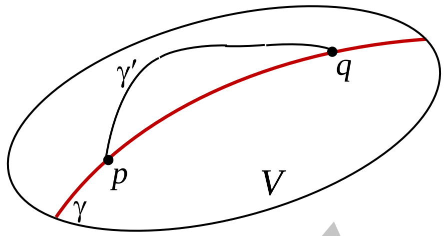

fig. 49

This theorem tells us, that geodesic line is the shortest “in small”, i.e. on its small sections. At large it may not be the shortest line.

***Example* 23.** Let p, q be two points on the cylinder, which lies on one circle and are close enough (figure 48). We delete a point s, which belongs to the shortest arc of the circle connecting p and q. Let γ_1 be the longest arc of the circle connecting p and q. Then γ_1 is the only one geodesic line from Ω_{pq}, but it is not the shortest one. Line γ_o depicted on figure 50 is shorter, then γ_o.

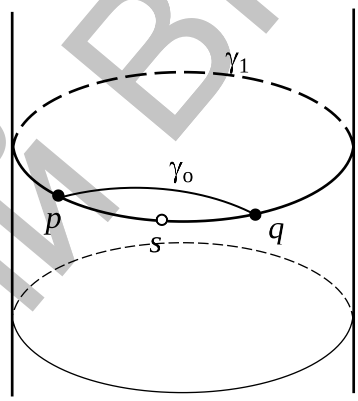

fig. 50

Consider a function $\rho: M\times M\to \boldsymbol{R}$

$$\rho(p,q)=\inf\{L(c)\,|\,c\in\Omega_{pq}\},$$

where $L(c)$ is the length of path c (is the length of the shortest curve, that connects p and q).

***Theorem* 9.** (M,ρ) *is metric space*.

***Proof*.** We need to check, that the following axioms are true.

1. $\rho(p,q)=\rho(q,p)$;

2. $\rho(p,r)+\rho(r,q)\geq\rho(p,q)$ (figure 51);

3. $\rho(p,q)\geq 0$ и $\rho(p,q)=0 \Leftrightarrow p=q$.

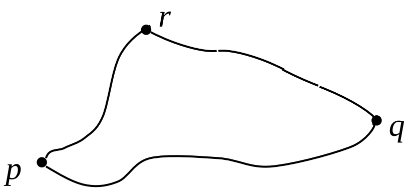

*fig.*51

Axiom 1 is obvious from the definition.

Suppose, that axiom 2 fails. Consider value

$$\varepsilon=\rho(p,q)-(\rho(p,r)+\rho(r,q))>0. \qquad (25)$$

Let $c_1\in\Omega_{pr}$ and $c_2\in\Omega_{rq}$ are such paths, that

$$L(c_1)-\rho(p,r)<\varepsilon/4,\ L(c_2)-\rho(r,q)<\varepsilon/4.$$

Then gluing together paths c_1 and c_2 we get such path c, that $L(c)=L(c_1)+L(c_1)$ and

$$0<L(c)-(\rho(p,r)+\rho(r,q))<\varepsilon/2. \qquad (26)$$

Let’s make difference (25) – (26). We get

$$\rho(p,q) - L(c) > \varepsilon/2 \quad \Rightarrow \quad L(c) < \rho(p,q).$$

It is contradiction and it means that our supposition is false.

Let's prove 3. If $p = q$ then the path $c(t) \equiv p$ connects p and q and $L(c) = 0 \;\Rightarrow\; \rho(p,q) = 0$.

Conversely, let $\rho(p,q) = 0$. It means that $\forall \varepsilon > 0$ there is $c \in \Omega_{pr}$ such that $L(c) < \varepsilon$. Therefore points p and q are into a sufficiently small neighborhood, where the geodesic line $d \in \Omega_{pq}$ is the shortest one. Thus $L(d) < \varepsilon \quad \forall \varepsilon > 0$ $\Rightarrow \quad L(d) = 0$. ■

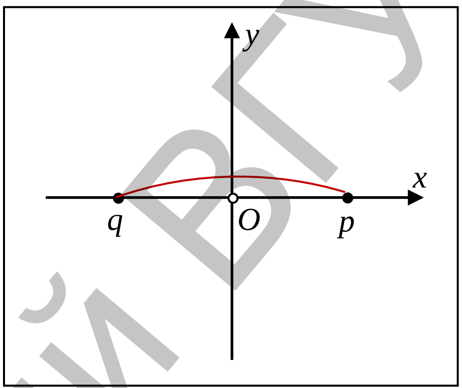

fig.52

The question arises: why in the definition we have infinum and not minimum? Consider the plane with the deleted origin: $\boldsymbol{R}^2 \backslash \{O\}$. Let $p = (1,0)$, $p = (-1,0)$ (figure 52). According to the definition $\rho(p,q) = 2$, but there is no the shortest path in Ω_{pq}, i.e. minimum cannot be reached. However there are paths with length as close to 2 as you like.

***Theorem* 10.** *Topology of the metric space* (M,ρ) *defined by the function* ρ *coincide with the topology of the differential manifold* M.

In order to prove the theorem we must prove, that open set in the manifold M is open in metric space (M,ρ) and vice versa.

If we fix point $p \in M$, then we get a function on manifold M $f(q) = \rho(p,q)$, $f : M \to \boldsymbol{R}$.

***Theorem* 11.** *Function* $f^2(q)$ *is differentiable.*

Immediately note that function $f(q)$ itself is not differentiable.

***Example* 24.** Let $M = \boldsymbol{R}$, $p(0)$, $q(x)$. Then $f(q) = |x|$ (figure 53). It is well-known, that this function is not differentiable at point p. Analogously, if p is the north pole of the sphere S^2 and r is the south pole, then out function is not differentiable both at p and r.

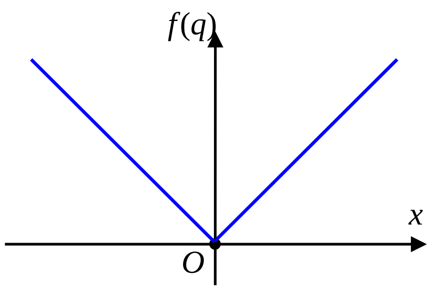

fig.53

The property of radial isometricity of mapping $\exp_p : T_pM \to M$ leads to the fact that for δ small enough it maps the ball $B(0,\delta) \subset T_pM$ onto the ball $B(p,\delta) \subset M$.

***Theorem* 12.** (Extremal property of geodesic lines II) *Let* M *be Riemannian manifold. For any point* $p \in M$ *there is a neighborhood* W, *that has the following property. For any points* q_1, q_2 *there is the only one geodesic line* *connecting the points and lying entirely in U. This geodesic line is the shortest among all the curves from Ω_{pq}.*

Before we prove the theorem we need some definitions.

***Definition* 69.** Open subset G of Riemannian manifold M is called simple, if for any two points $p, q \in M$ there is no more than one geodesic path $c:[0,1] \to G$ with ends $p = c(0)$, $q = c(1)$.

It is obvious, that any open subset in G is simple, if G is simple.

***Definition* 70.** Subset G of a Riemannian manifold M is called convex, if for any two points $p, q \in M$ there is a geodesic path c with length $L(\mathrm{c}) = \rho(p, q)$. Subset G is called strongly convex if it is convex, and moreover all the balls $B(p,\delta) \subset G$ are convex.

Theorem 12 now can be formulated as follows. Each point $p \in M$ has a convex neighborhood. As the matter of fact, it has strongly convex neighborhood, but we are not going to prove that fact.

***Proof of theorem* 12.** Step I. Let $p \in M$ and let V be its neighborhood from theorem 8. Let $S \subset T_pM$ be such subset, that $(\exp_p)|_S$ is one-to-one mapping and $V_1 = \exp_p(S) \subset V$. Then point p can be connected with each point $q \in V_1$ with no more than one geodesic line. Let

$$\delta(p) = \sup\{\delta \mid B(p,\delta) \subset V_1\}.$$

It is radius of the biggest ball in V_1 with center p. We will call δ the radius of injectivity of mapping $\exp_p$.

Denote $W = \exp_p(B(p,\delta(p)))$. Then $\overline{W}$ is closed bounded set and therefore $\overline{W}$ is compact. We except without poof, that $\delta(p)$ is a continuous function. Thus it reaches its minimum on $\overline{W}$.

Denote $\delta_o = \min_{r \in \overline{W}} \delta(r)$. Then ball $U = B(p, \delta_o/2)$ (figure 54) is a simple set.

In fact, $\forall q_1, q_2 \in U$ holds

$$\rho(q_1, q_2) \le \rho(q_1, p) + \rho(p, q_2) \le \frac{\delta_o}{2} + \frac{\delta_o}{2} = \delta_o.$$

It means, that $\delta(q_1) \ge \rho(q_1, q_2) \Rightarrow q_2 \in B(q_1, \delta(q_1))$. Thus $\exp_{q_1}$ maps injectively $B(0,\delta(q_1))$ on $B(q_1,\delta(q_1))$, i.e. there can't be more than one geodesic line connecting q_1 and q_2 inside $B(q_1,\delta(q_1))$. But $B(p,\delta_o/2) \subset B(q_1,\delta(q_1))$ and therefore $B(p,\delta_o/2)$ has the same property.

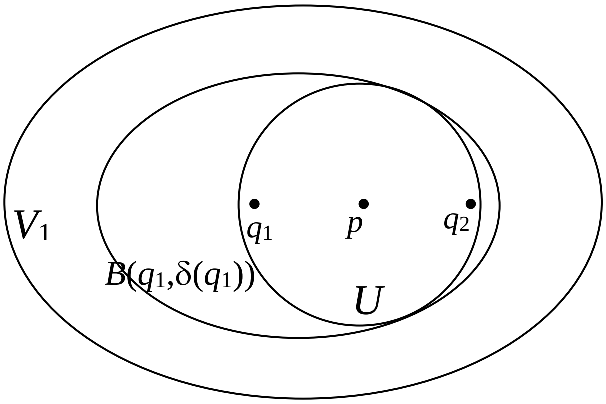

fig.54

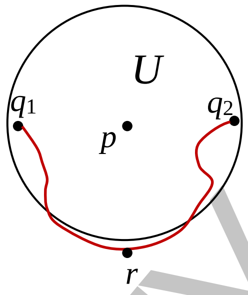

fig.55

Step II. We except without prove, that unique geodesic line γ, connecting q_1 and q_2 inside V doesn't leave U (figure 55).

Step III. is the shortest among all the lines from $\Omega_{q_1 q_2}$ that don't leave V. In order to complete the proof of the theorem we must prove, that any line which leaves V is longer than γ.

***Exersise* 9.** Prove it independently, using figure 56.

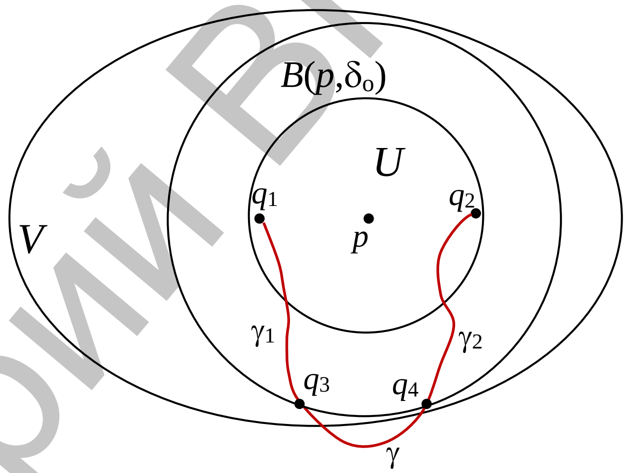

fig.56

§5. *Compaction*

***Definition* 71.** Let (M,ρ) be metric space. We say, that sequence of points $x_1, x_2, \ldots, x_n, \ldots$ *converges to point* x, if $\forall \varepsilon > 0$ there is such number N, that $\{x_{N+1}, x_{N+2}, \ldots\} \subset B(x,\varepsilon)$, that is, whatever the ball has center at point x, starting from certain number N all the points of the sequence fall into this ball.

***Definition* 72.** Let (M,τ) be topological space. We say, that sequence of points $x_1, x_2, \ldots, x_n, \ldots$ *converges to point* x, if for any neighborhood V of point x there is such number N, that $\{x_{N+1}, x_{N+2}, \ldots\} \subset V$.

***Definition* 73.** Set W in topological space (M,τ) is called *precompact*, if from any sequence of points $x_1, x_2, \ldots, x_n, \ldots$ we can choose convergent subsequence $\{x_{i_1}, x_{i_2}, \ldots\}$. If this subsequence converges to point x, which always belongs to W, then W is called *compact*.

***Example* 25.** 1) Ray $[0,+\infty)$ in topological space $\boldsymbol{R}$ is not precompact. We can't choose convergent subsequence from sequence $1,2,3,\ldots$.

2) Interval $(0,1)$ in topological space $\boldsymbol{R}$ is precompact, but not compact. Any subsequence from sequence $\{1/2, 1/3, 1/4, \ldots\}$ converges to point 0, but $0 \notin (0,1)$.

3) Segment $[0,1]$ in topological space $\boldsymbol{R}$ is compact.

4) The sphere and the torus in three-dimensional space are compact.

***Definition* 74.** Set W in metric space (M,ρ) is called *bounded*, if there is such ball $B(p,r)$, that $W \subset B(p,r)$. It is equivalent to the fact, that the diameter of this set is finite: $d(W) < \infty$.

***Theorem* 13.** *Set W in the Euclidean metric space $\boldsymbol{R}^n$ is precompact if and only if it is bounded.* (The sketch of the proof for $\boldsymbol{R}^2$ will be given on lectures).

***Definition* 75.** We say, that aggregate of sets $\{U_\alpha\}_{\alpha \in I}$ forms a cover (an overlapping) of set W if $W \subset \bigcup\limits_{\alpha \in I} U_\alpha$.

***Theorem* 14.** *Set W in metric space (M,ρ) is compact if and only if it is possible to choose it's finite subcover from any it's infinite open cover.*

In other words, W is compact $\Leftrightarrow$ from infinite number of open sets, that cover W we can choose finite number, that also cover W.

***Example* 26.** 1) Sets $(-1,1)$, $(0,2)$, $(1,3)$ … cover ray $[0,+\infty)$. We can't choose finite number of sets, that cover $[0,+\infty)$.

2) Sets … $(1/2^n,1)$ … $(1/8,1)$, $(1/4,1)$, $(1/2,1)$ forms cover of interval $(0,1)$. We can't choose finite number of sets, that cover $(0,1)$.

3) Plane $\boldsymbol{R}^2$ can be covered by open balls of radius 2 with centers at integral points (figure 57). It is obvious, that it is impossible to choose finite cover from these sets.

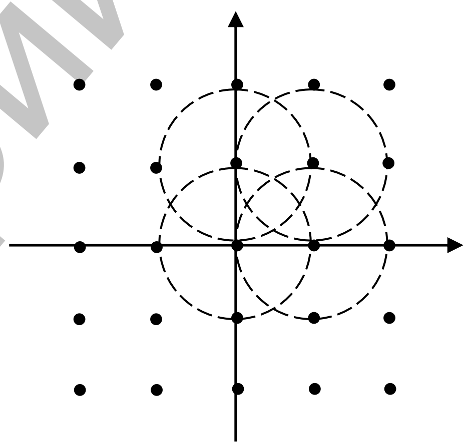

*fig.*57

***Definition* 76.** Set W in topological space (M,τ) is called bicompact if it is possible to choose it's finite subcover from any it's infinite open cover.

For metric spaces notions of compact and bicompact sets are coincide, nut for arbitrary topological space they are different.

Why theorem 12 is not true for an arbitrary metric space? Consider $\boldsymbol{R}^2 \backslash \{0\}$ – the plane with the deleted point 0. Let $B(0,r)\backslash\{0\}$ be an open ball with the deleted center. This set is bounded, but it is not precompact, because any sequence, that converges to 0 in $\boldsymbol{R}^2$ does't converge in our space. For instance, sequence $(1/i,0)$ does't converge (figure 58). This is because that our space is not complete, one point is missing in it.

By the way, $B(0,r)\backslash\{0\}$ is homeomorphic to infinite cylinder or to open ring $B(0,2)\backslash\overline{B}(0,1)$ (figure 59).

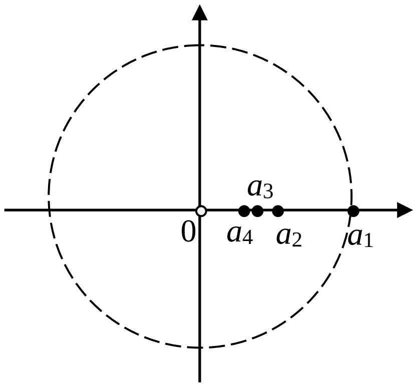

fig.58

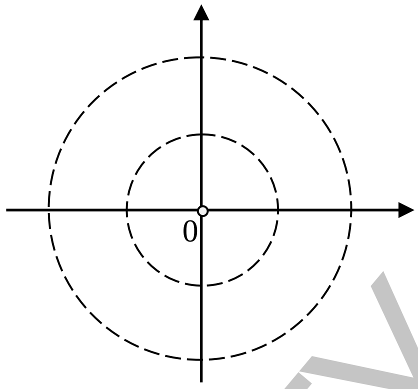

fig.59

***Theorem* 14.** *Any continuous function* $f: W \to \boldsymbol{R}$ *defined on a compact set* W *attains its maximal and minimal values on W.*

***Example* 27.** $f(x) = e^x$ is defined on $\boldsymbol{R}$. $\boldsymbol{R}$ is not compact, and this function doesn't attain the largest and least values. We can only say that $\inf\limits_{\boldsymbol{R}} f(x) = 0, \sup\limits_{\boldsymbol{R}} f(x) = +\infty$. But if we consider the same function as defined on $[0,1]$, then $\max f(x) = e$, $\min f(x) = 0$ (figure 60).

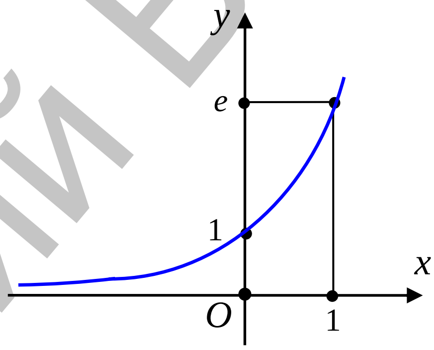

fig. 60

***Example* 28.** The sphere and the torus are compact. Therefore any continuous function $f: S^2 \to \boldsymbol{R}$ or $f: T^2 \to \boldsymbol{R}$ attains its maximal and minimal values.

§6. *Complete Riemannian manifold*

Remember which examples we used in order to show, that not always two points of connected Riemannian manifold M can be connected by a geodesic line, that not any geodesic line can be infinitely prolonged and that exponential mapping can be defined not on the entire T_pM. We took the cylinder and deleted a point from it, or we have considered the semiplane (half-plane). These manifolds are as if "not complete". It is only "a half of plane" or "manifold with a hole".

***Definition* 77.** Riemannian manifold M is called *complete*, if metric space (M,ρ) is complete, where ρ is the function of distance of the manifold.

Suppose, that Levi-Civitta connectivity is introduced on manifold M.

***Theorem* 15.** (of Hopf-Rinov) *The following conditions are equivalent*:

1) M *is complete*;

2) $\forall\, p \in M$ *mapping* $\exp_p$ *is defined on the entire* T_pM;

3) *any geodesic line defined on some interval* I *can be prolonged to geodesic line defined on* $\boldsymbol{R}$;

4) *every closed and bounded with respect to* ρ *subset in* M *is compact.*

The following statement is an implication of each of the statements 1-4:

5) Any two points $p, q \in M$ can be connected by a geodesic line of the length $\rho(p,q)$.

§7. *Comparison theorems. Connection with curvature and topological structure*

***Theorem* 16.** (Toponogov's angle comparison theorem) *Let M be a complete Riemannian manifold, $m = \dim M \geq 2$. Suppose, that for all $p \in M$ and for all two-dimensional spaces $\sigma \in T_pM$ holds $K_\sigma \geq \kappa = \text{const} > 0$. If $\kappa > 0$ we denote $\widetilde{M} = S_r^m$, $r = \frac{1}{\sqrt{\kappa}}$ and if $\kappa = 0$ we denote $\widetilde{M} = \boldsymbol{R}^m$.*

Let $\Delta = (c_o, c_1, c_2)$ be a geodesic triangle in M with angles $(\alpha_o, \alpha_1, \alpha_2)$. Then there is triangle $\widetilde{\Delta} = (\tilde{c}_o, \tilde{c}_1, \tilde{c}_2)$ in $\widetilde{M}$ with angles $(\tilde{\gamma}_o, \tilde{\gamma}_1, \tilde{\gamma}_2)$ such that

$$L(c_i) = L(\tilde{c}_i) \qquad (27)$$

$$\gamma_i \geq \tilde{\gamma}_i,\ i = 0, 1, 2. \qquad (28)$$

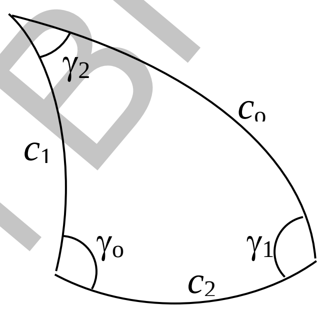

*fig.*56

We shell note, that for $\boldsymbol{R}^m$ holds $K_\sigma \equiv 0$ and for S_r^m holds $K_\sigma \equiv \kappa = \frac{1}{r^2}$. Therefore the theorem can be reformulated as follows. If the sectional curvature of manifold M is greater or equal to then the sectional curvature of manifold $\widetilde{M}$, then a geodesic triangle on M has angles greater or equal than corresponding geodesic triangle with equal sides on $\widetilde{M}$.

***Theorem* 17.** (of Hadamard-Cartan) *Let M be a complete Riemannian manifold, $n = \dim M \geq 2$ and $K_\sigma \leq 0$ for all $\sigma \subset T_pM$ and for all $p \in M$. Then for any point $p \in M$ the mapping $\exp_p : T_pM \to M$ is diffeomorphism. In particular, M is diffeomorphic to $\boldsymbol{R}^n$.*

Thus, a space of non-positive sectional curvature is diffeomorphic to $\boldsymbol{R}^n$; therefore, it is covered by only one map (M, φ). As φ we can take $\exp_p^{-1} : M \to T_pM$. Другими словами, M это есть $\boldsymbol{R}^n$ на котором вместо стандартной метрики введена некоторая другая метрика. In other words, M is $\boldsymbol{R}^n$ on which some other metric is introduced instead of the standard metric.

Denote $\rho_M = \sup\limits_{p,q \in M} \rho(p,q)$ – the diameter of a manifold M with respect to the internal metrics ρ.

***Theorem* 18.** (of Meyers) *Let M be complete Riemannian manifold, and $K_\sigma \geq \kappa = \text{const} \geq 0$ for all $\sigma \subset T_pM$ and for all $p \in M$. Then for all $p, q \in M$ holds the inequality*

$$\rho(p,q) \le \rho_M \le \frac{\pi}{\sqrt{\kappa}}.$$

This means that a manifold with positive sectional curvature separated from zero has a bounded diameter. Why do we write $K_\sigma \ge \kappa > 0$ instead of $K_\sigma > 0$? The second means that the curvature can tend to zero in some directions, and the first notation means that K_σ is delimited from zero by a constant. Recall that the diameter of a sphere of radius r with respect to the intrinsic metric is equal $\pi r = \frac{\pi}{\sqrt{\kappa}}$ (figure 28).

***Theorem* 19.** *Let M be a complete Riemannian manifold, and* $K_\sigma \ge \kappa = \text{const} \ge 0$ *for all $\sigma \subset T_p M$ and for all $p \in M$. If $\rho_M = \frac{\pi}{\sqrt{\kappa}}$, then M is isometrically diffeomorphic to the sphere S_r^n of the curvature* (i.e. $r = \frac{\pi}{\sqrt{\kappa}}$).

In other words, if M has the same diameter as the sphere, and its sectional curvature is not less than that of the sphere, then M is isometric to the sphere. Basically, this means that M is the sphere.

***Theorem* 20.** (The theorem on sphere) *Let M be complete Riemannian manifold, $n = \dim M \ge 2$. Let the sectional curvature of the manifold M is* δ*-bounded with $\delta > \frac{1}{4}$ i.e.*

$$\frac{1}{4} < \delta < K_\sigma \le 1$$

for all $\sigma \subset T_p M$ and for all $p \in M$. Then M is homeomorphic to the sphere (*i.e. M is topologically arranged, like a sphere, and with a special specification of the metric, it can serve as a model for M*).