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AN ANALOG OF SHEMETKOV'S CONJECTURE FOR FISCHER CLASSES OF FINITE GROUPS

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Abstract: We describe some methods for constructing Fischer classes of finite groups by means of the operators defined by given properties of Hall π-subgroups. It is in particular proved that, for a Fischer class $\mathfrak{F}$ and a set of primes π, the class of all finite π-soluble $C_\pi\mathfrak{F}$-groups, i.e., of all groups whose Hall π-subgroups belong to $\mathfrak{F}$, is a Fischer class.

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1. Introduction

We consider only finite groups. Many results in group theory are connected with the study of the properties of the groups defined by given properties of Hall π-subgroups (for example, see [1, § 16; 2, IX, 4; 3, 2.3 and 2.4] and also [4–15]). Recall that if n is a natural number then $\sigma(n)$ denotes the set of all prime divisors of n; if G is a group then $\sigma(G) = \sigma(|G|)$. Let π be a subset in the set P of all primes and let $\pi' = P/\pi$. A natural number n is called a *π-number* (a *π'-number*) if $\sigma(n) \subseteq \pi(\sigma(n) \subseteq \pi')$. Suppose that $H \le G$ and $|H|$ is a π-number. Then H is called a *π-subgroup* in G. If d is a divisor of n and $\sigma(d) \subseteq \pi$ then d is called a *π-divisor* of n. Denote by n_π the greatest π-divisor of n. A subgroup of order $|G|_\pi$ is called a *Hall π-subgroup* in G is denoted by G_π.

Following [4], call a group G an *E_π-group* if in G there is at least one Hall π-subgroup and a *C_π-group* if G is an E_π-group and every two Hall π-subgroups in G are conjugate. We denote by E_π and C_π the classes of all E_π-groups and all C_π-groups respectively. Let $\mathfrak{X}$ be a class of groups. Denote by $E_\pi\mathfrak{X}$ the class of all groups having at least one Hall π-subgroup belonging to $\mathfrak{X}$ and denote by $C_\pi\mathfrak{X}$ the class of groups $C_\pi \cap E_\pi\mathfrak{X}$.

Recall that a class of groups is called a *formation* if it is closed under homomorphic images and finite subdirect products. A formation $\mathfrak{F}$ is called *saturated* or *local* if $\mathfrak{F}$ is closed under Frattini extensions; i.e., from $G/\Phi(G) \in \mathfrak{F}$ it follows that $G \in \mathfrak{F}$ (here $\Phi(G)$ is the Frattini subgroup of a group G, i.e., the intersection of all maximal subgroups in G).

It is proved in [6] that if $\mathfrak{F}$ is a saturated formation then the class of all soluble $C_\pi\mathfrak{F}$-groups is also a saturated formation. Later in [7] (also see [1, Theorem 16.2]) it was proved that the class of all π-isolated $C_\pi\mathfrak{F}$-groups is a saturated formation for any saturated formation $\mathfrak{F}$.

In [1], Shemetkov formulated the following

Conjecture [1, Problem 19]. *Suppose that π is a set of primes and $\mathfrak{F}$ is a saturated formation. Then $C_\pi\mathfrak{F}$ is a saturated formation.*

This conjecture was refuted in [10], where it was proved that, for $\pi = \{3, 5\}$ and the saturated formation $\mathfrak{N}$ of all nilpotent groups, the formation $C_\pi\mathfrak{N}$ is not saturated. Moreover, in [10, Theorem 2], for a saturated formation $\mathfrak{F}$ such that the class $C_\pi\mathfrak{F}$ is a formation, a criterion for the saturation of $C_\pi\mathfrak{F}$, and in [15, Theorems 1–3], it was proved that the class $C_\pi\mathfrak{F}$ is a formation for every formation $\mathfrak{F}$ and criteria are found for its partial saturation.

The objects dual to formations and saturated formations are Fitting classes and local Fitting classes defined in [16, 17]. Recall that a class of groups $\mathfrak{F}$ is called a Fitting class if $\mathfrak{F}$ is closed under normal

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subgroups and products of normal $\mathfrak{F}$-subgroups. In the theory of Fitting classes, the result exactly dual to [6] was obtained in [11], where it was proved that, for every set of primes π and every local Fitting class $\mathfrak{F}$, the class of all soluble $C_\pi\mathfrak{F}$-groups is also a local Fitting class. In [12, Theorem 3.3], this result was validated for the class of all π-soluble $C_\pi\mathfrak{F}$-groups and an analog of Shemetkov's Conjecture was formulated for local Fitting classes [12, Problem, p. 382]. Later in [18], it was proved that the indicated results of [11, 12] also hold for the case when the Fitting class $\mathfrak{F}$ is partially local.

The present article is devoted to finding the conditions under which an analog of Shemetkov's Conjecture is valid for the class of all $C_\pi\mathfrak{F}$-groups in the case when $\mathfrak{F}$ is a Fischer class. A Fitting class $\mathfrak{F}$ is called a *Fischer class* [19] if $\mathfrak{F}$ is closed under subgroups of the form PN, where P is a Sylow p-subgroup in a group $G \in \mathfrak{F}$ and N is its normal subgroup. We showed that every local Fitting class is a Fischer class, though the converse in general fails (see Theorem 3.2). One of the main results of the article is the proof of the fact that, for any set of primes π and any Fischer class $\mathfrak{F}$, the class of all π-soluble $C_\pi\mathfrak{F}$-groups is a Fischer class (Theorem 4.4). In Section 5 conditions were found under which, for a Fischer class $\mathfrak{F}$, the class of all $C_\pi\mathfrak{F}$-groups G such that G has an $\mathfrak{F}$-injector containing a Hall π-subgroup G is a Fischer class (Theorem 5.4).

We follow [1, 2] in terminology and notation.

2. Preliminaries

We will use the well-known result of group theory:

Theorem 2.1 (Chunikhin [20]). *Suppose that $\pi \subseteq P$ and G is a π-soluble group. Then*

(1) *G has at least one Hall π-subgroup;*

(2) *every two Hall π-subgroups in G are conjugate;*

(3) *every π-subgroup of G lies in its Hall π-subgroup.*

A class of groups $\mathfrak{F}$ is called a *Fitting class* if the following hold:

(i) from $G \in \mathfrak{F}$ and $N \trianglelefteq G$ it follows that $N \in \mathfrak{F}$;

(ii) if N_1, $N_2 \trianglelefteq G$ and N_1, $N_2 \in \mathfrak{F}$ then $N_1N_2 \in \mathfrak{F}$.

Condition (ii) in this definition implies that, for every nonempty Fitting class $\mathfrak{F}$, every group G has the greatest normal subgroup belonging to $\mathfrak{F}$, which is called the $\mathfrak{F}$-*radical* of G and denoted by $G_\mathfrak{F}$.

The *product* $\mathfrak{F}\mathfrak{H}$ of Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ is the class of groups $(G : G/G_\mathfrak{F} \in \mathfrak{H})$. It is well known that the product of every two Fitting classes is a Fitting class and the multiplication operation of Fitting classes is associative (for example, see [2, IX.1.12(a),(c)]).

Let $\mathfrak{F}$ be a Fitting class. Then a subgroup V in a group G is called

(1) $\mathfrak{F}$-*maximal in* G if $V \in \mathfrak{F}$ and from $V \le U \le G$, $U \in \mathfrak{F}$ it follows that $U = V$;

(2) an $\mathfrak{F}$-*injector of* G if $V \cap K$ is an $\mathfrak{F}$-maximal subgroup in K for every subnormal subgroup K in G.

Denote the (possibly empty) set of all $\mathfrak{F}$-injectors in a group G by $\mathrm{Inj}_\mathfrak{F}(G)$. The definition of $\mathfrak{F}$-injector immediately implies the properties that we will use:

Lemma 2.2. *Let G be a group and let $\mathfrak{F}$ be a Fitting class. Then*

(1) *if $V \in \mathrm{Inj}_\mathfrak{F}(G)$ and $K \trianglelefteq G$ then $V \cap K \in \mathrm{Inj}_\mathfrak{F}(K)$;*

(2) *if V is an $\mathfrak{F}$-maximal subgroup in G and $V \cap M \in \mathrm{Inj}_\mathfrak{F}(M)$ for every maximal normal subgroup M in G then $V \in \mathrm{Inj}_\mathfrak{F}(G)$.*

Denote by $\mathfrak{S}$ ($\mathfrak{S}^\pi$) the class of all soluble (all π-soluble) groups respectively and by $\mathfrak{E}_\pi$, the class of all π-groups. In particular, if π coincides with the set P of all primes then $\mathfrak{E}_\pi$ is equal to the class $\mathfrak{E}$ of all groups. Given a Fitting class $\mathfrak{F}$, denote by $\sigma(\mathfrak{F})$ the union of the sets of all prime divisors of the order of all groups in $\mathfrak{F}$. Note that, by Theorem 2.1, $\mathfrak{S}^\pi \cap C_\pi\mathfrak{F} = \mathfrak{S}^\pi \cap E_\pi\mathfrak{F}$ and these classes consist of all π-soluble groups for which the Hall π-subgroups belong to $\mathfrak{F}$. If G belongs to E_π, in particular, if it is Hall π-soluble, then the $\mathfrak{E}_\pi$-injectors of G are Hall π-subgroups in G (see [3, p. 328]).

In the theory of soluble groups, an elegant generalization in terms of Fitting classes of Hall's and Sylow's fundamental theorems was obtained by Fischer, Gaschütz, and Hartley [16], who proved that, for every Fitting class $\mathfrak{F}$ in any soluble group G, there exist $\mathfrak{F}$-injectors and every two of its $\mathfrak{F}$-injectors are conjugate. This result was first generalized by Shemetkov [21] and Sementovskiĭ [22]. These results, necessary for us in the sequel, are the contents of

Theorem 2.3 [21, Theorems 2.1 and 2.2; 22, Theorem, p. 168]. *Suppose that $\mathfrak{F}$ is a Fitting class and $\pi = \sigma(\mathfrak{F})$. Then*

(1) *if $G \in \{\mathfrak{S}^\pi, \mathfrak{F}\mathfrak{S}\}$ then G has a unique conjugacy class of an $\mathfrak{F}$-injector;*

(2) *if $G \in \mathfrak{S}^\pi$ then a subgroup V in G is an $\mathfrak{F}$-injector in G if and only if G has a subnormal series $G = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_t = 1$ each factor in which is either a π'-group or a nilpotent π-group such that $V \cap G_i$ is an $\mathfrak{F}$-maximal subgroup in G_i for each $i \in \{0, 1, \dots, t\}$.*

For constructing examples of Fitting classes of $C_\pi\mathfrak{F}$-groups, we will also use the operators * and $_*$, defined by Lockett [23]. Recall that, to every Fitting class $\mathfrak{F}$, the operator * assigns the least Fitting class $\mathfrak{F}^*$ containing $\mathfrak{F}$ such that $(G \times H)_{\mathfrak{F}^*} = G_{\mathfrak{F}^*} \times H_{\mathfrak{F}^*}$ for all groups G and H, and the operator $_*$ assigns to a Fitting class $\mathfrak{F}$ the Fitting class $\mathfrak{F}_* = \cap\{\mathfrak{X} : \mathfrak{X}$ is a Fitting class and $\mathfrak{X}^* = \mathfrak{F}^*\}$. A Fitting class $\mathfrak{F}$ is called a *Lockett class* if $\mathfrak{F} = \mathfrak{F}^*$. The properties of the Lockett operators are described by

Lemma 2.4 [2, X.1.15]. *If $\mathfrak{F}$ is a Fitting class then $(\mathfrak{F}_*)_* = \mathfrak{F}_* = (\mathfrak{F}^*)_* \subseteq \mathfrak{F} \subseteq \mathfrak{F}^* = (\mathfrak{F}_*)^* = (\mathfrak{F}^*)^*$.*

3. Local and Partially Hereditary Fitting Classes

Recall that each mapping $f : P \to \{\text{Fitting classes}\}$ is called a *Hartley function* or an *H-function.*

Suppose that $\pi = \mathrm{Supp}(f) = \{p \in P : f(p) \neq \varnothing\}$ and $LR(f) = \mathfrak{E}_\pi \cap \left(\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{E}_{p'}\right)$. A Fitting class $\mathfrak{F}$ is called *local* [17] if there exists an H-function f such that $\mathfrak{F} = LR(f)$. Note that the family of these classes is large since, by the theorem of [24], every soluble hereditary Fitting class, i.e., a Fitting class closed under taking subgroups, is local.

In [19], there are defined partially hereditary Fitting classes, which, by Hartley's suggestion, were named *Fischer classes.*

Definition 3.1 [25]. A Fitting class $\mathfrak{F}$ is called a *Fischer class* if, from the fact that $G \in \mathfrak{F}$, $K \trianglelefteq G$, $K \le H \le G$, and H/K is a p-group for some prime p, it follows that $H \in \mathfrak{F}$.

Obviously, each hereditary Fitting class is a Fischer class. In particular, the set of all Fischer classes contains all soluble hereditary local Fitting classes. The interrelation between local Fitting classes and Fischer classes is disclosed by

Theorem 3.2. *The following hold:*

(1) *every local Fitting class is a Fischer class;*

(2) *there exist Fischer classes that are not local Fitting classes.*

Proof. (1) Suppose that $\mathfrak{F} = LR(f)$ for some H-function f with support π. Then $\mathfrak{F} = \mathfrak{E}_\pi \cap \mathfrak{D}$, where $\mathfrak{D} = \bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{E}_{p'}$. Since the Fitting class $\mathfrak{E}_\pi$ is hereditary and the intersection of each set of Fischer classes is a Fischer class, for proving (1) it suffices to find out that, for each prime $p \in \pi$, the Fitting class $\mathfrak{X}_p = f(p)\mathfrak{N}_p\mathfrak{E}_{p'}$ is a Fischer class. Assume that $G \in \mathfrak{X}_p$, $K \trianglelefteq G$, $K \le H \le G$, and H/K is a q-group for some prime q. Prove that $H \in \mathfrak{X}_p$. Consider the two open possibilities:

1. $p \neq q$. In this case $H/K \in \mathfrak{E}_{p'}$. Since $K \in \mathfrak{X}_p$; therefore, $K \le H_{\mathfrak{X}_p}$. Consequently, by the isomorphism $(H/K)/(H_{\mathfrak{X}_p}/K) \cong H/H_{\mathfrak{X}_p}$, we have $H/H_{\mathfrak{X}_p} \in \mathfrak{E}_{p'}$. This, with account taken of the associativity of the multiplication of Fitting classes, implies that $H \in \mathfrak{X}_p\mathfrak{E}_{p'} = \mathfrak{X}_p$, and (1) of the theorem holds. It remains to consider the case

2. $p = q$. Let H_p be a Sylow p-subgroup in H. Since G is an $\mathfrak{X}_p$-group, the quotient group $G/G_{f(p)\mathfrak{N}_p}$ belongs to $\mathfrak{E}_{p'}$. Therefore, a Sylow p-subgroup in G is also a Sylow p-subgroup in $G_{f(p)\mathfrak{N}_p}$. Consequently, by Sylow's Theorem, $H_p \le G_{f(p)\mathfrak{N}_p}$; thus, $[K, H_p] \le K \cap G_{f(p)\mathfrak{N}_p} = K_{f(p)\mathfrak{N}_p}$. Hence, $H_pK_{f(p)\mathfrak{N}_p}$ is a normal subgroup in H_pK. From $H/K \in \mathfrak{N}_p$ we obtain $H_pK = H$. Moreover, $H_pK_{f(p)\mathfrak{N}_p}/K_{f(p)\mathfrak{N}_p} \in \mathfrak{N}_p$. So, $H_pK_{f(p)\mathfrak{N}_p} \in (f(p)\mathfrak{N}_p)\mathfrak{N}_p = f(p)\mathfrak{N}_p$, and hence $H_pK_{f(p)\mathfrak{N}_p} \le H_{f(p)\mathfrak{N}_p}$. Note also that $H_pK/H_pK_{f(p)\mathfrak{N}_p} = H_pK_{f(p)\mathfrak{N}_p}K/H_pK_{f(p)\mathfrak{N}_p} \cong K/K \cap H_pK_{f(p)\mathfrak{N}_p} = K/K_{f(p)\mathfrak{N}_p}K_p$. Therefore, $H/H_pK_{f(p)\mathfrak{N}_p} \in \mathfrak{E}_{p'}$. Then, by the isomorphism $(H/H_pK_{f(p)})/(H_{f(p)\mathfrak{N}_p}/H_pK_{f(p)}) \cong H/H_{f(p)\mathfrak{N}_p}$, we obtain $H \in \mathfrak{X}_p$, and (1) is proved.

(2) For proving the existence of Fischer classes that are not local Fitting classes, use the constructions of classes of groups that were studied by Hauck in the class $\mathfrak{S}$ of all soluble groups in [26, 27] (also see [2, IX.2.5(a),(b)]). Let E_z be the operation assigning to every class of groups $\mathfrak{X}$ the class of groups $\mathrm{E}_z\,\mathfrak{X} = (G \in \mathfrak{S} : \exists N \trianglelefteq G,\ N \le Z_\infty(G)$, and $G/N \in \mathfrak{X})$. Then, by Lemma 2.2 in [26], E_z is a closure operation, i.e., $\mathfrak{X} \subseteq \mathrm{E}_z\,\mathfrak{X}$, from $\mathfrak{X} \subseteq \mathfrak{Y}$ it follows that $\mathrm{E}_z\,\mathfrak{X} \subseteq \mathrm{E}_z\,\mathfrak{Y}$, and $\mathrm{E}_z(\mathrm{E}_z\,\mathfrak{X}) = \mathrm{E}_z\,\mathfrak{X}$.

Let p be a prime and let $\mathfrak{F}$ be the least of E_z-closed Fitting classes such that $\mathfrak{N}_p\mathfrak{F} = \mathfrak{S}$. As was established in [26, Subsection 4.3] (also see Theorem 5.9(1,2) in [27]), $\mathfrak{F}$ is a Fischer class and $\mathfrak{F} = (G \in \mathfrak{S} : G/C_G(O_p(G)) \in \mathfrak{N}_p)$. Show that the Fitting class $\mathfrak{F}$ is not local. Suppose on the contrary that $\mathfrak{F}$ is a local Fitting class. Then $\mathfrak{F} = \mathfrak{S}_\pi \cap \left(\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'}\right)$ for some H-function f with support π. Suppose that, for some $q \in \pi$, the value of the H-function $f(q)$ is such that $\omega = \sigma(f(q)) \neq P$. Hence $\mathfrak{F} \subseteq \mathfrak{S}_\omega\mathfrak{N}_q\mathfrak{E}_{q'}$ and $\mathfrak{S}_\omega\mathfrak{N}_q\mathfrak{E}_{q'} \neq \mathfrak{S}$. Choose a group G of the least order in the class $\mathfrak{S} \setminus \mathfrak{S}_\omega\mathfrak{N}_q\mathfrak{E}_{q'}$. Let $\Gamma = Z_r \wr G$ be the regular wreath product of a cyclic group of order $r \in \pi$ and G with $r \neq p \in \pi$. Then $\Gamma = N \leftthreetimes G$, where $N = \underbrace{Z_r \times \ldots \times Z_r}_{|G|}$.

Since $O_p(\Gamma) = 1$, we obviously have $\Gamma \in \mathfrak{F}$. Consequently, G is a homomorphic image of an $\mathfrak{F}$-group and $\Gamma/N \cong G \in \mathfrak{S}_\omega\mathfrak{N}_q\mathfrak{S}_{\ q'}$. The so-obtained contradiction is strengthened by the fact that each value of the H-function f is a Fitting class of full characteristic. Hence, $\mathfrak{N} \subseteq f(p)$ for each prime $p \in \pi$; therefore, $\mathfrak{N}^2 \subseteq \mathfrak{D} = \bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'}$. Suppose that E_p^r is the group obtained as the extension of a minimal normal p-subgroup by a group of order r. Then $E_p^r \in \mathfrak{S}_\pi \cap \mathfrak{D} = \mathfrak{F}$, which is impossible. The so-obtained contradiction proves (2). The theorem is proved.

4. Fischer Classes of π-Soluble $C_\pi\mathfrak{F}$-Groups

Recall that if π is a subset in the set of all primes P then the symbols G_π and $\mathrm{Hall}_\pi(G)$ stand for a Hall π-subgroup of a group G and the set of all Hall π-subgroups in G respectively.

Lemma 4.1. *Suppose that K_1 and K_2 are normal subgroups of a group $G \in \mathfrak{S}^\pi$ and $G_\pi \in \mathrm{Hall}_\pi(G)$. Then*

(1) $K_1K_2 \cap G_\pi = (K_1 \cap G_\pi)(K_2 \cap G_\pi)$;

(2) $K_1G_\pi \cap K_2G_\pi = (K_1 \cap K_2)G_\pi$.

Proof. (1) Obviously, $(K_1 \cap G_\pi)(K_2 \cap G_\pi) \le K_1K_2 \cap G_\pi$. Since $K_i \cap G_\pi$ is a Hall π-subgroup in K_i, we infer that $|K_i \cap G_\pi| = |K_i|_\pi$ for $i \in \{1, 2\}$. Analogously, $|K_1K_2 \cap G_\pi| = |K_1K_2|_\pi$.

Consequently, $|(K_1 \cap G_\pi)(K_2 \cap G_\pi)| = |K_1|_\pi|K_2|_\pi/|K_1 \cap K_2|_\pi = |K_1K_2|_\pi = |K_1K_2 \cap G_\pi|$. This implies the validity of equality in (1).

Item (2) follows from (1) by Lemma A.1.2 in [2]. The lemma is proved.

The construction of Fitting classes of π-soluble $C_\pi\mathfrak{F}$-groups is described by

Lemma 4.2. *For every set of primes π and every Fitting class $\mathfrak{F}$, the class of all π-soluble $C_\pi\mathfrak{F}$-groups is a Fitting class.*

Proof. Let $\mathfrak{C} = (G \in \mathfrak{S}^\pi : \mathrm{Hall}_\pi(G) \subseteq \mathfrak{F})$. If $\mathfrak{F} = \varnothing$ then $\mathfrak{C} = \varnothing$, and the lemma is obvious. Furthermore, for the cases when $\pi = \varnothing$ and $\pi = P$, we have $\mathfrak{C} = \mathfrak{S}^\pi$ and $\mathfrak{C} = \mathfrak{F}$ respectively. Consequently, $\mathfrak{C}$ is a Fitting class in these cases.

Suppose that $\mathfrak{F} \neq \varnothing$ and $\varnothing \subset \pi \subset$P. Assume that $G \in \mathfrak{C}$ and $K \trianglelefteq G$. Then, by Theorem 2.1, the π-solubility of G implies that in G there exists a Hall π-subgroup G_π which is an $\mathfrak{F}$-subgroup in G. Since $G_\pi \cap K \in \mathrm{Hall}_\pi(K)$ and $K = G_\pi \cap K \trianglelefteq G_\pi$, we have $K_\pi \in \mathfrak{F}$. Therefore, $G \in \mathfrak{C}$ by the conjugacy of Hall π-subgroups in K.

Let $G = NM$, where $N \trianglelefteq G$ and $M \trianglelefteq G$, while N and M are $\mathfrak{C}$-subgroups in G. Prove that $G \in \mathfrak{C}$. Since $N \in \mathfrak{S}^\pi$, $M \in \mathfrak{S}^\pi$, and $\mathfrak{S}^\pi$ is a Fitting class, $G \in \mathfrak{S}^\pi$. Consequently, by Theorem 2.1, in G there exists a Hall π-subgroup G_π. Then $G_\pi \cap N \in \mathrm{Hall}_\pi(N)$ and $G_\pi \cap M \in \mathrm{Hall}_\pi(M)$. Moreover, $G_\pi \cap N$ and $G_\pi \cap M$ are normal $\mathfrak{F}$-subgroups in G_π. Consequently, by Lemma 4.1, $(G_\pi \cap N)(G_\pi \cap M) = G_\pi \cap MN = G_\pi$ and $G_\pi \in \mathfrak{F}$. By the conjugacy of Hall π-subgroups in G, we have $G \in \mathfrak{C}$. The lemma is proved.

Lemma 4.3. *For every set of primes π and every Fitting class $\mathfrak{F}$, the Fitting class $\mathfrak{C}$ of all π-soluble $C_\pi\mathfrak{F}$-groups is π-saturated, i.e., $\mathfrak{C}\mathfrak{E}_{\pi'} = \mathfrak{C}$.*

Proof. If $\mathfrak{F}$ is an empty Fitting class or $\pi \in \{\varnothing,\mathrm{P}\}$ then, obviously, the Fitting class $\mathfrak{C}$ is π-saturated. Assume that $\mathfrak{F} \neq \varnothing$, $\varnothing \subset \pi \subset \mathrm{P}$, and $G \in \mathfrak{C}\mathfrak{E}_{\pi'}$. Observe that $G_\mathfrak{C}$ and $G/G_\mathfrak{C}$ are π-soluble. Consequently, G is π-soluble.

Let H be a Hall π-subgroup in $G_\mathfrak{C}$. Since $|G : G_\mathfrak{C}|$ is a π'-number, H is a Hall π-subgroup in G. But $G_\mathfrak{C} \in \mathfrak{C}$. Therefore, $H = G_\pi \in \mathfrak{F}$. Consequently, $G \in \mathfrak{C}$, and $\mathfrak{C}\mathfrak{E}_{\pi'} \subseteq \mathfrak{C}$. The reverse inclusion is obvious. The lemma is proved.

The validity of the analog of Shemetkov's Conjecture (see the Introduction) for the class of all π-soluble $C_\pi\mathfrak{F}$-groups in the case when $\mathfrak{F}$ is a Fischer class is confirmed by

Theorem 4.4. *Let $\pi \subseteq \mathrm{P}$. Then for every Fischer class $\mathfrak{F}$ the class of all π-soluble $C_\pi\mathfrak{F}$-groups is a Fischer class.*

Proof. If $\mathfrak{F} = \varnothing$ or $\pi \in \{\varnothing, \mathrm{P}\}$ then the theorem is obvious. Suppose that $\varnothing \subset \pi \subset \mathrm{P}$, $\mathfrak{F} \neq \varnothing$, and $\mathfrak{C} = C_\pi\mathfrak{F} \cap \mathfrak{S}^\pi$.

Assume that $G \in \mathfrak{C}$, $K \trianglelefteq G$, $K \le H \le G$, and H/K is a p-group for some prime p. To establish the membership of H in $\mathfrak{C}$, consider the two possible cases.

Case 1. $p \in \pi'$. Since, by Lemma 4.2, $\mathfrak{C}$ is a Fitting class, $K \in \mathfrak{C}$. Then $K \trianglelefteq H$ implies that $K \le H_\mathfrak{C}$. By hypothesis, $H/K \in \mathfrak{N}_p \subseteq \mathfrak{E}_{\pi'}$, and the class of groups $\mathfrak{E}_{\pi'}$ is a formation. Consequently, $H/H_\mathfrak{C} \in \mathfrak{E}_{\pi'}$ in view of the isomorphism $(H/K)/(H_\mathfrak{C}/K) \cong H/H_\mathfrak{C}$. Since, by Lemma 4.3, $\mathfrak{C}$ is π-saturated, $H \in \mathfrak{C}$, and the theorem holds in Case 1.

Case 2. $p \in \pi$. In this case $H/K \in \mathfrak{N}_p \subseteq \mathfrak{S}_\pi$. Since H is π-soluble, by Theorem 2.1, in H there exists a Hall π-subgroup H_π. Consequently, $H_\pi K/K \in \mathrm{Hall}_\pi(H/K)$ and $H = H_\pi K$. Hence, $H/K \cong H_\pi/H_\pi \cap K$, and $H_\pi/H_\pi \cap K$ is a p-group. Involving the normality of K in G and H, we obtain the equality $G_\pi \cap K = H_\pi \cap K$ valid for some Hall π-subgroup G_π in G. Moreover, from $G \in \mathfrak{C}$ it follows that $G_\pi \in \mathfrak{F}$. Form the conditions $G_\pi \in \mathfrak{F}$, $H_\pi \cap K \trianglelefteq G_\pi$, $H_\pi \cap K \le H_\pi \le G_\pi$, and $H_\pi/H_\pi \cap K \in \mathfrak{N}_p$ we infer that $H_\pi \in \mathfrak{F}$. Hence, $H \in \mathfrak{C}$. The theorem is proved.

Recall that a soluble Fitting class $\mathfrak{F}$ is called normal if an $\mathfrak{F}$-injector of any group $G \in \mathfrak{S}$ is a normal subgroup in G. For an arbitrary Fitting class $\mathfrak{F}$, the class of all soluble $C_\pi\mathfrak{F}$-groups is not a Fischer class. This is confirmed by the following

Example 4.5. Suppose that $\pi = \mathrm{P}$ and $\mathfrak{F} = \mathfrak{S}_*$ is the least of the nonunit normal Fitting classes. Then

$$\mathfrak{C} = C_\pi(\mathfrak{S}_*) = (G \in \mathfrak{S} : \mathrm{Hall}_\pi(G) \subseteq \mathfrak{S}_*) = \mathfrak{S}_*.$$

Suppose on the contrary that $\mathfrak{C}$ is a Fischer class. Then, by [2,Theorem X.1.25], $\mathfrak{C}$ is a Lockett class, i.e., $\mathfrak{C}^* = \mathfrak{C}$. Granted the properties of the Lockett operators, by Lemma 2.4 we have

$$\mathfrak{C}^* = (\mathfrak{S}_*)^* = \mathfrak{S}^* = \mathfrak{S} = \mathfrak{C} = \mathfrak{S}_*.$$

But it was proved in [28] that the three-symbol symmetric group S_3 belongs to $\mathfrak{S}\backslash\mathfrak{S}_*$. By the obtained contradiction, $\mathfrak{C}$ is not a Fischer class.

5. On Fischer Classes of $D^\pi_\mathfrak{F}$-Groups

Let $\mathfrak{F}$ be a Fitting class. By analogy with the notation in [4], we say that a group G *has Property* $E_\mathfrak{F}$ if G has at least one $\mathfrak{F}$-injector and call G an *$E_\mathfrak{F}$-group* in this case. If G is an $E_\mathfrak{F}$-group and every two $\mathfrak{F}$-injectors G are conjugated then we say that G has *Property* $C_\mathfrak{F}$ and call G a $C_\mathfrak{F}$- group. The symbols $E_\mathfrak{F}$ and $C_\mathfrak{F}$ will be also used for denoting the classes of $E_\mathfrak{F}$- and $C_\mathfrak{F}$-groups respectively. If $\mathfrak{F}$ is equal to the Fitting class $\mathfrak{E}_\pi$ of all π-groups then, by [3, Theorem 7.2.33], $E_\mathfrak{F}$ is equal to the Fitting class $\mathfrak{E}$ of all groups though E_π is different from $\mathfrak{E}$ and is not a Fitting class (see [29]).

Definition 5.1. Suppose that $\mathfrak{F}$ is a Fitting class and $\pi \subseteq P$. Define the *class of groups* $D^\pi_\mathfrak{F}$ as follows: $G \in D^\pi_\mathfrak{F}$ if and only if G has properties C_π and $C_\mathfrak{F}$ simultaneously and every $\mathfrak{F}$-injector in G contains a Hall π-subgroup of G.

Let $\mathfrak{X}$ be a class of groups. Then denote by $\mathfrak{X}^\pi$ the class of π-soluble $\mathfrak{X}$-groups, i.e., $\mathfrak{X}^\pi = \mathfrak{X} \cap \mathfrak{S}^\pi$.

Lemma 5.2. *Suppose that $\mathfrak{F}$ is a Fitting class, $\omega = \sigma(\mathfrak{F})$, and $\pi \subseteq P$. Then the following hold:*
(1) $D^\pi_\mathfrak{F} \cap (\mathfrak{F}\mathfrak{S})^\pi$ is a Fitting class;
(2) if $\pi \subseteq \omega$ then the class of all ω-soluble $D^\pi_\mathfrak{F}$-groups is a Fitting class.

Proof. Suppose that $\mathfrak{D}_1 = \mathfrak{D}^\pi_\mathfrak{F} \cap (\mathfrak{F}\mathfrak{S})^\pi$ and $\mathfrak{D}_2 = \mathfrak{D}^\pi_\mathfrak{F} \cap \mathfrak{S}^\omega$ for $\pi \subseteq \omega$. Using Theorems 2.1 and 2.3, we easily see that each of the classes $\mathfrak{D}_1$ and $\mathfrak{D}_2$ consists of those and only those groups the indices of whose $\mathfrak{F}$-injectors are π'-numbers.

Assume that $G \in \mathfrak{D}_i$ and $K \trianglelefteq G$, where $i \in \{1, 2\}$. Then, by Theorem 2.3, in G there exists an $\mathfrak{F}$-injector V and the index $|G : V|$ is a π'-number. By Lemma 2.2(1), $V \cap G \in \mathrm{Inj}_\mathfrak{F}(K)$. Since $|VK| = |V| \cdot |K|/|V \cap K|$, we have $|K : V \cap K| = |G : V|/|G : VK|$. Consequently, $|K : V \cap K|$ is a π'-number and $K \in \mathfrak{D}_i$.

Let K_1 and K_2 be normal $\mathfrak{D}_i$-subgroups in G and $G = K_1K_2$. Since, for $i \in \{1, 2\}$, K_i belongs to the Fitting classes $(\mathfrak{F}\mathfrak{S})^\pi$ and $\mathfrak{S}^\omega$, the group G belongs to each of these classes. Hence, by Theorem 2.3, in G there exists an $\mathfrak{F}$-injector V. Then, by Lemma 2.2(1), the subgroups $V \cap K_1$ and $V \cap K_2$ are $\mathfrak{F}$-injectors in K_1 and K_2 respectively. Show that $|G : V|$ is a π'-number. Indeed,

$$k = |G : V| = |K_1| \cdot |K_2|/|V| \cdot |K_1 \cap K_2| = |K_1 : V_1| \cdot |K_2 : V_2| \cdot |V_1| \cdot |V_2|/|V| \cdot |K_1 \cap K_2|.$$

Hence,

$$\begin{aligned} k &= |K_1 : V_1| \cdot |K_2 : V_2| \cdot |V_1V_2| \cdot |V_1 \cap V_2|/|V| \cdot |K_1 \cap K_2| \\ &= |K_1 : V_1| \cdot |K_2 : V_2| \cdot |(V_1 \cap K_1)(V_2 \cap K_2)| \cdot |V \cap (K_1 \cap K_2)|/|V| \cdot |K_1 \cap K_2|. \end{aligned}$$

Consequently,

$$k = |K_1 : V_1| \cdot |K_2 : V_2|/|V : (V_1 \cap K_1)(V_2 \cap K_2)| \cdot |(K_1 \cap K_2) : (V \cap (K_1 \cap K_2))|.$$

Since by hypothesis the indices $|K_1 : V_1|$ and $|K_2 : V_2|$ are π'-numbers, k is a π'-number and $G \in \mathfrak{D}_i$. The lemma is proved.

For proving the main result of this section, we first establish a test for an $\mathfrak{F}$-injector in a π-soluble group.

Lemma 5.3. *Suppose that $\mathfrak{F}$ is a Fitting class, $\pi = \sigma(\mathfrak{F})$, and $G \in \mathfrak{S}^\pi$. If a normal subgroup K in G and an $\mathfrak{F}$-subgroup V in G are such that $V \cap K \in \mathrm{Inj}_\mathfrak{F}(K)$ and $VK = G$ then $V \in \mathrm{Inj}_\mathfrak{F}(G)$.*

Proof. Since G is π-soluble, there exists a series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G$ in which $G_k = K$ for some k and each factor G_i/G_{i-1} is either a π'-group or a nilpotent π-group for $i \in \{1, 2, \dots, n\}$. Therefore, by Theorem 2.3(2), for proving the lemma, it suffices to check that $G_j \cap V$ is an $\mathfrak{F}$-maximal subgroup in G_j for each $j \in \{0, 1, \dots, n\}$. Since $V \cap K$ is an $\mathfrak{F}$-injector in K, we see that $V \cap G_j$ is an $\mathfrak{F}$-maximal subgroup in G_j for $0 \le j \le k$.

Assume that $k < j \le n$. Suppose that $G_j \cap V < F \le G$ and $F \in \mathfrak{F}$. Since $G = VK$, by the Dedekind identity we have $G_j = G_j \cap KV = K(G_j \cap V)$. So, applying the Dedekind identity once again, we infer

$$(G_j \cap V)(F \cap K) = F \cap K(G_j \cap V) = F \cap G_j = F.$$

Since by assumption $G_j \cap V < F$, we have $V \cap K \le F \cap K$. Consider the two cases:

Case 1. $V \cap K = F \cap K$. In this case $F = (G_j \cap V)(V \cap K) = G_j \cap V$. We get a contradiction to the assumption $G_j \cap V < F$. It remains to settle

Case 2. $V \cap K < F \cap K$. Since $F \cap K \trianglelefteq F$ and $F \in \mathfrak{F}$, $F \cap K$ is an $\mathfrak{F}$-subgroup containing $V \cap K$. This contradicts the fact that $V \cap K$ is an $\mathfrak{F}$-injector in K, and hence is $\mathfrak{F}$-maximal in K. Therefore, $V \cap G_j$ is an $\mathfrak{F}$-maximal subgroup in G_j for $k < j \le n$. The lemma is proved.

The construction of Fischer classes by means of the class of $D^\pi_\mathfrak{F}$-groups is given by

Theorem 5.4. *Suppose that $\mathfrak{F}$ is a Fischer class, $\omega = \sigma(\mathfrak{F})$, and $\pi \subseteq P$. Then*
(1) $D^\pi_\mathfrak{F} \cap (\mathfrak{F}\mathfrak{S})^\pi$ is a Fischer class;
(2) if $\pi \subseteq \omega$ then the class of all ω-soluble $D^\mathfrak{F}_\pi$-groups is a Fischer class.

Proof. Let $\mathfrak{F}_1 = D^{\pi}_{\mathfrak{F}} \cap (\mathfrak{F}\mathfrak{S})^{\pi}$ and let $\mathfrak{F}_2$ be the class of all ω-soluble $D^{\pi}_{\mathfrak{F}}$-groups, where $\pi \subseteq \omega$. By Lemma 5.2, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are Fitting classes.

Suppose that $G \in \mathfrak{F}_i$ and $K \trianglelefteq G$, where $i \in \{1,2\}$. Prove that if $K \leq H \leq G$ and H/K is a p-group for some prime p then $H \in \mathfrak{F}_i$.

Let us prove (1). Observe first that $H/K \in \mathfrak{N}_p \subset \mathfrak{S}$. Since by Lemma 5.2 $\mathfrak{F}_1$ is a Fitting class, $K \in \mathfrak{F}_1$. Therefore, $K \leq H_{\mathfrak{F}_1}$. Involving the isomorphism $(H/K)/(H_{\mathfrak{F}_1}/K) \cong H/H_{\mathfrak{F}_1}$, we infer that $H/H_{\mathfrak{F}_1} \in \mathfrak{S}$. Moreover, $H_{\mathfrak{F}_1} \in \mathfrak{F}\mathfrak{S}$, and so $H_{\mathfrak{F}_1} \leq H_{\mathfrak{F}\mathfrak{S}}$. In view of the isomorphism $(H/H_{\mathfrak{F}_1})/(H_{\mathfrak{F}\mathfrak{S}}/H_{\mathfrak{F}_1}) \cong H/H_{\mathfrak{F}\mathfrak{S}}$ and the associativity of multiplication of Fitting classes, we infer $H \in (\mathfrak{F}\mathfrak{S})\mathfrak{S} = \mathfrak{F}\mathfrak{S}$. Moreover, $G \in \mathfrak{F}_1$, and so $H \in \mathfrak{S}^{\pi}$. Consequently, $H \in (\mathfrak{F}\mathfrak{S})^{\pi}$, and by Theorems 2.3 and 2.1 in H there exist a unique class of $\mathfrak{F}$-injectors and a unique class of conjugate Hall π-subgroups. Consider the two possibilities:

1.1. $p \in \pi'$. In this case $H/K \in \mathfrak{S}_{\pi'}$. Let F be an $\mathfrak{F}$-injector in H. Then, by Lemma 2.2(1), the subgroup $F \cap K$ is an $\mathfrak{F}$-injector in K. Since $K \in \mathfrak{F}_1$, K is a $D^{\pi}_{\mathfrak{F}}$-group; therefore, $F \cap K$ contains a Hall π-subgroup in K. Consequently, there is a Hall π-subgroup H_π in H such that $H_\pi = H_\pi \cap K \leq F \cap K \leq F$. Hence, $H \in \mathfrak{F}_1$, and the theorem holds in case 1.1.

1.2. $p \in \pi$. Since $G \in \mathfrak{F}_1$, there exists an $\mathfrak{F}$-injector V in G containing a Hall π-subgroup of G. Consequently, by Theorem 2.1(3), V also contains a Hall π-subgroup H_π of H. Since H/K is a p-group, by the isomorphism $(V \cap H)K/K \cong (V \cap H)/(V \cap H \cap K)$ we infer that $(V \cap H)/(V \cap K) \in \mathfrak{N}_p$. Moreover, $V \cap K \trianglelefteq V$. Thus, $V \in \mathfrak{F}$, $V \cap K \leq V \cap H \leq V$, and $(V \cap H)/(V \cap K) \in \mathfrak{N}_p$. Consequently, $V \cap H \in \mathfrak{F}$.

Prove the $\mathfrak{F}$-maximality of the subgroup $V \cap H$ in H. Suppose that $V \cap H \leq R \leq H$ and $R \in \mathfrak{F}$. Then $V \cap H \cap K \leq R \cap K$. Since $R \cap K \trianglelefteq R$, the intersection $R \cap K$ is an $\mathfrak{F}$-subgroup in K. Furthermore, the subgroup $(V \cap H) \cap K = V \cap K$ is an $\mathfrak{F}$-injector in K, and hence $V \cap H$ is an $\mathfrak{F}$-maximal subgroup in K. Consequently, $(V \cap H) \cap K = R \cap K$. Since H/K is a π-group, $H = H_\pi K$. Now, $V \geq H_\pi$ implies that $H = (V \cap H)K$. Therefore, applying the Dedekind identity, we obtain $R = R \cap (V \cap H)K = (V \cap H)(R \cap K)$ and $R = (V \cap H)((V \cap H)K) = V \cap H$. This proves the $\mathfrak{F}$-maximality of $V \cap H$ in H.

Thus, the following conditions are fulfilled for $H, V \cap H$ and K:

(i) $H \in \mathfrak{F}\mathfrak{S}$ and $H/K \in \mathfrak{N}$;

(ii) the subgroup $V \cap H$ is $\mathfrak{F}$-maximal in H and $(V \cap H) \cap K \in \mathrm{Inj}_{\mathfrak{F}}(K)$.

Hence, by Lemma 2.5 in [30], the subgroup $V \cap H$ is an $\mathfrak{F}$-injector in H. Moreover, $V \cap H \geq H_\pi$. Consequently, $H \in \mathfrak{F}_1$, and the first claim of the theorem is proved.

(2) Since $\pi \subseteq \omega$, any group in $\mathfrak{F}_2$ is π-soluble. Arguing as in the case of the proof of (1), we conclude that the following conditions are fulfilled for the $\mathfrak{F}$-injector V of G and its subgroups K and H:

(j) $(V \cap H)K = H \in \mathfrak{S}^{\omega}$;

(jj) $V \cap H$ is an $\mathfrak{F}$-subgroup H and $(V \cap H) \cap K \in \mathrm{Inj}_{\mathfrak{F}}(K)$.

By Lemma 4.3, $V \cap H \in \mathrm{Inj}_{\mathfrak{F}}(H)$. Following the proof of (1), we have $V \cap H \geq H_\pi$. Therefore, $H \in \mathfrak{F}_2$. The theorem is proved.

If a Fitting class $\mathfrak{F}$ is not a Fischer class then the Fitting class of all soluble $\mathfrak{D}^{\pi}_{\mathfrak{F}}$-groups is not a Fischer class in general. This is confirmed by

Example 5.5. Let $\mathfrak{F} = (G \in \mathfrak{S} : \mathrm{Soc}_3(G) \leq Z(G))$, where $\mathrm{Soc}_3(G)$ is the 3-socle of G, i.e., the product of all its minimal normal 3-subgroups. Then, by [2, Theorem IX.2.8], $\mathfrak{F}$ is a Fitting class and is not a Fischer class (see Example IX.3.7(a) in [2]). Suppose that $\pi = \{2\}$. Then, by Lemma 5.2(1), the class of groups $\mathfrak{H} = \mathfrak{D}^{\pi}_{\mathfrak{F}}$ is a Fitting class. In this case, as was shown in Example IX.3.15 of [2], there exists a group $G \in \mathfrak{S}$ the Sylow 2-subgroup of an $\mathfrak{H}$-injector of which is not a Sylow 2-subgroup for every normal subgroup in G. This means that the Fitting class $\mathfrak{H}$ is not normally embedded (see [2, Definition IX.3.3]). Consequently, by [2, Theorem IX.3.4(a)], the Fitting class $\mathfrak{H}$ is not a Fischer class.

In conclusion, we observe that every hereditary Fitting class is a Fischer class. Nevertheless, the heredity of a Fischer class $\mathfrak{F}$ in general does not imply the heredity of the class $\mathfrak{H}$ of all soluble $\mathfrak{D}^{\pi}_{\mathfrak{F}}$-groups. Indeed, if $\mathfrak{F}$ is equal to the class $\mathfrak{N}$ of all nilpotent groups and $\pi = \{3\}'$ then $\mathfrak{F}$ is a hereditary Fischer class and, by Theorem 5.4, the class of groups $\mathfrak{H}$ is a Fischer class. But $\mathfrak{H}$ is not hereditary (see a remark to Theorem IX.3.8(a) in [2]).

References

1. *Shemetkov L. A.*, Formations of Finite Groups [in Russian], Nauka, Moscow (1978).
2. *Doerk K. and Hawkes T.*, Finite Soluble Groups, Walter de Gruyter, Berlin and New York (1992).
3. *Ballester-Bolinches A. and Ezquerro L. M.,* Classes of Finite Groups, Springer-Verlag, Dordrecht (2006) (Math. Appl.; V. 584).
4. *Hall P.*, "Theorems like Sylow's," Proc. London Math. Soc., **3**, No. 22, 286–304 (1956).
5. *Lockett P.,* "On the theory of Fitting classes of finite soluble groups," Math. Z., Bd 131, Heft 3, 103–115 (1973).
6. *Blessenohl D.,* "Über Formationen und Halluntergruppen endlicher auflösbaren Gruppen," Math. Z., Bd 142, Heft 3, 299–300 (1975).
7. *Slepova L. M.*, "On formations of $E^{\mathfrak{F}}$-groups," Dokl. Akad. Nauk BSSR, **21**, No. 7, 587–589 (1977).
8. *Doerk K. and Hawkes T.,* "Ein Beispiel aus der Theorie der Schunkklassen," Arch. Math. (Basel), Bd 30, 539–544 (1978).
9. *Hauck P.*, "Eine Bemerkung zur kleinsten normalen Fittingklasse," J. Algebra, **53**, No. 4, 395–401 (1978).
10. *Vasil′ev A. F. and Shemetkov L. A.*, "Nonlocal formations of finite groups," Dokl. NAN Belarusi, **39**, No. 4, 5–8 (1995).
11. *Zagurskiĭ V. N. and Vorob′ev N. T.*, "Fitting classes with given properties of Hall subgroups," Math. Notes, **78**, No. 2, 213–218 (2005).
12. *Guo W. and Li B.*, "On the Shemetkov problem for Fitting classes," Beiträge Algebra Geom., **48**, No. 1, 281–289 (2007).
13. *Mechovich A. P., Vorob′ev N. N., and Vorob′ev N. T.*, "Hall operators on the set of formations of finite groups," Algebra Discrete Math., **9**, No. 1, 72–78 (2010).
14. *Vdovin E. P. and Revin D. O.*, "A conjugacy criterion for Hall subgroups in finite groups," Siberian Math. J., **51**, No. 3, 402–409 (2010).
15. *Vdovin E. P., Revin D. O., and Shemetkov L. A.*, "Formations of finite C_π-groups," St. Petersburg Math. J., **24**, No. 1, 29–37 (2012).
16. *Fischer B., Gaschütz W., and Hartley B.,* "Injektoren endlicher auflösbarer Gruppen," Math. Z, Bd 102, Heft 5, 337–339 (1967).
17. *D'Arcy P.*, "Locally defined Fitting classes," J. Austral. Math. Soc., **20**, No. 1, 25–31 (1975).
18. *Vit′ko E. A. and Vorob′ev N. T.,* "On classes of Fitting classes and Hall's subgroups of finite π-soluble groups," Vestsi NAN Belarusi Ser. Fiz.-Mat., No. 1, 37–42 (2011).
19. *Fischer B.,* Klassen konjugierter Untergruppen in endlichen auflösbaren Gruppen. Habilitationschrift, Univ. Frankfurt (M), Frankfurt (1966).
20. *Chunikhin S. A.*, Subgroups of Finite Groups [in Russian], Nauka i Tekhnika, Minsk (1964).
21. *Shemetkov L. A.,* "On subgroups of π-soluble groups," in: Finite Groups [in Russian], Nauka i Tekhnika, Minsk, 1975, pp. 207–212.
22. *Sementovskiĭ V. G.,* "Injectors of finite groups," in: Study of Normal and Subgroup Structure of Finite Groups [in Russian], Nauka i Tekhnika, Minsk, 1984, pp. 166–170.
23. *Lockett F. P.,* "The Fitting class $\mathfrak{F}^*$," Math. Z., Bd 137, Heft 2, 131–136 (1974).
24. *Vorob′ev N. T.*, "Locality of solvable subgroup-closed Fitting classes," Math. Notes, **51**, No. 3, 221–225 (1992).
25. *Hartley B.*, "On Fischer's dualization of formation theory," Proc. London Math. Soc. (3), **19**, 193–207 (1969).
26. *Hauck P.*, "On products of Fitting classes," J. London Math. Soc., **20**, No. 3, 423–434 (1979).
27. *Hauck P.,* Zur Theorie der Fittingklassen endlicher auflösbarer Gruppen. Habilitationschrift, Johannes Gutenberg-Univ. Mainz, Mainz (1977).
28. *Camina A. R.,* "A note on Fitting classes," Math. Z., Bd 136, Heft 4, 351–352 (1974).
29. *Revin D. O.*, "The class of E_π-groups is nonradical," in: Proceedings of the International Conference on Algebra Dedicated to 100th Anniversary of S. M. Chernikov, Kyiv, 2012, p. 127.
30. *Guo W. and Li B.*, "On the injectors of finite groups," J. Group Theory, **10**, No. 6, 849–852 (2008).

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