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On modularity property of the lattice of partially composition Fitting classes

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Abstract – In this paper the sufficient conditions of the modularity of the lattice of all partially composition Fitting classes were found.

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Contents

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1. Introduction

Throughout this paper, all groups are finite. All unexplained notations and terminologies are standard. The reader is referred to [1,5,6,16,19,21,22,26] if necessary.

The symbols $\mathfrak{J}$ and $\mathfrak{S}_\omega$ denote, respectively, the class of all simple groups and the class of all soluble ω-groups, where ω is some non-empty set of primes.

Let $\mathfrak{X}$ be a non-empty collection of groups. We write $(\mathfrak{X})$ to denote the intersection of all classes of groups which contain $\mathfrak{X}$. In particular, (G) is the class of all groups which are isomorphic to the group G; $\mathcal{K}(G)$ is the class of all simple groups which are isomorphic to composition factors of G, $\mathcal{K}(\mathfrak{X}) = \bigcup\limits_{G\in\mathfrak{X}} \mathcal{K}(G)$.

If $\mathfrak{T}$ is a class of simple groups, then $\mathrm{E}\mathfrak{T}$ denotes the class of all groups for which all composition factors belong to $\mathfrak{T}$. By definition, all identity groups belong to $\mathrm{E}\mathfrak{T}$. If A is a simple group, then $\mathrm{E}(A')$ denotes the class of all groups G such that $A \notin \mathcal{K}(G)$.

Recall that, for an arbitrary class of groups $\mathfrak{F} \supseteq (1)$, where (1) is the class of all identity groups, the symbol $G^{\mathfrak{F}}$ denotes the intersection of all normal subgroups N such that $G/N \in \mathfrak{F}$, and $G_{\mathfrak{F}}$ denotes the product of all normal $\mathfrak{F}$-subgroups of G. We use $\mathfrak{G}_{cp}$ to denote the class of all groups G such that $C_G(H/K) = G$ for every p-chief factor H/K of G. Let $\mathfrak{L}$ be a non-empty class of simple groups and let A be a simple group. Clearly, $\mathrm{E}\mathfrak{L}$, $\mathrm{E}(A')$, $\mathfrak{S}_\omega$ and $\mathfrak{G}_{cp}$ are Fitting classes and formations (see [7]). Further we write

$$O^{\mathfrak{L}}(G) = G^{\mathrm{E}\mathfrak{L}},\ R^{\omega}(G) = G^{\mathfrak{S}_\omega},\ C_A(G) = G^{\mathrm{E}(A')},\ C_{Z_p}(G) = G^{\mathfrak{G}_{cp}}.$$

Note that if $A = Z_p$ is a group of prime order p, then the subgroup $C_{Z_p}(G)$ is also denoted by $C_p(G)$.

Let $\mathfrak{L}$ be a non-empty class of simple groups, $\mathfrak{L}' = \mathfrak{J}\backslash\mathfrak{L}$. Let f be a function of the form

$$f : \mathfrak{L} \cup \{\mathfrak{L}'\} \to \{\text{Fitting classes}\}. \tag{1}$$

According to [26] we consider the class of groups

$$CR_{\mathfrak{L}}(f) = \Big(G \mid O^{\mathfrak{L}}(G) \in f(\mathfrak{L}') \text{ and } C_A(G) \in f(A) \text{ for all } A \in \mathfrak{L} \cap \mathcal{K}(G)\Big).$$

If $\mathfrak{F}$ is a Fitting class such that $\mathfrak{F} = CR_{\mathfrak{L}}(f)$ for a function f of the form (1), then $\mathfrak{F}$ is said to be an $\mathfrak{L}$-*composition Fitting class.* The function f is called an $\mathfrak{L}$-*composition Hartley function* (at L. A. Shemetkov's suggestion), or an $\mathfrak{L}$-*composition H-function* (see [21,22]). In this paper we find a composition Fitting class formula by means of its composition H-function values. The analogous formulas for local and composition formations are well-known (see [4,15,21,22]).

First we give an $\mathfrak{L}$-composition Fitting class formula. Let $\mathfrak{L}^+$ be a collection of all abelian groups of $\mathfrak{L}$, $\mathfrak{L}^-$ be a collection of all non-abelian groups of $\mathfrak{L}$. Then

$$\mathrm{Supp}(f) = \{A \in \mathfrak{L} \cup \{\mathfrak{L}'\} \mid f(A) \neq \varnothing\},$$

$$\mathrm{Supp}_+(f) = \{A \in \mathfrak{L}^+ \cup \{\mathfrak{L}'\} \mid f(A) \neq \varnothing\} \text{ and}$$

$$\mathrm{Supp}_-(f) = \{A \in \mathfrak{L}^- \cup \{\mathfrak{L}'\} \mid f(A) \neq \varnothing\}.$$

Let f be a function of the form (1), $\mathfrak{T}_1 = \mathfrak{L} \cap \mathrm{Supp}_+(f)$, $\mathfrak{T}_2 = \mathfrak{L}^+ \setminus \mathfrak{T}_1$, $\mathfrak{B}_1 = \mathfrak{L} \cap \mathrm{Supp}_-(f)$ and $\mathfrak{B}_2 = \mathfrak{L}^- \setminus \mathfrak{B}_1$. Clearly,

$$CR_{\mathfrak{L}}(f) = \left(\bigcap_{A \in \mathfrak{T}_2} \mathrm{E}(A') \right) \cap \left(\bigcap_{A \in \mathfrak{T}_1} f(A)\mathfrak{G}_{cp} \right) \cap$$

$$\cap \left(\bigcap_{B \in \mathfrak{B}_1} f(B)\mathrm{E}(B') \right) \cap \left(\bigcap_{B \in \mathfrak{B}_2} \mathrm{E}(B') \right) \cap f(\mathfrak{L}')\mathrm{E}\mathfrak{L}.$$

In 1986 A. N. Skiba [20] proved that the lattice of all (local) formations is modular. This result was developed by a number of other research. A series of modular and distributive lattices of group formations was found (see [2,3,9–14,17–25,27–31]). On the other hand we know nothing about the modularity of the lattice of all Fitting classes. In "The Kourovka Notebook" Kamornikov and Skiba posed the following

Problem 1.1 (see [8], Problem 14.47). *Is the lattice of all soluble Fitting classes of finite groups modular?*

In this paper the sufficient conditions of the modularity of the lattice of all partially composition Fitting classes were found. We prove the following theorem:

Theorem 1.2. *Let $\mathfrak{X}$, $\mathfrak{H}$ and $\mathfrak{F}$ be Fitting classes and x, y and f the minimal ω-composition H-functions of $\mathfrak{X}$, $\mathfrak{H}$ and $\mathfrak{F}$ respectively. If $x \leq f$ and $x(a) \vee y(a) = \mathrm{S}_n\{G \mid G = G_{x(a)}G_{y(a)}\}$ for all $a \in \mathrm{Supp}(x) \cap \mathrm{Supp}(y)$, then:*

$$(\mathfrak{X} \vee_{\omega_c} \mathfrak{H}) \cap \mathfrak{F} = \mathfrak{X} \vee_{\omega_c} (\mathfrak{H} \cap \mathfrak{F}).$$

2. Preliminaries

Recall that a *Fitting class* is a group class with the following properties:

(1) Every normal subgroup of an $\mathfrak{F}$-group is an $\mathfrak{F}$-group.

(2) If M and N are $\mathfrak{F}$-groups and M and N are normal in G, then MN belongs to $\mathfrak{F}$.

For any set of groups $\mathfrak{X}$ we denote by $\mathrm{Com}(\mathfrak{X})$ the class of all simple abelian groups A such that $A \cong H/K$, where H/K is a composition factor of $G \in \mathfrak{X}$. Recall also some known facts that are necessary for proving the main result.

Lemma 2.1. *Let $N \lhd G$ and A be a simple group. Then:*
(1) If $A \notin \mathcal{K}(G/N)$, then $C_A(G) = C_A(N)$.
(2) If $G/N \in \mathtt{E}\mathfrak{L}$, then $O^{\mathfrak{L}}(G) = O^{\mathfrak{L}}(N)$.
(3) If $G/N \in \mathfrak{S}_\omega$, then $R^\omega(G) = R^\omega(N)$.

Let $\{f_i \mid i \in I\}$ be a collection of $\mathfrak{L}$-composition H-functions. By $\bigcap\limits_{i\in I} f_i$ we denote the $\mathfrak{L}$-composition H-function f such that $f(A) = \bigcap\limits_{i\in I} f_i(A)$ for all $A \in \mathfrak{L} \cup \{\mathfrak{L}'\}$. The following result is not hard to obtain.

Lemma 2.2. *Let $\mathfrak{F} = \bigcap\limits_{i\in I} \mathfrak{F}_i$, where $\mathfrak{F}_i = CR_{\mathfrak{L}}(f_i)$. Then $\mathfrak{F} = CR_{\mathfrak{L}}(f)$, where $f = \bigcap\limits_{i\in I} f_i$.*

By $c_{\mathfrak{L}}$ we denote the complete lattice of all $\mathfrak{L}$-composition Fitting classes. The symbol $c_{\mathfrak{L}}\mathrm{fit}(\mathfrak{X})$ denotes the intersection of all $\mathfrak{L}$-composition Fitting classes containing a collection $\mathfrak{X}$ of groups. In particular, we write $\mathrm{fit}(\mathfrak{X})$ to denote the intersection of all Fitting classes containing a collection $\mathfrak{X}$ of groups. Let $\mathfrak{X}$ be an arbitrary collection of groups and let A be a simple group. Put

$$\mathfrak{X}(C_A) = \begin{cases} \mathrm{fit}(C_A(G) \mid G \in \mathfrak{X}) & \text{if } A \in \mathcal{K}(\mathfrak{X}), \\ \varnothing & \text{if } A \notin \mathcal{K}(\mathfrak{X}). \end{cases}$$

Let $\{f_i \mid i \in I\}$ be the collection of all $\mathfrak{L}$-composition H-functions of a Fitting class $\mathfrak{F}$. Since the lattice $c_{\mathfrak{L}}$ is complete, using Lemma 2.2, we conclude that $f = \bigcap\limits_{i\in I} f_i$ is an $\mathfrak{L}$-composition H-function of $\mathfrak{F}$. The H-function f is called *minimal*.

The following lemma gives a description of the minimal $\mathfrak{L}$-composition H-function of a Fitting class $\mathfrak{F} = c_{\mathfrak{L}}\mathrm{fit}(\mathfrak{X})$, where $\mathfrak{X}$ is a non-empty collection of groups.

LEMMA 2.3. *Let $\mathfrak{X}$ be a non-empty collection of groups, $\mathfrak{F} = c_{\mathfrak{L}}\mathrm{fit}(\mathfrak{X})$, and let f be the minimal $\mathfrak{L}$-composition H-function of $\mathfrak{F}$. Then:*
(1) $f(\mathfrak{L}') = \mathrm{fit}(O^{\mathfrak{L}}(G) \mid G \in \mathfrak{X})$.
(2) $f(A) = \mathrm{fit}(C_A(G) \mid G \in \mathfrak{X})$ *for all* $A \in \mathfrak{L} \cap \mathcal{K}(\mathfrak{X})$.
(3) $f(A) = \varnothing$ *for all* $A \in \mathfrak{L} \setminus \mathcal{K}(\mathfrak{X})$.

PROOF. Let t be an $\mathfrak{L}$-composition H-function such that

$$t(A) = \begin{cases} \mathrm{fit}(O^{\mathfrak{L}}(G) \mid G \in \mathfrak{X}) & \text{if } A = \mathfrak{L}', \\ \mathrm{fit}(C_A(G) \mid G \in \mathfrak{X}) & \text{if } A \in \mathfrak{L} \cap \mathcal{K}(\mathfrak{X}), \\ \varnothing & \text{if } A \in \mathfrak{L} \setminus \mathcal{K}(\mathfrak{X}). \end{cases}$$

We show that $t = f$. Let $\mathfrak{M} = CR_{\mathfrak{L}}(t)$. First we show that $\mathfrak{F} = \mathfrak{M}$. Let $T \in \mathfrak{X}$. Hence

$$O^{\mathfrak{L}}(T) \in \mathrm{fit}(O^{\mathfrak{L}}(G) \mid G \in \mathfrak{X}) = t(\mathfrak{L}'),$$

$$C_A(T) \in \mathrm{fit}(C_A(G) \mid G \in \mathfrak{X}) = t(A)$$

for all $A \in \mathfrak{L} \cap \mathcal{K}(\mathfrak{X})$. Consequently, $T \in \mathfrak{M}$. Hence $\mathfrak{X} \subseteq \mathfrak{M}$. It follows that $\mathfrak{F} \subseteq \mathfrak{M}$.

We prove the converse inclusion. Let f_1 be an $\mathfrak{L}$-composition H-function of $\mathfrak{F}$. First we prove $t \leq f_1$. Let $N \in \mathfrak{X}$. Then

$$O^{\mathfrak{L}}(N) \in f_1(\mathfrak{L}'),\ C_A(N) \in f_1(A)$$

for all $A \in \mathfrak{L} \cap \mathcal{K}(\mathfrak{X})$. Consequently,

$$t(\mathfrak{L}') = \mathrm{fit}(O^{\mathfrak{L}}(G) \mid G \in \mathfrak{X}) \subseteq f_1(\mathfrak{L}'),$$

$$t(A) = \mathrm{fit}(C_A(G) \mid G \in \mathfrak{X}) \subseteq f_1(A)$$

for all $A \in \mathfrak{L} \cap \mathcal{K}(\mathfrak{X})$. Hence $t \leq f_1$. Therefore, $\mathfrak{M} \subseteq \mathfrak{F}$. Thus, $\mathfrak{M} = \mathfrak{F}$ and $t = f$. The lemma is proved.

□

LEMMA 2.4. *Let $\mathfrak{F} = CR_{\mathfrak{L}}(f)$, where $f(A) = \mathfrak{F}$ for all $A \in \mathfrak{L}^- \cup \{\mathfrak{L}'\}$ and let $G \notin \mathfrak{F}$. Then either $O^{\mathfrak{L}}(G) \nsubseteq G_{\mathfrak{F}}$ or there exists $A \in \mathfrak{L} \cap \mathcal{K}(G/G_{\mathfrak{F}})$ such that $C_A(G) \notin f(A)$.*

Proof. Let $O^{\mathfrak{L}}(G) \subseteq G_{\mathfrak{F}}$ and $C_A(G) \in f(A)$ for all $A \in \mathfrak{L} \cap \mathcal{K}(G/G_{\mathfrak{F}})$. Since $G_{\mathfrak{F}} \in \mathfrak{F}$, it follows $O^{\mathfrak{L}}(G) \in \mathfrak{F} = f(\mathfrak{L}')$. Let $A \in (\mathfrak{L} \cap \mathcal{K}(G)) \setminus \mathcal{K}(G/G_{\mathfrak{F}})$. Then, by Lemma 2.1, $C_A(G) \in \mathfrak{F} = f(A)$. Consequently, $C_A(G) \in f(A)$ for all $A \in \mathfrak{L} \cap \mathcal{K}(G)$. Hence $G \in \mathfrak{F}$. A contradiction. This proves the lemma. □

3. Proof of Theorem 1.2

Recall that a group G is called *comonolithic* if $G \neq 1$ and G has exactly one maximal normal subgroup (the *comonolith* of G).

We now prove the following Proposition, which plays an essential role in the proof of our main Theorem. Note that Proposition is independently interesting. Based on this result, we consider an ω-composition Fitting class concept (ω is a non-empty set of primes).

Proposition 3.1. *Let $\mathfrak{F}$ be a Fitting class. Then the following statements hold.*

(1) *If $\varnothing \neq \mathfrak{T} \subseteq \mathfrak{L}$ and $\mathfrak{F}$ is an $\mathfrak{L}$-composition Fitting class, then $\mathfrak{F}$ is $\mathfrak{T}$-composition.*

(2) *If $\mathfrak{L} = \bigcup_{i \in I} \mathfrak{L}_i$ and $\mathfrak{F}$ is an $\mathfrak{L}_i$-composition Fitting class for all $i \in I$, then $\mathfrak{F}$ is $\mathfrak{L}$-composition.*

Proof. (2) Let f be an $\mathfrak{L}$-composition H-function such that $f(A) = \mathfrak{F}$ for all $A \in \mathfrak{L}^- \cup \{\mathfrak{L}'\}$ and $f(Z_p) = \mathfrak{F}(C_p)$ for all $Z_p \in \mathfrak{L}^+$. Let $\mathfrak{M} = CR_{\mathfrak{L}}(f)$. We show that $\mathfrak{F} = \mathfrak{M}$.

Let $\mathfrak{F} \nsubseteq \mathfrak{M}$ and let G be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{M}$. Then G is a comonolithic group with the comonolith $R = G_{\mathfrak{M}}$. Suppose that G/R is an $\mathrm{E}(\mathfrak{L}')$-group. Then $O^{\mathfrak{L}}(G) = G$. Therefore, $O^{\mathfrak{L}}(G) \in \mathfrak{F} = f(\mathfrak{L}')$. By Lemma 2.1, $C_A(G) = C_A(R)$ for all $A \in \mathfrak{L} \cap \mathcal{K}(G)$. Since $R \in \mathfrak{M}$, it follows that $G \in \mathfrak{M}$, a contradiction. Hence G/R is an $\mathrm{E}A$-group, where $A = Z_p \in \mathfrak{L}^+$. Consequently, $Z_p \in \mathfrak{L}_i^+$ for some $i \in I$. Let f_i be the minimal $\mathfrak{L}_i$-composition H-function of $\mathfrak{F}$. By Lemma 2.3,

$$f_i(Z_p) = \mathfrak{F}(C_p) = f(Z_p).$$

Since $G \in \mathfrak{F}$ and $\mathfrak{F}$ is an $\mathfrak{L}_i$-composition Fitting class, we have $C_p(G) \in f(Z_p)$. This contradicts Lemma 2.4. Analogously, we have a contradiction in the case $A \in \mathfrak{L}^-$. Thus, $\mathfrak{F} \subseteq \mathfrak{M}$.

Now we show that $\mathfrak{M} \subseteq \mathfrak{F}$. Let $\mathfrak{M} \nsubseteq \mathfrak{F}$ and let G be a group of minimal order in $\mathfrak{M} \setminus \mathfrak{F}$. Then G is a comonolithic group with the comonolith $R = G_{\mathfrak{F}}$.

Evidently, G/R is an $\mathrm{E}A$-group for some group $A = Z_p \in \mathfrak{L}^+$. Since $Z_p \in \mathfrak{L}_i^+$ and f_i is the minimal $\mathfrak{L}_i$-composition H-function of $\mathfrak{F}$, it follows that

$$C_p(G) \in f(Z_p) = \mathfrak{F}(C_p) = f_i(Z_p).$$

This again contradicts Lemma 2.4. Thus, $\mathfrak{M} \subseteq \mathfrak{F}$. Therefore, $\mathfrak{M} = \mathfrak{F}$.

The assertion (1) can be proved analogously. The proposition is proved.

□

Corollary 3.2. *A Fitting class $\mathfrak{F}$ **is** $\mathfrak{L}$-composition if and only if $\mathfrak{F}$ is $\mathfrak{L}^+$-composition.*

Remark 3.3. By Proposition 3.1 and Corollary 3.2 we can study an $\mathfrak{L}$-composition Fitting class in the case $\mathfrak{L} = \mathfrak{L}^+$, where $\mathfrak{L}^+$ is a class of abelian simple groups. If $\mathfrak{L} = \mathfrak{L}^+$ and $\omega = \pi(\mathrm{Com}(\mathfrak{L}))$, then we can replace the concept "$\mathfrak{L}$-composition Fitting class" by the concept "ω-composition Fitting class" and replace the symbol $f(\mathfrak{L}')$ by the symbol $f(\omega')$.

Let f be a function of the form

$$f : \omega \cup \{\omega'\} \to \{\text{Fitting classes}\}. \tag{2}$$

We consider the class of groups

$$CR_\omega(f) = \Big(G \mid R^\omega(G) \in f(\omega') \text{ and } C_p(G) \in f(p) \text{ for all } p \in \omega \cap \pi(\mathrm{Com}(G))\Big).$$

If $\mathfrak{F}$ is a Fitting class such that $\mathfrak{F} = CR_\omega(f)$ for a function f of the form (2), then $\mathfrak{F}$ is said to be ω-*composition* and f is said to be an ω-*composition H-function* of $\mathfrak{F}$. We denote by $\pi(G)$ the set of all prime divisors of the order of a group G. For any collection $\mathfrak{X}$ of groups we denote by $\pi(\mathfrak{X})$ the union of the sets $\pi(G)$ for all groups G of $\mathfrak{X}$.

Remark 3.4. Let f be a function of the form (2), $\pi_1 = \mathrm{Supp}(f)$ and $\pi_2 = \omega \setminus \pi_1$, where $\mathrm{Supp}(f) = \{p \in \omega \mid f(p) \neq \varnothing\}$. We can easily see that

$$CR_\omega(f) = \left(\bigcap_{p\in\pi_2} \mathrm{E}(Z_p')\right) \cap \left(\bigcap_{p\in\pi_1} f(p)\mathfrak{G}_{cp}\right) \cap f(\omega')\mathfrak{G}_\omega.$$

If $\omega = \mathbb{P}$, then $\pi_1 = \mathrm{Supp}(f)$ and $\pi_2 = \mathbb{P} \setminus \pi_1$. In this case we have

$$CR_{\mathbb{P}}(f) = CR(f) = \left(\bigcap_{p\in\pi_2} \mathrm{E}(Z_p')\right) \cap \left(\bigcap_{p\in\pi_1} f(p)\mathfrak{G}_{cp}\right).$$

Using Lemma 2.2 and Corollary 3.2, we obtain

Lemma 3.5. *Let* $\mathfrak{F} = \bigcap_{i \in I} \mathfrak{F}_i$, *where* $\mathfrak{F}_i = CR_\omega(f_i)$. *Then* $\mathfrak{F} = CR_\omega(f)$, *where* $f = \bigcap_{i \in I} f_i$.

By c_ω we denote the complete lattice of all ω-composition Fitting classes. The symbol $c_\omega\mathrm{fit}(\mathfrak{X})$ denotes the intersection of all ω-composition Fitting classes containing a collection $\mathfrak{X}$ of groups.

Lemma 3.6. *Let* $\mathfrak{X}$ *be a non-empty collection of groups,* $\mathfrak{F} = c_\omega\mathrm{fit}(\mathfrak{X})$. *Let* $\pi = \omega \cap \pi(\mathrm{Com}(\mathfrak{X}))$, *and* f *the minimal* ω*-composition* H*-function of* $\mathfrak{F}$. *Then:*

(1) $f(\omega') = \mathrm{fit}(R^\omega(G) \mid G \in \mathfrak{X})$.
(2) $f(p) = \mathrm{fit}(C_p(G) \mid G \in \mathfrak{X})$ *for all* $p \in \pi$.
(3) $f(p) = \varnothing$ *for all* $p \in \omega \setminus \pi$.
(4) $\mathfrak{F} = CR_\omega(h)$, *where* $h(\omega') = \mathfrak{F}$ *and* $h(p) = f(p)$ *for all* $p \in \omega$.

Proof. (1)–(3) See the proof of Lemma 2.3 (assertions (1)–(3)).

(4) Let h be an ω-composition H-function such that $h(\omega') = \mathfrak{F}$ and $h(p) = f(p)$ for any $p \in \omega$. Let $\mathfrak{H} = CR_\omega(h)$. We show that $\mathfrak{F} = \mathfrak{H}$.

If $G \in \mathfrak{F}$, then $R^\omega(G) \in \mathfrak{F} = h(\omega')$ and $C_p(G) \in f(p) = h(p)$ for any $p \in \omega \cap \pi(\mathrm{Com}(G))$. Consequently, $G \in \mathfrak{H}$. Therefore, $\mathfrak{F} \subseteq \mathfrak{H}$.

Suppose $\mathfrak{H} \not\subseteq \mathfrak{F}$. Let G be a group of minimal order in $\mathfrak{H} \setminus \mathfrak{F}$. Then G is a comonolithic group and $R = G_{\mathfrak{F}}$ is the comonolith of G. Since $G \in \mathfrak{H} = CR_\omega(h)$, we have

$$R^\omega(G) \in h(\omega') = \mathfrak{F}.$$

Hence $G/R \in \mathfrak{S}_\omega$ and, by Lemma 2.1,

$$R^\omega(G) = R^\omega(G_{\mathfrak{F}}) \in f(\omega').$$

Besides,

$$C_p(G) \in h(p) = f(p)$$

for any $p \in \omega \cap \pi(\mathrm{Com}(G))$. Consequently, $G \in \mathfrak{F}$, a contradiction. Hence $\mathfrak{H} \subseteq \mathfrak{F}$. Thus, $\mathfrak{H} = \mathfrak{F}$, as desired. □

We denote by $\vee(f_i \mid i \in I)$ an H-function f such that

$$f(a) = \mathrm{fit}\left(\bigcup_{i \in I} f_i(a)\right)$$

for all $a \in \omega \cup \{\omega'\}$.

For any collection of ω-composition Fitting classes $\{\mathfrak{F}_i \mid i \in I\}$ we write

$$\vee_{\omega_c}(\mathfrak{F}_i \mid i \in I) = c_\omega \mathrm{fit}\left(\bigcup_{i\in I} \mathfrak{F}_i\right).$$

Lemma 3.7. *Let f_i be the minimal ω-composition H-function of a Fitting class $\mathfrak{F}_i$, $i \in I$. Then $\vee(f_i \mid i \in I)$ is the minimal ω-composition H-function of the Fitting class $\mathfrak{F} = \vee_{\omega_c}(\mathfrak{F}_i \mid i \in I)$.*

Proof. Let

$$\pi = \omega \cap \pi\left(\bigcup_{i\in I} \mathrm{Com}(\mathfrak{F}_i)\right) = \omega \cap \left(\bigcup_{i\in I} \pi\Big(\mathrm{Com}(\mathfrak{F}_i)\Big)\right) = \omega \cap \pi\Big(\mathrm{Com}(\mathfrak{F})\Big).$$

Let $f = \vee(f_i \mid i \in I)$ and let h be the minimal ω-composition H-function of $\mathfrak{F}$. If $p \in \omega \setminus \pi$, then $f_i(p) = \varnothing$ for any $i \in I$. Hence $f(p) = \varnothing$. Clearly, $h(p) = \varnothing$.

Let $p \in \pi$. Then there exists $i \in I$ such that $f_i(p) \neq \varnothing$. Using Lemma 3.6, we have

$$h(p) = \mathrm{fit}\Big(C_p(G) \mid G \in \bigcup_{i\in I} \mathfrak{F}_i\Big) =$$

$$= \mathrm{fit}\Big(\bigcup_{i\in I} \mathrm{fit}(C_p(G) \mid G \in \mathfrak{F}_i)\Big) =$$

$$= \mathrm{fit}\Big(\bigcup_{i\in I} f_i(p)\Big) = \Big(\vee\,(f_i \mid i \in I)\Big)(p) = f(p).$$

Moreover, using Lemma 3.6, we have

$$h(\omega') = \mathrm{fit}\Big(R^\omega(G) \mid G \in \bigcup_{i\in I} \mathfrak{F}_i\Big) =$$

$$= \mathrm{fit}\Big(\bigcup_{i\in I} \mathrm{fit}(R^\omega(G) \mid G \in \mathfrak{F}_i)\Big) =$$

$$= \mathrm{fit}\Big(\bigcup_{i\in I} f_i(\omega')\Big) = \Big(\vee\,(f_i \mid i \in I)\Big)(\omega') = f(\omega').$$

Thus, $\vee(f_i \mid i \in I)$ is the minimal ω-composition H-function of $\mathfrak{F} = \vee_{\omega_c}(\mathfrak{F}_i \mid i \in I)$. This proves the lemma. □

Let $\mathfrak{X}$ be a class of groups. We denote by s_n an operation on $\mathfrak{X}$ such that

$$\mathrm{s}_n\mathfrak{X} = (G \mid G \text{ is a subnormal subgroup of a group } H \in \mathfrak{X}).$$

Let f be a function of the form (2). Then

$$\mathrm{Supp}(f) = \{a \in \omega \cup \{\omega'\} \mid f(a) \neq \varnothing\}.$$

Proof of Theorem 1.2. First we prove the modular law for minimal ω-composition H-functions x, y and f. We note the inclusion

$$x \vee (y \cap f) \leq (x \vee y) \cap f$$

is obvious.

We show that $(x \vee y) \cap f \leq x \vee (y \cap f)$ for all prime $a \in \omega \cup \{\omega'\}$.

If $f(a) = \varnothing$ or $x(a) \vee y(a) = \varnothing$ the inclusion is trivial. Thus, the Fitting classes $x(a) \vee y(a)$ and $f(a)$ are non-empty. Consider the case either $x(a) = \varnothing$ or $y(a) = \varnothing$.

Let $x(a) = \varnothing$. Then

$$\Big(x(a) \vee y(a)\Big) \cap f(a) = y(a) \cap f(a) = x(a) \vee \Big(y(a) \cap f(a)\Big),$$

the modular law is true.

Let $y(a) = \varnothing$. Then

$$\Big(x(a) \vee y(a)\Big) \cap f(a) = x(a) \cap f(a) = x(a).$$

On the other hand,

$$x(a) \vee \Big(y(a) \cap f(a)\Big) = x(a) \vee \varnothing = x(a),$$

the modular law is true.

Thus, we can further suppose that every Fitting class $x(a)$, $y(a)$ and $f(a)$ is non-empty.

Let K be a group in $\Big(x(a) \vee y(a)\Big) \cap f(a)$, where $a \in \omega \cup \{\omega'\}$. By hypothesis, there exists a group $G = G_{x(a)}G_{y(a)}$ such that $K \triangleleft G$.

Therefore,

$$K \triangleleft G_{f(a)} = G \cap G_{f(a)} = G_{x(a)}G_{y(a)} \cap G_{f(a)} =$$

$$= G_{x(a)}\Big(G_{y(a)} \cap G_{f(a)}\Big) = G_{x(a)}G_{y(a) \cap f(a)}.$$

Hence

$$K \in x(a) \vee \Big(y(a) \cap f(a)\Big).$$

Consequently,

$$(x \vee y) \cap f \leq x \vee (y \cap f).$$

Thus, the modular law is true for minimal ω-composition H-functions x, y and f.

We show that

$$(\mathfrak{X} \vee_{\omega_c} \mathfrak{H}) \cap \mathfrak{F} = \mathfrak{X} \vee_{\omega_c} (\mathfrak{H} \cap \mathfrak{F}).$$

Let G be a group in $\mathfrak{X}$. Then

$$R^{\omega}(G) \in x(\omega') \subseteq f(\omega'),\ C_p(G) \in x(p) \subseteq f(p)$$

for any $p \in \omega \cap \pi(\mathrm{Com}(G))$. Hence $\mathfrak{X} \subseteq \mathfrak{F}$.

By Lemma 3.7, $x \vee y$ is an ω-composition H-function of $\mathfrak{X} \vee_{\omega_c} \mathfrak{H}$. Hence, by Lemma 3.5, we have

$$(\mathfrak{X} \vee_{\omega_c} \mathfrak{H}) \cap \mathfrak{F} = CR_{\omega}\Big((x \vee y) \cap f\Big).$$

By Lemma 3.5, it follows

$$\mathfrak{H} \cap \mathfrak{F} = CR_{\omega}(y \cap f).$$

Using Lemma 3.7, we have

$$\mathfrak{X} \vee_{\omega_c} (\mathfrak{H} \cap \mathfrak{F}) = CR_{\omega}\Big(x \vee (y \cap f)\Big).$$

From above, we have

$$(\mathfrak{X} \vee_{\omega_c} \mathfrak{H}) \cap \mathfrak{F} = \mathfrak{X} \vee_{\omega_c} (\mathfrak{H} \cap \mathfrak{F}).$$

This proves the theorem.

□

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