AUTOISOMETRIES AND AUTOSIMILARITIES OF LIE ALGEBRA $\mathcal{A}(1)\oplus\mathcal{R}^2$

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In [2], all methods of defining the Lorentzian scalar product on the four-dimensional Lie algebra $\mathcal{H}_3\oplus\mathcal{R}$ were found, for which this Lie algebra admits a one-parameter group of self-similarities or autoisometries. The purpose of this work is to find all the autosimilarities and autoisometries of the four-dimensional Lie algebra $\mathcal{A}(1)\oplus\mathcal{R}^2$, belonging to the VI Bianchi type (subtype VI_1), provided that the Lorentzian scalar product is specified on it. We will also find conditions for the existence of such one-parameter groups. In [2], a similar problem was solved for the case of the Euclidean scalar product.

Material and methods. The object of our research is the Lie algebra $\mathcal{A}(1)\oplus\mathcal{R}^2$ endowed with a Lorentz scalar product. Methods of linear algebra and analytical geometry are applied.

Results and its discussion. A linear transformation of a Lie algebra $F\colon\mathcal{G}\to\mathcal{G}$ is called *an auto-morphism* if it preserves the bracket operation: $[FX, FY]=F[X, Y]\ \forall X, Y\in\mathcal{G}$. Let Euclidean or Lorentz scalar product $\langle,\rangle$ is introduced in the Lie algebra $\mathcal{G}$. Transformation $F\colon\mathcal{G}\to G$ is called *the similarity* with the coefficient e^{μ}, if $\langle FX, FY\rangle=e^{2\mu}\langle X, Y\rangle\ \forall X, Y\in\mathcal{G}$. In the case $\mu=0$ transformation F is called isometry.

A transformation that is both a similarity and an automorphism will be called an *autosimilarity*. A transformation that is an isometry and an automorphism will be called an *autoisometry*.

The problem of finding the one-parameter autoisometry groups of the Lie algebra is of interest for the following reason. Its solution will allow in the future to find all self-similar Riemannian or Lorentzian homogeneous manifolds of the given Lie group.

In a suitable basis (E_1, E_2, E_3, E_4) the commutation relations of Lie algebra $\mathcal{G}_4=\mathcal{A}(1)\oplus\mathcal{R}^2$ are defined by one equality: $[E_1, E_2]=E_2$. We will call such a basis canonical. This Lie algebra contains a three-dimensional commutative ideal $\mathcal{L}$, which is the linear span of the vectors E_2, E_3, E_4, and also the two-dimensional center $\mathcal{Z}$, which is the linear span of the vectors (E_3, E_4). One-dimensional subspace $\boldsymbol{R}E_2$ is a derived Lie algebra: $\boldsymbol{R}E_2=\mathcal{G}_4^{(2)}=[\mathcal{G}_4, \mathcal{G}_4]$.

All the vector subspaces indicated above must be invariant under Lie algebra automorphisms. Therefore, any automorphism of the Lie algebra $F\colon\mathcal{G}_4\to\mathcal{G}_4$ in the canonical basis is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ \gamma & 0 & \varepsilon & \mu \\ \delta & 0 & \lambda & \nu \end{pmatrix},\ \alpha\neq 0,\ \begin{vmatrix} \varepsilon & \mu \\ \lambda & \nu \end{vmatrix}\neq 0,$$

and all other coefficients can take any values. Thus, the full group of automorphisms of Lie algebra G_4 is eight-dimensional.

In the next theorem, the signature characterizes the metric that is induced on the indicated ideal, the Gram matrix and the matrix of the transformation group are indicated in the canonical basis; in all cases $\varepsilon>0$, $\nu\neq 0$, $\mu\neq 0$ and $t\in\boldsymbol{R}$. If we put $\mu=0$, for the autosimilarities, then we get the autoisometry group.

Theorem. *Let a Lorentz scalar product of signature* $(+,+,+,-)$ *be given on the Lie algebra. Then this Lie algebra admits one-parameter groups of auto-similarities and autoisometries only in the cases indicated in the following table.*

Condition	*Gram matrix*	*Matrix defining a one-parameter group*
	Autoisometries	
1. $\mathcal{L}$: (+,+,+), $E_2 \in \mathcal{Z}^{\perp}$.	$\Gamma_1 = \begin{pmatrix} -\varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$F_1(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix}$
2. $\mathcal{L}$: (+,+,–), E_2 is space-like and $E_2 \in \mathcal{Z}^{\perp}$.	$\Gamma_2 = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$F_2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix}$
3. $\mathcal{L}$: (+,+,–), E_2 is timelike and $E_2 \in \mathcal{Z}^{\perp}$.	$\Gamma_3 = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$F_3(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{ch}\,t & \mathrm{sh}\,t \\ 0 & 0 & \mathrm{sh}\,t & \mathrm{ch}\,t \end{pmatrix}$
4. $\mathcal{L}$: (+,+,–), $\mathcal{Z}$: (+,0), E_2 is isotropic.	$\Gamma_4 = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$F_4(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\nu t} & 0 & 0 \\ 0 & 0 & e^{-\nu t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
	Autosimilarities	
5. $\mathcal{L}$: (+,+,0), $\mathcal{Z}$: (+,0), E_2 is not isotropic, $E_2 \in \mathcal{Z}^{\perp}$.	$\Gamma_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$F_5(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\mu t} & 0 & 0 \\ 0 & 0 & e^{2\mu t} & 0 \\ 0 & 0 & 0 & e^{\mu t} \end{pmatrix}$
6. $\mathcal{L}$: (+,+,0), $\mathcal{Z}$: (+,0), E_2 is not isotropic, $E_2 \in \mathcal{Z}^{\perp}$.	$\Gamma_6 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$F_6(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t^2/2 & 0 & 1 & t \\ t & 0 & 0 & 1 \end{pmatrix}$
7. $\mathcal{L}$: (+,+,0), $\mathcal{Z}$: (+,0), E_2 is not isotropic, $E_2 \notin \mathcal{Z}^{\perp}$.	$\Gamma_7 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1+\varepsilon \\ -1 & 0 & 0 & 0 \\ 0 & 1+\varepsilon & 0 & \varepsilon \end{pmatrix}$	$F_7(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 0 \\ t^2/2 & 0 & 1 & t \\ t & 0 & 0 & 1 \end{pmatrix}$
8. $\mathcal{L}$: (+,+,0), $\mathcal{Z}$: (+,+), E_2 is isotripic.	$\Gamma_8 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$F_8(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{2\mu t} & 0 & 0 \\ 0 & 0 & e^{\mu t}\cos t & -e^{\mu t}\sin t \\ 0 & 0 & e^{\mu t}\sin t & e^{\mu t}\cos t \end{pmatrix}$

Conclusion. In this paper, we found the complete group of automorphisms of the four-dimensional Lie algebra $\mathcal{A}(1) \oplus \mathcal{R}^2$, determined the canonical forms of the Lorentzian metrics in this Lie algebra, for which it admits a one-parameter group of autoisometries or self-similarities, and indicated the matrices that define transformation groups. The obtained results will be used to construct self-similar homogeneous Lorentzian manifolds of Lie group $A^{+}(1) \times (R^{+})^2$.

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2. Podoksenov M.N., Autoisometries of Lie algebra $\mathcal{A}(1) \oplus \mathcal{R}^2$ / M.N.Podoksenov, A.K. Guts // Science – to education, production, economy: materials of the 72nd Regional Scientific and Practical Conference of Teachers, Researchers and Postgraduates, Vitebsk, February 20, 2020 / Vitebsk: VSU named after P.M. Masherov, 2020. – PP.27-29.