Communications in Algebra®, 36: 3200–3208, 2008

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870802103560

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ON INJECTORS OF FINITE SOLUBLE GROUPS

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In this article, we give the description of the $\mathfrak{H}$-injectors of a finite soluble group, for a Hartley class $\mathfrak{H}$.

Key Words: Fitting classes; Hartley classes; Injectors.

2000 Mathematics Subject Classification: 20D10; 20D20.

1. INTRODUCTION

All groups in this article are finite and soluble.

A class $\mathfrak{F}$ of groups is called a *Fitting class* if $\mathfrak{F}$ is closed with respect to normal subgroups and if the conditions $G = AB$, $A, B \trianglelefteq G$, and $A, B \in \mathfrak{F}$ always imply $G \in \mathfrak{F}$.

Let $\mathfrak{F}$ be a Fitting class. A subgroup V of a group G is said to be an *$\mathfrak{F}$-injector of G* if $V \cap N$ is a $\mathfrak{F}$-maximal subgroup of N for every normal subgroup N of G.

It is well known that Fitting classes play an important role in the theory of Classes of Groups. The importance of the theory of Fitting classes can be seen in the following theorem proved by Fischer et al. (1967).

Theorem A. *Let $\mathfrak{F}$ be a Fitting class. Then a group G has at least an $\mathfrak{F}$-injector, and any two $\mathfrak{F}$-injectors of G are conjugate in G.*

Concerning Fitting classes and injectors, the following problem naturally arose (see Hartley, 1969).

Problem. Let $\mathfrak{F}$ be a local Fitting class. Could we describe the $\mathfrak{F}$-injectors of a group?

Received October 31, 2006; Revised January 31, 2007. Communicated by M. Ferrero.
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Following Vorob'ev (1996), a Fitting class $\mathfrak{F}$ is local, if

$$\mathfrak{F} = \mathfrak{S}_{\pi(\mathfrak{F})} \bigcap_{p \in \pi(\mathfrak{F})} f(p)\mathfrak{N}_p \mathfrak{S}_{p'}$$

for some Hartley function f, i.e., $f(p)$ is a nonempty Fitting class for all p.

For the local Fitting class $\mathfrak{N}$ of all nilpotent groups, Fischer (1966) has proved that the set of all $\mathfrak{N}$-injectors of a soluble group G is exactly the set of all nilpotent maximal subgroups of G containing the Fitting subgroup $F(G)$.

In order to develop the important result of Fischer (1966), Hartley (1969) introduced the following concept.

Let $\Sigma = \{\pi_i \mid i \in I\}$ be the set of pairwise disjoint subsets of $\mathbb{P}$, the set of all prime numbers, and $\mathbb{P} = \bigcup_{i \in I} \pi_i$. Then, a function $h : \Sigma \longrightarrow$ {nonempty Fitting classes} is called an *H-function*. Let

$$LH(h) = \bigcap_{i \in I} h(\pi_i)\mathfrak{S}_{\pi_i'}\mathfrak{S}_{\pi_i},$$

where $\mathfrak{S}_\pi$ is the class of all soluble π-groups and $h(\pi_i)\mathfrak{S}_{\pi_i'}\mathfrak{S}_{\pi_i}$ is the usual Fitting product. A Fitting class $\mathfrak{H}$ is a Hartley class if $\mathfrak{H} = LH(h)$, for some H-function h. In this case, we also say that $\mathfrak{H}$ is defined by the local H-function h or h is a local H-function of the Hartley class $\mathfrak{H}$.

It is clear that every local Fitting class is a Hartley class.

Hartley (1969) proved that if a local Fitting class $\mathfrak{H} = \mathfrak{X}\mathfrak{N}$, where $\mathfrak{X}$ is any nonempty Fitting class and $\mathfrak{N}$ is the Fitting class of all nilpotent groups, then the set of all $\mathfrak{H}$-injectors of a group G is exactly the set of the subgroups V of G such that $V/G_{\mathfrak{X}}$ is a nilpotent injector of $G/G_{\mathfrak{X}}$. D'Arcy (1975) also proved that for the local Fitting class $\mathfrak{D} = \bigcap_{i \in I} \mathfrak{S}_{\pi_i'}\mathfrak{S}_{\pi_i}$, a subgroup V of a group G is a $\mathfrak{D}$-injector of G if V is a $\mathfrak{D}$-maximal subgroup of G containing the $\mathfrak{D}$-radical of G.

It is easy to see that the class $\mathfrak{N}$ of all nilpotent groups, the class $\mathfrak{H} = \mathfrak{X}\mathfrak{N}$, where $\mathfrak{X}$ is any nonempty Fitting class, and the class $\mathfrak{D} = \bigcap_{i \in I} \mathfrak{S}_{\pi_i'}\mathfrak{S}_{\pi_i}$ are Hartley classes.

The purpose of this article is to describe the $\mathfrak{H}$-injectors associated with a Hartley class $\mathfrak{H}$.

Theorem. *Let G be a group, V a subgroup of G, and $\mathfrak{H} = LH(h)$ a Hartley class. Then*:

(1) *V is an $\mathfrak{H}$-injector of G if and only if V/G_h is a $\mathfrak{D}$-injector of G/G_h, where $\mathfrak{D} = \bigcap_{i \in I} \mathfrak{S}_{\pi_i'}\mathfrak{S}_{\pi_i}$*;

(2) *The set of all $\mathfrak{H}$-injectors of G is exactly the set of the $\mathfrak{H}$-maximal subgroups of G containing the $\mathfrak{H}$-radical of G.*

All unexplained notations and terminologies are standard. The reader is referred to Doerk and Hawkes (1992) *and Guo* (2000) *if necessary.*

2. PRELIMINARIES

A class of groups $\mathfrak{F}$ is called a *formation* if $\mathfrak{F}$ is closed under homomorphic images and every group G has a smallest normal subgroup with quotient in $\mathfrak{F}$. This smallest normal subgroup is called the *$\mathfrak{F}$-residual of G* and is denoted by $G^{\mathfrak{F}}$.

Let π be a nonempty set of primes and π' the complement of π in the set of all prime numbers. It is clear that the class $\mathfrak{S}_\pi$ of all soluble π-groups is a Fitting formation.

From the definition of Fitting class, we see that for any nonempty Fitting class $\mathfrak{F}$, every group G has a unique $\mathfrak{F}$-maximal normal subgroup, which is called the *$\mathfrak{F}$-radical of G* and denoted by $G_{\mathfrak{F}}$.

For two classes of groups $\mathfrak{F}$ and $\mathfrak{H}$, their product is defined as $\mathfrak{F}\mathfrak{H} = \{G :$ G has a normal subgroup $N \in \mathfrak{F}$ with $G/N \in \mathfrak{H}\}$.

For two Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$, their product is defined as $\mathfrak{F}\Diamond\mathfrak{H} = (G \mid G/G_{\mathfrak{F}} \in \mathfrak{H})$. It is easy to see that if the Fitting class $\mathfrak{H}$ is closed under homomorphic image, then $\mathfrak{F}\Diamond\mathfrak{H} = \mathfrak{F}\mathfrak{H}$ (cf. Doerk and Hawkes, 1992, IX, (1.11)).

If $\mathfrak{X}$ is class of groups, then $\mathrm{Fit}(\mathfrak{X})$ is the Fitting class generated by $\mathfrak{X}$, that is, $\mathrm{Fit}(\mathfrak{X})$ is the smallest Fitting class containing $\mathfrak{X}$.

Let $\Sigma = \{\pi_i \mid i \in I\}$ be the set of pairwise disjoint subsets of $\mathbb{P}$ such that $\mathbb{P} = \bigcup \pi_i$. Then, a function $f : \Sigma \longrightarrow \{\text{classes of groups}\}$ is called a *class function*. Obviously, an H-function is a class function. For two class functions f and h, if $f(\pi_i) \subseteq h(\pi_i)$ for all $i \in I$, then we write that $f \leq h$.

Let f be a local H-function of a Hartley class $\mathfrak{F}$, f is called *integrated* if $h(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$.

In order to prove our main result, we need the following lemmas. The first one is elementary.

Lemma 2.1. *Every Hartley class can be defined by a local integrated H-function.*

Lemma 2.2. *Every Hartley class $\mathfrak{H}$ can be defined by a local integrated H-function h such that $h(\pi_i) \subseteq h(\pi_j)\mathfrak{S}_{\pi_j'}$ for all $i \neq j \in I$.*

Proof. Let $\mathfrak{H}$ be a Hartley class. By Lemma 2.1, $\mathfrak{H} = LH(h_1)$, for some integrated H-function h_1. We define

$$\psi(\pi_i) = \{G \mid G \simeq H^{\mathfrak{S}_{\pi_i'}}, \text{ for some } H \in h_1(\pi_i)\},$$

for all $i \in I$.

Let X be a group in $\psi(\pi_i)$ $(i \in I)$. Then $X \simeq Y^{\mathfrak{S}_{\pi_i'}}$, for some group $Y \in h_1(\pi_i)$. Since every Fitting class is closed with respect to normal subgroup, $Y^{\mathfrak{S}_{\pi_i'}} \in h_1(\pi_i)$ and so $X \in h_1(\pi_i)$. This shows that $\psi \leq h_1$, and consequently

$$\psi(\pi_i)\mathfrak{S}_{\pi_i'} \subseteq h_1(\pi_i)\mathfrak{S}_{\pi_i'}.$$

If $Y_1 \in h_1(\pi_i)\mathfrak{S}_{\pi_i'}$, then $Y_1/(Y_1)_{h_1(\pi_i)} \in \mathfrak{S}_{\pi_i'}$. Since $(Y_1^{\mathfrak{S}_{\pi_i'}})^{\mathfrak{S}_{\pi_i'}} = Y_1^{\mathfrak{S}_{\pi_i'}}$, we have $Y_1^{\mathfrak{S}_{\pi_i'}} \in \psi(\pi_i)$, that is, $Y_1 \in \psi(\pi_i)\mathfrak{S}_{\pi_i'}$. Therefore, we obtain the following equation:

$$\psi(\pi_i)\mathfrak{S}_{\pi_i'} = h_1(\pi_i)\mathfrak{S}_{\pi_i'}. \tag{1}$$

Now, let h be a function such that $h(\pi_i) = \mathrm{Fit}(\psi(\pi_i))$, for all $i \in I$. Let

$$\mathfrak{M} = \bigcap_{i \in I} h(\pi_i)\mathfrak{S}_{\pi_i'}\mathfrak{S}_{\pi_i}.$$