(ознакомительный фрагмент)

COMMUNICATIONS IN ALGEBRA®
2018, VOL. 46, NO. 1, 217–229
https://doi.org/10.1080/00927872.2017.1319475

On $\mathcal{F}$-injectors of fitting set of a finite group

Nanying Yang[a], W. Guo[b], and N. T. Vorob'ev[c]

[a]School of Science, Jiangnan University, Wuxi, P. R. China; [b]Department of Mathematics, University of Science and Technology of China, Hefei, P. R. China; [c]Department of Mathematics, Masherov Vitebsk State University, Vitebsk, Belarus

ABSTRACT
Let G be some generalized π-soluble groups and $\mathcal{F}$ be a Fitting set of G. In this paper, we prove the existence and conjugacy of $\mathcal{F}$-injectors of G and give a description of the structure of the injectors.

ARTICLE HISTORY
Received 29 June 2016
Communicated by V. A. Artamonov

KEYWORDS
Finite group; fitting class; fitting set; injector; π-soluble group

2010 MATHEMATICS SUBJECT CLASSIFICATION
20D10

1. Introduction

Throughout this paper, all groups are finite and p is a prime. G always denotes a group, $|G|$ is the order of G, $\sigma(G)$ is the set of all primes dividing $|G|$, π denotes a set of some primes. Let $\mathbb{P}$ be the set of all primes and $\pi' = \mathbb{P} \setminus \pi$. For any set $\mathcal{X}$ of subgroups of G, we let $\sigma(\mathcal{X}) = \bigcup_{G\in\mathcal{X}} \sigma(G)$.

Recall that a class $\mathfrak{F}$ of groups is called a Fitting class if $\mathfrak{F}$ is closed under taking normal subgroups and products of normal $\mathfrak{F}$-subgroups. As usual, we denote by $\mathfrak{E}$, $\mathfrak{S}$, $\mathfrak{N}$ the classes of all groups, all soluble groups, all nilpotent groups, respectively; $\mathfrak{E}_\pi$, $\mathfrak{S}_\pi$, $\mathfrak{N}_\pi$ denote the classes of all π-groups, all soluble π-groups, all nilpotent π-groups, respectively; and $\mathfrak{S}^\pi$ and $\mathfrak{N}^\pi$ to denote the class of all π-soluble groups and the class of all π-nilpotent groups, respectively. It is well known that all the above classes are Fitting classes.

Following Anderson [1] (see also [6, VIII, (2.1)]), a nonempty set $\mathcal{F}$ of subgroups of G is called a Fitting set of G if the following three conditions are satisfied: (i) If $T \trianglelefteq S \in \mathcal{F}$, then $T \in \mathcal{F}$; (ii) If $S, T \in \mathcal{F}$ and $S, T \trianglelefteq ST$, then $ST \in \mathcal{F}$; (iii) If $S \in \mathcal{F}$ and $x \in G$, then $S^x \in \mathcal{F}$.

For a set $\mathcal{X}$ of subgroups of G, the join $G_\mathcal{X}$ of all normal $\mathcal{X}$-subgroups of G is called the $\mathcal{X}$-radical of G. From the definitions of Fitting class $\mathfrak{F}$ and Fitting set $\mathcal{F}$ of G, for each nonempty Fitting class $\mathfrak{F}$ (nonempty Fitting set $\mathcal{F}$ of G, resp.) of G, $G_\mathfrak{F}$ ($G_\mathcal{F}$, resp.) is the unique maximal normal $\mathfrak{F}$-subgroup ($\mathcal{F}$-subgroup resp.) of G. In particular, $G_\mathfrak{N} = F(G)$ is the Fitting subgroup of G.

It is well known that the theory of Fitting classes plays an important role in the theory of groups. The importance of the theory of Fitting classes can first be seen in the following theorem proved by Fischer et al. [8]: *For every Fitting class $\mathfrak{F}$, every soluble group possesses exactly one conjugacy class of $\mathfrak{F}$-injectors.* This result is in fact a graceful generalization of the classical Sylow theorem and Hall theorem (the Hall theorem shown that every soluble group possesses a Hall π-subgroup and any two Hall π-subgroups are conjugated in G for any $\pi \subseteq \sigma(G)$).

Recall that, for any class $\mathfrak{F}$ of groups, a subgroup V of G is said to be $\mathfrak{F}$-maximal if $V \in \mathfrak{F}$ and $U = V$ whenever $V \le U \le G$ and $U \in \mathfrak{F}$. A subgroup V of a group G is said to be an $\mathfrak{F}$-injector of G if $V \cap K$ is an $\mathfrak{F}$-maximal subgroup of K for every subnormal subgroup K of G. For a Fitting set $\mathcal{F}$ of G, the $\mathcal{F}$-injector of G is similarly defined (see [6, VIII, (2.5)]). Clearly, every $\mathfrak{F}$-injector (resp. $\mathcal{F}$-injector) of G

CONTACT W. Guo wbguo@ustc.edu.cn Department of Mathematics, University of Science and Technology of China, Hefei 230026, P. R. China.

is an $\mathfrak{F}$-maximal (resp. $\mathcal{F}$-maximal) subgroup of G. Note that if $\mathfrak{F} = \mathfrak{N}_p$ is the Fitting class of all p-groups, then the $\mathfrak{F}$-injectors of a group G are Sylow p-subgroups of G; if $\mathfrak{F} = \mathfrak{E}_\pi$ and G has a Hall π-soluble group, then the $\mathfrak{F}$-injectors of G are Hall π-subgroups of G (see [10, p.68, Ex.1] or [2, p.328]).

As a development of the theorem of Fischer et al. [8], Shemetkov [16], and Anderson [1] proved that if G is a π-soluble group (resp. soluble group) and $\mathcal{F}$ is a Fitting set of G, then G possesses exactly one conjugacy class of $\mathcal{F}$-injectors, where $\pi = \sigma(\mathcal{F})$.

If $\mathfrak{F}$ is a Fitting class, then the set $\{H \leq G : H \in \mathfrak{F}\}$ is a Fitting set, which is denoted by $Tr_{\mathfrak{F}}(G)$ and called the trace of $\mathfrak{F}$ in G (see [6, p.537]). Note that for a Fitting class $\mathfrak{F}$, the $\mathfrak{F}$-injectors and $Tr_{\mathfrak{F}}(G)$-injectors of G coincide (see below Lemma 2.1). But by [6, VIII, Examples (2.2)(c)], not every Fitting set of G is the trace of a Fitting class. Hence, in view of the theorem of Shemetkov in [16] and the theorem of Fischer et al. [8], the following question naturally arises:

Problem 1.1. For an arbitrary Fitting set $\mathcal{F}$ of a non-soluble group G (in particular, some generalized π-soluble groups G, for example, the groups in Theorem A below), when G possesses an $\mathcal{F}$-injector and any two $\mathcal{F}$-injectors are conjugate?

It is clear that for a Fitting class $\mathfrak{F}$, every $\mathfrak{F}$-injector of G contains the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ of G. In [7] Fischer gave the characterization the $\mathfrak{N}$-injectors by means of the nilpotent radical $G_{\mathfrak{N}} = F(G)$. He proved that the set of $\mathfrak{N}$-injectors is exactly the set of all maximal nilpotent subgroups of G containing $F(G)$.

The product $\mathfrak{F}\mathfrak{H}$ of two Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ is the class $(G \mid G/G_{\mathfrak{F}} \in \mathfrak{H})$. It is well known that the product of any two Fitting classes is also a Fitting class and the multiplication of Fitting classes satisfies associative law (see [6, IX, (1.12)(a)(c)]).

Hartley [13] proved that, for the Fitting class of type $\mathfrak{X}\mathfrak{N}$ (where $\mathfrak{X}$ is a nonempty Fitting class), a subgroup V of a soluble group G is an $\mathfrak{X}\mathfrak{N}$-injector of G if and only if $V/G_{\mathfrak{X}}$ is a nilpotent subgroup of G. As a further improvement, the authors in [12] proved that for a Hartley class $\mathfrak{H}$, a subgroup V of soluble group G is $\mathfrak{H}$-injector of G if and only if V/G_h is a D-injector of G/G_h, where $D = \bigcap_{i\in I} \mathfrak{S}_{\pi_i'}\mathfrak{S}_{\pi_i}$ and $G_h = \prod_{i\in I} G_{h(\pi_i)}$. Moreover, in [12], it was proved that the set of $\mathfrak{H}$-injectors coincide with the set of all $\mathfrak{H}$-maximal subgroups containing the $\mathfrak{H}$-radical of G.

Following [2, Section 7.2], a Fitting set $\mathcal{F}$ is said to be injective if every group G possesses at least one $\mathcal{F}$-injector.

In connection with above, the following more general question naturally arises.

Problem 1.2. Let $\mathcal{F}$ be a injective Fitting set of G in some universe. What is the structure of the $\mathcal{F}$-injector?

In order to resolve the Problems 1.1 and 1.2, we need to develop and extend the local method of Hartley [13] (which is for Fitting classes and in the universe of soluble groups) to for Fitting sets and in a more general universe (not necessary in the universe of soluble groups).

First, for a Fitting set $\mathcal{F}$ of G and a Fitting class $\mathfrak{X}$, we call the set $\{H \leq G|H/H_{\mathcal{F}} \in \mathfrak{X}\}$ of subgroups of G the product of $\mathcal{F}$ and $\mathfrak{X}$ and denote it by $\mathcal{F} \circ \mathfrak{X}$.

Following [17], a function $f : \mathbb{P} \longrightarrow$ {Fitting sets of G} is called a Hartley function (or in brevity, an H-function).

Definition 1.3. Let $\emptyset \neq \pi \subseteq \mathbb{P}$ and $SLR(f) = \bigcap_{p\in\pi} f(p) \circ \mathfrak{E}_{p'}$. A Fitting set $\mathcal{F}$ of G called: *π-semilocal* if $\mathcal{F} = SLR(f)$ for some H-function f; *π-local* if $\mathcal{F} = SLR(f)$ for some H-function f such that $f(p) \circ \mathfrak{N}_p = f(p)$ for all $p \in \pi$. In this case when $\mathcal{F} = SLR(f)$, $\mathcal{F}$ is said to be defined by the H-function f or f is an H-function of $\mathcal{F}$.

In particular, if $\pi = \mathbb{P}$, then a π-semilocal Fitting set (a π-local Fitting set, respectively) is said to be a semilocal Fitting set (a local Fitting set, respectively).

Definition 1.4. Let $\mathcal{F} = SLR(f)$ be a π-semilocal Fitting set of G. Then f is said to be
1) integrated if $f(p) \subseteq \mathcal{F}$ for all $p \in \pi$;
2) full if $f(p) = f(p) \circ \mathfrak{N}_p$ for all $p \in \pi$;
3) invariable if $f(p) = f(q)$ for all $p, q \in \pi$.

For a π-semilocal Fitting set $\mathcal{F} = SLR(f)$ of G, the subgroup $G_f = \Pi_{p\in\pi} G_{f(p)}$ is said the f-radical of G.

The following two theorems resolved the Problems 1.1 and 1.2 for some generalized π-soluble groups and some π-semilocal Fitting sets of G.

Theorem A. *Let $\mathcal{F}$ be a Fitting set of G. Then G possesses an $\mathcal{F}$-injector and any two $\mathcal{F}$-injectors are conjugate if one of the following conditions:*
(1) $G \in \mathcal{F} \circ \mathfrak{S}^{\pi}$, where $\pi = \sigma(\mathcal{F})$;
(2) $\mathcal{F}$ is π-semilocal and $G \in \mathfrak{S}^{\pi}$;
(3) $\mathcal{F}$ is π-semilocal and $G \in \mathcal{F} \circ \mathfrak{S}^{\pi}$, where $\sigma(G_{\mathcal{F}}) \subseteq \pi$.
Moreover, the index of every $\mathcal{F}$-injector in G is a π-number in the case (2).

Theorem B. *Let $\mathcal{F}$ be a π-semilocal Fitting set of G defined by a full and invariable H-function f and G_f the f-radical of G. If G/G_f is π-soluble, then the following statements hold:*
(1) A subgroup V of G is an $\mathcal{F}$-injector of G if and only if V/G_f is a Hall π'-subgroup of G/G_f;
(2) G possesses an $\mathcal{F}$-injector and any two $\mathcal{F}$-injectors are conjugate in G;
(3) A subgroup V of G is an $\mathcal{F}$-injector of G if and only if V is an $\mathcal{F}$-maximal subgroup of G and $G_{\mathcal{F}} \leq V$;
(4) If $G \in \mathfrak{S}^{\pi}$, then every $\mathcal{F}$-injector of G is of the type $G_{\pi'}G_f$, where $G_{\pi'}$ is some Hall π'-subgroup of G.

Theorem A gives the new theory of $\mathcal{F}$-injectors for the generalized π-soluble group (in particular, π-soluble group). Theorem B describes the structures of the injectors. From Theorems A and B, a series of famous results can be directly follow. For example, Fischer, Gaschütz and Harley Theorem [8] (see also [6, VIII, Theorem (2.9) and IX, Theorem (1.4)]), Shemetkov [16, Theorem 2.2], Ballester-Bolinches and Ezquerro [2, Theorem 2.4.27], Sementovskii [15, Theorem], Bolado-Caballero, Martiner-Verduch [3, Theorem 2.2] and Guo [9] (see also [10, Theorem 2.5.3].

Remark 1.5.
(a) The statement of Theorem A is not true in general for any Fitting set and a non-soluble group, for example, let $S = A_7$ the alternating group of degree 7, $T = PSL(2, 11)$ the projective special linear group, a Fitting class $\mathfrak{F} = D_0(S, T, 1)$ and a group G such as in [2, Theorem 7.1.3]. Then for the Fitting set $\mathcal{F} = Tr_{\mathfrak{F}}(G)$, G has no $\mathcal{F}$-injector (see [2, Theorem 7.1.3]).
(b) It is well known that there exist examples of the sets π of primes and non-abelian groups G such that G possesses a Hall π-subgroup, but the Hall π-subgroups are not conjugate (see Revin and Vdovin [14, Theorem 1.1]). However, a Hall π-subgroups of a E_π-group G (Note that a group G is said to be an E_π-group if G has a Hall π-subgroup) is a $\mathfrak{E}_\pi$-injector of G (see [2, p. 328]). Therefore, $\mathfrak{E}_\pi$-injectors of a E_π-group G are not conjugate in general.

The following example shows that the statement of Theorem B(3) is not true if $\mathcal{F}$ is only a Fitting set even of a soluble G.

Example 1.6. Let a Fitting class $\mathfrak{F} = \mathfrak{N}\mathfrak{N}_3$, $S = S_3$ the symmetric group of degree 3 and M a faithful irreducible S-module over F_5, and let $T = M \rtimes S$, N be a faithful irreducible T-module over F_2, $G = N \rtimes T$ and $\mathcal{F} = Tr_{\mathfrak{F}}(G)$. Then $F(G) = N$ is a 2-group, and clearly $\mathcal{F}$ is not π-semilocal for any $\pi \subsetneq \mathbb{P}$. Similar as [6, IX, Example 4.4], we know that the set of $\mathcal{F}$-injectors coincide with the set of the subgroups of type