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TOPOLOGY

ТОПОЛОГИЯ

Методические рекомендации

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CHAPTER 1. FOUNDATIONS OF TOPOLOGY

§1. *Notion of the metric space. Distance between sets. Diameter of a set*

In the Euclidean space $\mathcal{E}^n$ distance between points $P(x_1, x_2,\dots x_n)$, $Q(y_1, y_2,\dots y_n)$ is calculated by formula

$$\rho(P,Q)=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+\ldots+(y_n-x_n)^2},$$

If coordinates of points are given in the orthonormal coordinate system. We can consider this distance as a function, which match a number $\rho(P,Q)$ to pair of point P and Q. Function ρ has the following properties:

1. $\rho(P,Q)=\rho(Q,P)$ (symmetry);

2. $\rho(P,Q)+\rho(Q,R)\geq\rho(P,R)$ (triangle inequality);

3. $\rho(P,Q)\geq 0$ и $\rho(P,Q)=0 \Leftrightarrow P=Q$.

***Definition* 1.1.** Let now M be an arbitrary set, elements of which we call points. Let function ρ be defined on set M, which match a number $\rho(P, Q)$ to two points $P, Q\in M$; we call this number the distance between P and Q. And let the axioms 1, 2, 3 are true. Then pair (M, ρ) is called *a metric space*, and function ρ is called *metrics*.

***Example* 1.1.** Let V be an arbitrary subset of Eucledean space $\mathcal{E}^n$. We will consider distance between $P, Q\in V$ the same as in the space. Then (V, ρ) is a metric space. Such metrics ρ is called *induced from* $\mathcal{E}^n$.

***Example* 1.2.** Let S^2 be a sphere in three-dimensional geometric space. Distance ρ_1 between $P, Q\in S^2$ is defines as the length of the shortest curve on the surface, which connects P and Q. I is well-known fact, that this curve is the arc of the big circle (fig. 1.1).

We can also define the distance as in example 1: $\rho(P, Q)$ is the length of chord PQ. Then (S^2, ρ_1) and (S^2, ρ) — are different metric spaces.

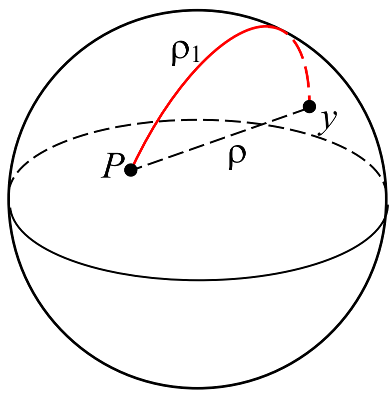

*fig.*1.1

***Example* 1.3.** Let's define the distance between two point $A(x_1, y_1)$, $B(x_2, y_2)$ on the plane by formula $\rho_2(A,B)=|x_2-x_1|+|y_2-y_1|$. In this case $\rho_2(A, B)$ is equal to the length of the broken line ACB, which is drawn in figure 1.2.

Exercise. 1. Check it yourself, that all the axioms of metric space are true for metrics ρ_2.

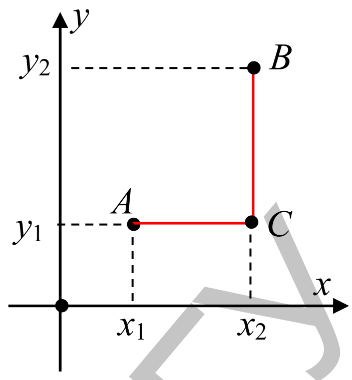

*fig.*1.2

***Definition* 1.2.** Set $U(P,\varepsilon)= =\{Q\in M\,|\,\rho(P,Q)<\varepsilon\}$ is called *an open ball* in metric space $(M,\ \rho)$. In particular, it can be an open circle on the plane, or an interval on the line.

***Definition* 1.3.** *Diameter of set* V in metric space $(M,\ \rho)$ is the exact upper boundary of distances between points of this set:

$$d(V)=\sup_{P,Q\in V}\rho(P,Q).$$

***Definition* 1.4.** *Distance between two sets* V, W is the exact lower boundary of distances between points of this sets:

$$\rho(V,W)=\inf_{P\in V,Q\in W}\rho(P,Q).$$

In particular, if one set consists of one point, then we get definition of *distance from point to set*.

Why do we have supremum, but not maximum, infimum, but not minimum? Let's see on the example.

***Example* 1.4.** Let U an be open (without boundary) circle of radius 1 on the plane with the center at the origin, and $W=Q(2,0)$ (figure 1.3). Then $d(U)=2$, although there are no such points in U distance between which is 2. Thereby the maximum is not reached. Analogously, $\rho(Q,U)=1$, although there is no such points $P\in U$, that holds $\rho(Q,\ P)=1$. Thereby the minimum is not reached.

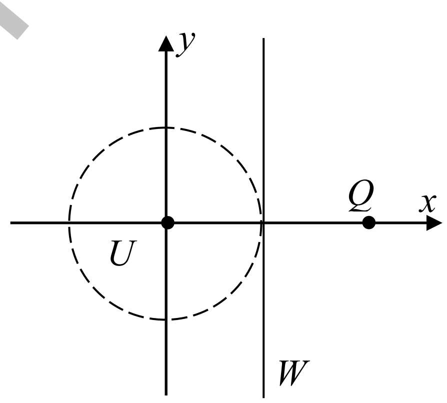

*fig.*1.3

We shell note that if sets intersect, then distance between them is equal zero. The converse is not true. For instance, if W is straight line $x=1$ (figure 1.3), then $\rho(U,W)=0$, but $U\cap W=\varnothing$.

***Definition* 1.3.** A set V in metric space (M,ρ) is said to be *bounded*, if $d(V)<\infty$. We shell note, that metric space itself can be bounded like (S^2,ρ_1), for instance.

Exercise. 2. What diameters of metric spaces (S^2,ρ_1) и (S^2,ρ) are equal to?

§2. *Open sets. Notion of topology space*

***Definition* 1.4.** Let V be some set in metric space (M, ρ). Point $P \in V$ is said to be <u>*an inner point*</u> of this set, if it is included in V along with some open ball containing it (figure 1.4), i.e. if there is suck $\varepsilon > 0$, that $U(P, \varepsilon) \subset V$.

***Definition* 1.5.** Set $V \subset (M, \rho)$ is said to be <u>*open*</u>, if all the points of this set are the inner points for this set. The empty set is considered open.

According to the definition metric space (M, ρ) itself is open.

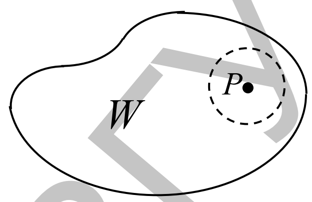

*fig.*1.4

***Definition* 1.6.** Set V in Euclidean space is called <u>*connected*</u>, if for any points $P, Q \in V$ there is a continuous curve $\gamma \subset V$, connecting P and Q.

This usual definition of a connected set множества has a significant drawback: we still don't know what «a continuous curve » is, and even we don't know what a curve is. Moreover, this definition is not suitable for an arbitrary metric space. Mathematically more precise definition requires some explanation.

***Definition* 1.7.** Set V in Euclidean space is called <u>*disconnected*</u>, if it can be represented as a union $V = V_1 \cup V_2$ of two disjoint sets, each of which is open in V (in the induced metrics).

Imagine that a set consists of two disjoint parts V_1 и V_2, which are not open the metric space, and P is a point on the boundary of V_1. Consider metric space (V, ρ) with metrics induced from M. Then ball $U(P, \varepsilon)$ in (V, ρ) looks like it is shown in figure 1.5. According to the definition point P turns out to be the inner point of set V_1. Similarly, this is true for an arbitrary point of the set V_1. Thereby, V_1 turns out to be open in V. This situation is impossible if V connected in an intuitive sense of the word.

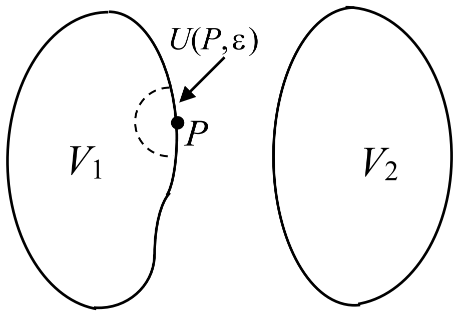

*fig.*1.5

***Definition* 1.8.** Set V in Euclidean space (M, ρ) is called <u>*connected*</u>, if it is not disconnected. An open connected set is called <u>*a domain*</u>. Any domain containing point P is called <u>*a neighborhood*</u> of this point.

***Theorem* 1.1. I.** *The union of any finite or infinite number of open sets is an open set.*

II. *The intersection of any finite number of open sets is an open set.*

Proof. **I.** Suppose that $V_1, V_2, V_3,\ldots$ – are open sets There is at least one of the sets $V_1, V_2, V_3,\ldots$ that contains P. Let it be V_i. This set is open. Therefore there is an open ball $U(P,\varepsilon)\subset V_i$. But in this case $U(P,\varepsilon)\subset V$. I.e. P is an inner point of V. Since P is an arbitrary point of V, this set is open.

II. Suppose that $V_1, V_2,\ldots, V_n$ are open sets and $V = V_1\bigcap V_2\bigcap\ldots\bigcap V_n$. Consider arbitrary point $P\in V$. Then $P\in V_i$ for all $i=1,2,\ldots,n$. Since all the sets $V_1, V_2,\ldots, V_n$ are open, there are such numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ that $U(P,\varepsilon_i)\subset V_i$. Let's take $\varepsilon=\min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$. Then $U(P,\varepsilon)\subset V_i$ for all $i=1,2,\ldots,n$. It means, that $U(P,\varepsilon)\subset V$. Since P is an arbitrary point of V, this set is open.

The following example shows that the intersection of an infinite number of open sets may not be open.

Example **1.5.** Let

$$V_1=(-2;2),\ V_2=(-1{,}5;1{,}5),\ V_3=\left(-\frac{4}{3};\frac{4}{3}\right),\ldots,\ V_i=\left(-1-\frac{1}{i};1+\frac{1}{i}\right),\ \ldots$$

Then $\bigcap\limits_{i=1}^{\infty} V_i=[-1,1]$.

Definition **1.9.** It is said that the system of all open subsets of a metric space (M,ρ) forms <u>*the topology*</u> of this space. This system is indicated by the letter $\boldsymbol{\tau}$.

We have found that the set of subsets $\boldsymbol{\tau}$ has the following properties:

I. $V_1, V_2, V_3,\ldots\in\boldsymbol{\tau}\ \Rightarrow\ \bigcup\limits_{i\in J} V_i\in\boldsymbol{\tau}$ (J – is the set of all indices);

II. $V_1, V_2, V_3,\ldots, V_n\in\boldsymbol{\tau}\ \Rightarrow\ \bigcap\limits_{i=1}^{n} V_i\in\boldsymbol{\tau}$;

III. $\varnothing\in\boldsymbol{\tau}$, $M\in\boldsymbol{\tau}$.

Definition **1.10.** Let M be an arbitrary set, on which a system of subsets $\boldsymbol{\tau}$, is given satisfying axioms I, II, III. Then a pair $(M,\boldsymbol{\tau})$ is called <u>*topological space*</u>, and $\boldsymbol{\tau}$ is called <u>*topology*</u>. The sets in $\boldsymbol{\tau}$ will be called <u>*open*</u>.

We see that any metric space is topological one. The topology defined on it by the metric ρ, is called <u>*the metric topology*</u>.

Let $(M,\boldsymbol{\tau})$ be topological space, and F be a subset in M. Then we can define a topology on F, i.e. turn F into a topological space as follows. A subset $V\subset F$ is called open, if there is set W, open in M, such that $V=W\cap F$. Such topology on F is called induced from $(M,\boldsymbol{\tau})$.

The most important case for us is when F is a surface in three-dimensional space (figure 1.6). It turns out that we can determine what an open set on a surface is.

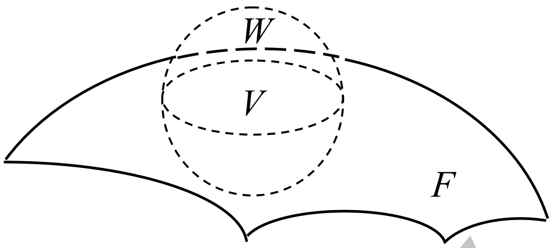

*fig.*1.6

§3. *Closed sets. Closure of set*

***Definition* 1.11.** Point P is called <u>*adherent point*</u> of set W, if any its neighborhood intersects W. This is equivalent to the fact that $\rho(P,W)=0$. Subset $W\subset(M,\tau)$ is called <u>*closed*</u>, if it contains all its adherent points.

***Definition* 1.12.** Subset W in topological space (M,τ) is called <u>*closed*</u>, if its complement $M\backslash W$ is open in M.

Let's prove, that these definitions are equivalent.

Suppose, that set W contains all its adherent points and $V=M\backslash W$. Let $P\in V$ be arbitrary point. Then P is not an adherent point, and it means that there is some neighborhood U of this point, that does not intersects W i.e. $U\subset V$. Since P is an arbitrary point of V, this set is open.

Conversely. Suppose that $V=M\backslash W$ is open set and P is an adherent point of set W. We shell prove, that $P\in W$. Suppose that is not true, i.e. $P\in V$. Since V is open, there is some neighborhood U of point P such that $U\subset V$. But this means, that P is not an adherent point of set W. We've got a contradiction. Thus $P\in W$. Since P is an arbitrary adherent point W, this set is closed.

Obviously, every point of the set V itself is its adherent point. But, if V is not closed, then there are additionally points, which do not belong to V, but are its adherent points.

***Definition* 1.13.** The set of all adherent points of set V is called <u>*the closure*</u> of set V.

We use the following notation for closure: $\overline{V}$.

***Definition* 1.14.** The set of all inner points of set V is called its interior and is denoted by $\mathring{V}$. The set $\overline{V}\backslash\mathring{V}$ is called <u>*the boundary*</u> of set V.

***Пример* 1.6.** Let $U(O,1)$ be open circular disk on the plane. Its closure is $B(O,1)=\overline{U}(O,1)=\{Q|\ \rho(O,Q)\le 1\}$ the closed circular disk, and its boundary is a circle $S^1=\{Q|\ \rho(O,Q)=1\}$.

Closure operation properties.

1. $\overline{V \cup W} = \overline{V} \cup \overline{W}$;
2. $\overline{V \cap W} \subseteq \overline{V} \cap \overline{W}$;
3. $V \subset \overline{V}$, $\overline{\overline{V}} = \overline{V}$;
4. $\overline{\varnothing} = \varnothing$.

***Example* 1.7.** Let $V=(-1, 0)$, $W=(0,1)$. Then $V \cap W = \varnothing \Rightarrow \overline{V \cap W} = \varnothing$. On the other hand, $\overline{V} = [-1, 0]$, $\overline{W} = [0,1]$, $\Rightarrow \overline{V} \cap \overline{W} = \{0\}$. This example shows, that equality in property **2** may not hold.

***Теорема* 1.2. I.** *The intersection of any finite or infinite number of closed sets is a closed set.*

II. *The union of any finite number of closed sets is a closed set.*

III. $\varnothing$ и M – are both closed.

We accept these properties and the theorem without proof.

§4. *Continuous mappigs. Homeomorphism*

Remind definition of continuous function from Mathematical Analysis.

***Definition* 1.15.** Function of one variable $f: \boldsymbol{R} \to \boldsymbol{R}$ is called <u>*continuous at point*</u> x_o, if $\forall \varepsilon > 0 \;\; \exists \delta > 0$ such that $|x - x_o| < \delta \Rightarrow |f(x) - f(x_o)| < \varepsilon$.

This definition can be reformulated in the language of open balls.

***Definition* 1.16.** Function of one variable $f: \boldsymbol{R} \to \boldsymbol{R}$ is called <u>*continuous at point*</u> x_o, если $\forall \varepsilon > 0 \;\; \exists \delta > 0$ such that $x \in U(x_o, \delta) \Rightarrow f(x) \in U(f(x_o), \varepsilon)$.

This definition is suitable also for mapping of two metric spaces $f: (M, \rho) \to (N, \rho_1)$. We can also formulate it in the following form.

***Definition* 1.17.** Let (M, ρ) and (N, ρ_1) – be two metric spaces. Mapping $f: M \to N$ is called <u>*continuous at point*</u> $P \in M$, if $\forall \varepsilon > 0 \;\; \exists \delta > 0$ such that $f(B(P, \delta)) \subset B(Q, \varepsilon)$, где $Q = f(P)$.

The meaning of this definition is as follows: points close to x_o after action of mapping turn out to be close to y_o. In other words, no matter how small the open ball with center in y_o is, there is a ball with center x_o, which is mapped into the first ball (figure 1.7).

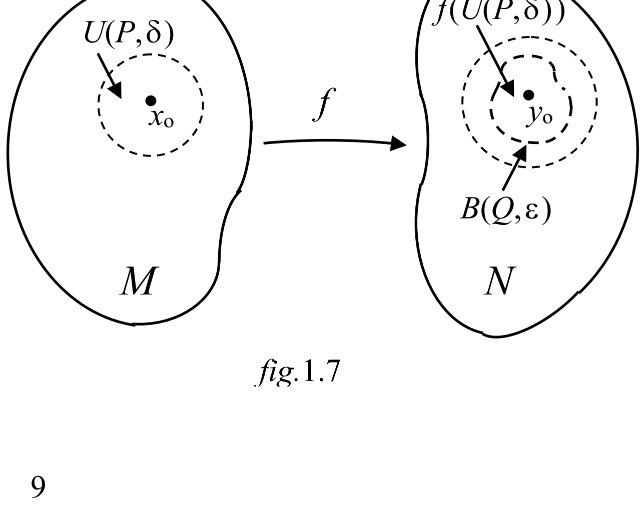

*fig.*1.7

In order to obtain a definition of continuous at a point mapping of two topological spaces, it is sufficient to replace open balls with arbitrary neighborhoods.

***Definition* 1.18.** Let $(X, \boldsymbol{\tau})$ and $(Y, \boldsymbol{\tau}_1)$ be two topological spaces. Mapping $f\colon X \to Y$ is called <u>*continuous at point*</u> $P \in X$, if for any neighborhood V of point $Q = f(P) \in Y$ there is such neighborhood U of point x_o, that $f(U) \subset V$.

The condition used in the definition: «for any neighborhood V of point $y_o = f(x_o) \in Y$ there is such neighborhood U of point x_o, that $f(U) \subset V$» is called <u>*Cauchy condition*</u>. Mapping $f\colon X \to Y$ called <u>*continuous*</u>, if it is continuous at every point $x_o \in X$.

***Theorem* 1.3**. *Mapping $f\colon X \to Y$ of two topological spaces is continuous if and only if preimage (inverse image) of every open set $V \subset Y$ is open set $U = f^{-1}(V)$ in X.*

Proof. Suppose that $f\colon X \to Y$ is continuous and $V \subset Y$ is open set. Let $U = f^{-1}(V)$, $P \in U$ be an arbitrary point and $Q = f(P)$. Then $Q \in V$. Since V is open, there is a neighborhood W of point Q such that $W \subset V$. Since f is continuous, There is neighborhood O of point P in X such that $f(O) \subset W \subset V$. This inclusion means that $O \subset U$. Thus P is an inner point. Since $P \in U$ is an arbitrary point, U is open.

Conversely. Suppose that preimage of every open set in Y is open set in X, $P \in X$ is an arbitrary point and $Q = f(P)$. Let V be an arbitrary neighborhood of point Q and $U = f^{-1}(V)$. Then U is open and $f(U) \subset V$. It means that our mapping is continuous at point P. Since $P \in U$ is an arbitrary point, f is continuous. ■

***Definition* 1.19.** Mapping $f\colon X \to Y$ of two topological spaces is called <u>*open*</u>, if image of any open set U in X is open set $V = f(U)$ in Y.

***Definition* 1.20.** Отображение $f\colon X \to Y$ of two topological spaces is called <u>*homeomorphism*</u> или <u>*topological mapping*</u>, if this mapping is

1) bijective (i.e. one-to-one mapping of set X on the entire set Y);
2) continuous;
3) open.

This definition is equivalent to the fact that f is invertible and both mappings f and f^{-1} are continuous.

It turns out that the topological maping $f\colon X \to Y$ sets up a one-to-one correspondence between the open sets of space $(X, \boldsymbol{\tau})$ and open sets of space $(Y, \boldsymbol{\tau}_1)$. Therefore, from the point of view of the topology, spaces $(X, \boldsymbol{\tau})$ and $(Y,$

τ_1) are arranged identically, if there is homeomorphism $f\colon X \to Y$. In this case the spaces are called <u>*homeomorphic*</u> or <u>*topologically equivallent*</u>.

***Example* 1.8.** Open interval $(-1, 1)$ and the entire numerical line $\boldsymbol{R}$ are homeomorphic. Topological mapping is set up by function $f\colon(-1,1)\to \boldsymbol{R},\ f(x)=\mathrm{tg}\frac{\pi}{2}x$ (figure 1.8).

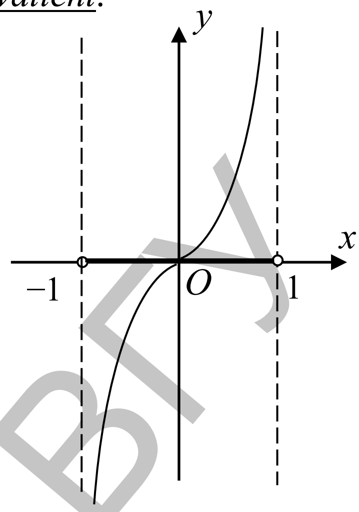

*fig.*1.8

***Example* 1.9.** The sphere S^2 and the plane $\boldsymbol{R}^2$ are not homeomorphic. However, if we delete one point out of the sphere, then the remaining set will be homeomorphic to the plane. Homeomorphism is set up by, so called, stereographic projection $p\colon S^2\backslash\{N\}\to\boldsymbol{R}^2$ (figure 1.9).

When we are talking about surfaces, homeomorphism can be visualized as follows. We can wrinkle, compress and stretch the surface as you like (like made of rubber), we can't only cut and glue. Everything that results will be homeomorphic to the original surface.

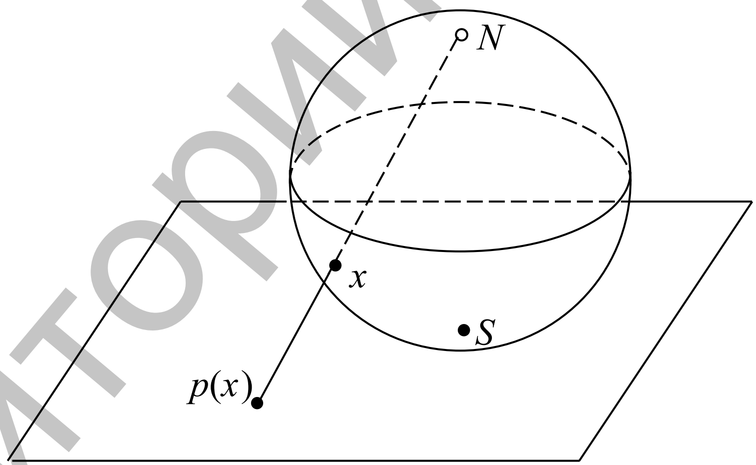

*fig.*1.9

***Example* 1.10.** A sphere and any convex polyhedron (tetrahedron, cube, ...) are homeomorphic. In order to construct a homeomorphism, we place the polyhedron inside the sphere so that the center of the sphere is inside the polyhedron and project it from the center onto the surface of the sphere (figure 1.10).

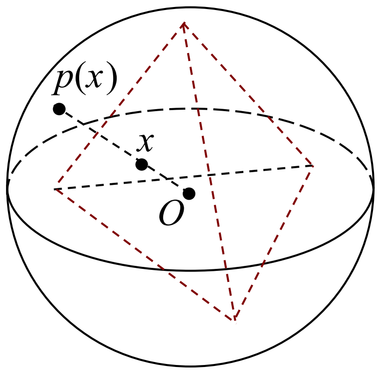

*fig.*1.10

In order to prove that surfaces or curves are homeomorphic, it is sufficient to construct a homeomorphism. If we failed to construct it, this does not mean that the homeomorphism does not exist. Therefore, to prove that two topological spaces are not homeomorphic, we must find quantities that are preserved under homeomorphism. They are called topological invariants. Consider one example.

Example **1.11.** The topological invariant for curves is the presence and number of dividing points. For example, removing one point from a line, we divide it into two disconnected sets. If one point is removed from the circle, then it remains connected (figure 1.11). Therefore, the circle S^1 and the line are not homeomorphic.

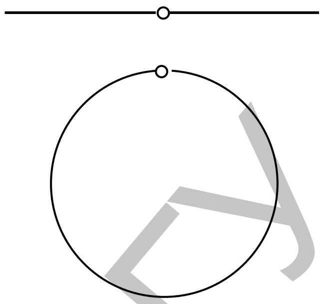

*fig.*1.11

Exercise **1.1.** Taking into account the previous example, find very simple topological invariant that proves that the sphere and the torus are not topologically equivalent.

§5. *The main task of the Topology*

A mapping is said to be continuous, if two points close to each other remain to be close after the action of the mapping.

Example **1.** Consider the circle F and two points A, B at a short distance to each other. We chose a point C on the arc AB and tear the circle at the point C. We obtain an arc Φ (figure 1.12). New positions of the points A, B we denote A', B'. Now A', B' are far from each other. The mapping we constructed $F \to \Phi$ is not continuous.

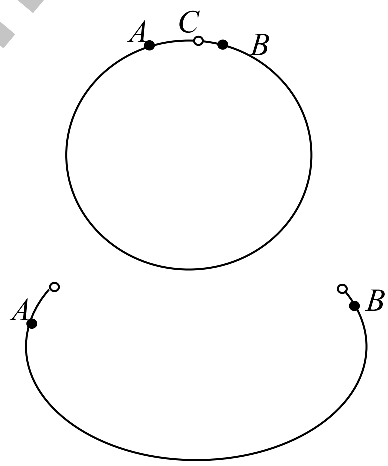

*fig.*1.12

Mapping f is called <u>*homeomorphism*</u> or the topology mapping, if it is bijection and it is continuous in both sides. It means, that f and f^{-1} are both continuous.

The topology studies the properties of figures, which are preserved under the action of topology mappings. Two figures F_1 and F_2 are said to be <u>*homeomorphic*</u> or <u>*topologically*</u> <u>*equivalent*</u>, if there exists a homeomorphism $f: F_1 \to F_2$.

One can imagine visually the topology mapping as follows: we can squeeze or stretch a set or crumple it, but we can't tear or paste it.

The idea of continuity is the main idea of proves.

***Theorem* 1.** *A square can be circumscribed about any closed curve.*

Proof. Consider a pair of parallel straight lines l and l', such that the curve γ is located in the strip between them. Then we move this lines continuously until they become tangent to γ. The lines we get are called the lines of support for γ.

Let's draw one more pair lines of support m and m', that are perpendicular to l. We obtain the rectangle $ABCD$. Lets prove, that $ABCD$ can be the square for some direction of line l.

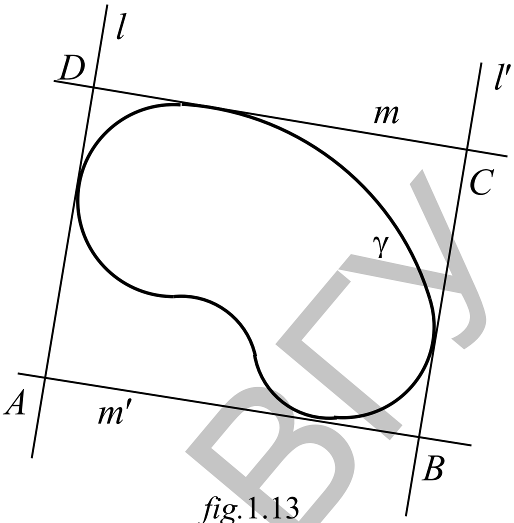

*fig.*1.13

Let AD be the side, that is parallel to l and AB be the side, that is perpendicular to l. Denote the length of AD as $h_1(l)$ and the length of AB as $h_2(l)$. The circumscribed rectangle is a square, if $h_1(l)-h_2(l)=0$.

Lets construct a rectangle starting with the pair of lines m and m'. It coincides with $ABCD$. So

$$h_1(m)-h_2(m)=-(h_1(l)-h_2(l)).$$

Lets turn the straight line l until it coincides with m. The circumscribed rectangle will be transformed continuously, so the value $h_1(l)-h_2(l)$ will vary continuously to the opposite value. Therefore there exists a position of the line l, when the value $h_1(l)-h_2(l)$ is equal to zero. It means, that there exists a position of the line l, when $ABCD$ is a square.

CHAPTER 2. FOUNDATIONS OF THE GRAPH THEORY

§1. *The most simple topological invariants*

We noted in the example 4, that a sphere is not homeomorphic to torus. But how we can prove this statement? If we couldn't manage to find homeomorphism, it doesn't mean, that this homeomorphism doesn't exist. In this situation we should find so called topological invariants. These are the numbers related to figures, that are equal for topologically equivalent figures. If these numbers are not equal, the figures are not topologically equivalent.

***Example* 5.** A hyperbola consists from two pieces. We call them the connected components. A parabola has one connected component. The number of connected components is a topological invariant.

***Example* 6.** Consider a figure-of-eight and the point X on it (figure 2.1). If we delete this point, the figure will become disconnected. Such point is called a <u>*splitting point*</u>. The number of splitting points is a topological invariant. The circle has no splitting points. We can delete any point from a circle and the figure remains connected. So the circle is not topologically equivalent to the figure-of-eight.

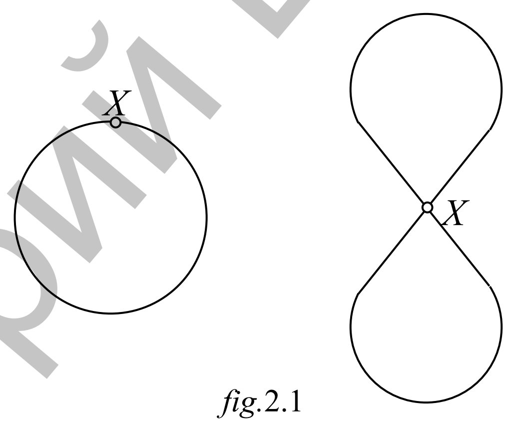

*fig.*2.1

***Definition*.** A figure, that consists from finite number points and finite number of arcs connecting this points is called a <u>*finite graph*</u>. The points are called <u>*vertexes*</u> and the arcs are called <u>*edges*</u>. A number of edges, that meet at the vertex A is called an <u>*index of this vertex*</u>.

There can be the edges that have the beginning and the end at the same vertex. Such vertexes are called <u>*loops*</u>.

***Definition*.** A connected graph is called <u>*a full graph*</u>, if it has no loops and any pare of vertexes is connected by the only one edge.

***Exercises* 1.** *Let a_k be the number of vertexes with the index k. Proof, that the total amount of edges in a connected graph is equal* $\frac{1}{2}(a_1+a_2+\ldots+a_k)$.

2. *Proof, that the total amount of edges in a finite graph, which have an odd index is even.*

Definition. A graph is said to be <u>*unicursal*</u>, if it can be drawn with a stroke of a pen, i.e. one can find a path through the entire graph that passes each edge only once.

The property of a graph to be unicursal is obviously a topological invariant.

Theorem 2. *A connected finite graph is unicursal if and only if it has no more than 2 vertexes of odd index.*

Proof. Sufficiency. **1.** Suppose first, that there are no vertexes with odd index in a graph G. Let A be an arbitrary vertex. We start our path at vertex A. Suppose, that we have come to vertex B for the first time. It means, that we have used one edge ending in the vertex B. The number of edges ending in B is even. So there is at least one more edge ending at B that we haven't used. Therefore we can leave B by this edge. Suppose, that we have come to the vertex B for the second time. It means, that we have used 3 edges ending at vertex B and there is at least one more edge ending at B that we haven't used. So we still have opportunity to leave B. The case, that we have no opportunity to leave a vertex can happen only if we have returned to the point A.

Suppose, that we have returned to the point A and we haven't used all the edges of our graph. Denote our path as G_1. It is a graph that has all the vertexes of even index. Therefore the graph $G \setminus G_1$ has all the vertexes of even index too. Let C be the first point on G_1 that has non-used edges. We start a new path G_2 at vertex C by graph $G \setminus G_1$. The opportunity that we can't leave a vertex can happen only if we came back to C. Now we construct a new path G^* as follows. We start by the path G_1 and stand by G_1 until we come to the vertex C. Then we continue our path by G_2 and we come back to C. After that we go by the remained part of G_1 to point A.

Suppose, that G^* still does not coincide with the whole G. Let D be the first vertex on G^* that has non-used edges. We add to G^* a new part of the path from D to D. And so on. Graph G is finite. So after the several procedures of adding the new parts we will exhaust all the edges of G and we will get the path from A to A along all the edges of G.

2. Suppose now, that there are 2 vertexes with odd index A_1 and A_2 in a graph G. The graph G is connected. So there is a path G_1 from A_1 to A_2. This path contains odd number of edges which end at A_1 and odd number of edges which end at A_2. Therefore the graph $G \setminus G_1$ has all the vertexes of even index and we can use the procedure of adding the new parts to G_1, described above. After the several procedures we will exhaust all the edges of G. ■

***Exercise* 3.** *Proof a necessity in this theorem.*

The following problem is classical problem and it was formulated by Leonard Euler in 18th century. There were 7 bridges in Koeningsberg over the Pregl river (figure 2.2). The question is: if it is possible to walk along all the bridges and only once along each of them? Denote the islands as A and B, the banks as C and D. So we can illustrate this map by a graph, that has 4 vertexes and 7 edges (figure 2.3). We see, that there are all the vertexes of odd index. Therefore this graph is not unicursal.

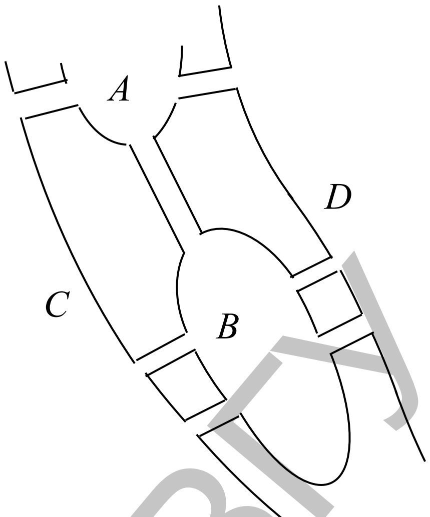

*fig.*2.2

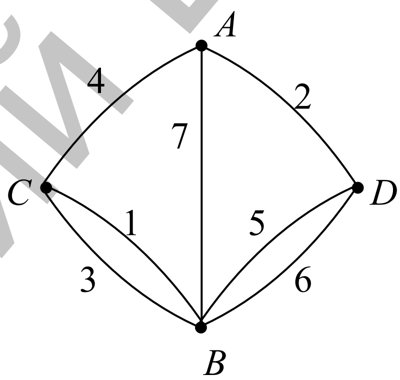

*fig.*2.3

***Exercises* 4.** *It is sufficient add a bridge anywhere in order to make this graph unicursal. Proof this statement.*

5. *Find the condition, for the full graph to be unicursal.*

§2. *Eulerian characteristics of a graph*

Any graph can be constructed gradually by adding one edge after another. For example, we can give the numbers to every edge and draw the edges in this order.

***Example* 7.** Consider the graph on the picture 8. If we draw the edges in the indicated order, the graph will be not connected first. If we draw the edges in the inverse order, the graph will be connected all the time.

***Theorem* 3** (*about drawing of a connected graph*). *Any connected graph can be drawn as follows. We take one edge, then add another edge so, that we get a connected graph; then we add one more edge so, that we get a connected graph again and so on.*

***Exercises* 5.** *Prove, that any connected graph can be drawn with a stroke of a pen, if it is allowed to pass each edge for several times.*

6. *Prove the theorem 3. You can use the previous exercise.*

7. *The sequence of edges is called to be* <u>*simple*</u>*, if its unit is homeomorphic to a closed interval. Prove that any 2 vertexes of the graph could be connected by simple sequence of edges.*

8. *Suppose, that any 2 vertexes of a graph can be connected by at least 2 simple sequences of edges. Prove that such graph has no vertexes of index 1. Is the inverse statement true?*

Definition. A closed sequence of edges of graph G is called to be <u>*a contour*</u>, if it is homeomorphic to a circle (figure 2.4). A connected graph G is called to be <u>*a tree*</u>, if it contains no contours (figure 2.5).

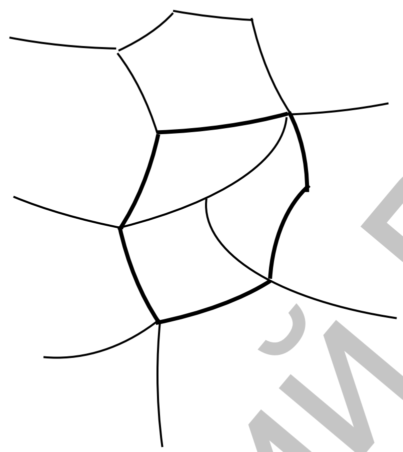

*fig.*2.4

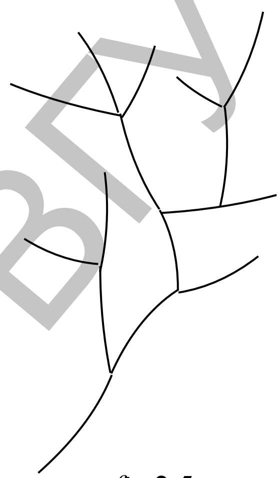

*fig.*2.5

Let's prove now, that for any tree with V vertexes and E edges the following formula is true:

$$V - E = 1\,. \qquad (1)$$

We use induction by the number of edges.

Let $E = 1$. Then the tree has 1 edge and 2 vertexes and the formula is true. Suppose, that the formula is proved for the number of edges $E = n$ and the graph G has $n+1$ edges. Graph G is a connected graph. Therefore we can construct G from some connected graph G' by adding of one edge r. Graph G' is also a tree. According to our assumption the formula (1) is true for G'. We note, that only one end of the edge r gives us a new vertex. So, adding of r brings us 1 more vertex and 1 more edge and the formula remains true. The situation, when adding of r brings us no new vertexes is impossible. We get a loop in this case (figure 2.6).

The induction we carried out, proves that formula (1) is true for any tree.

Definition. Let G be an arbitrary graph, V be the number of vertexes and E be the number of edges. The difference $V - E$ is called Eulerian characteristics of G and we denote it as $\chi(G)$.

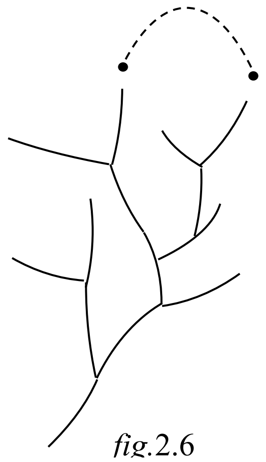

*fig.*2.6

We proved, that Eulerian characteristics of a tree is equal to 1.

***Exercises* 9.** *A graph, that contains no contours is called a wood. Let G be a wood and m be a number of trees in G (i.e. the number of connected components). Prove, that $\chi(G)=m$.*

10. *Prove, the following statement. If G is a tree, than two arbitrary vertexes in G can be connected by the unique simple sequence of edges. Is the inverse statement true?*

Let G be a connected graph, and it is not a tree. Then there is a loop in G. Let r_1 be an arbitrary edge of this contour. If we delete r_1 from G, we obtain a connected graph G', because the ends of r_1 are connected by the remaining part of the contour. Graph G' has the same vertexes as G. If G' is not a tree, it has a contour. We took an arbitrary edge r_2 from this contour and get a connected graph G''. It has the same vertexes as G. And so on. At the final step number k we delete the edge r_k and obtain a connected tree G^*, that has the same vertexes as G. We call G^* *the maximal tree* in G. The edges $r_1, r_2, \dots, r_k$ are said to be *cross-connection*. For graph G^* we have $V-E=1$. The graph G has $E+k$ edges. So

$$\chi(G)=V-(E+r)=1-k.$$

So we have proved the following theorem.

***Theorem* 4.** *For any connected graph G the inequality*

$$\chi(G)\le 1 \qquad (2)$$

is true and the equality $\chi(G)=1$ takes place only if G is a tree.

According to (2) the number of cross-connections we can calculate by the formula

$$k=1-\chi(G). \qquad (3)$$

***Exercises* 11.** *If a connected graph can be obtained from a tree by adding of several closed edges (one-edge loops), than it has an only one maximal tree. Is the inverse statement true?*

12. *Prove, that if a graph has l connected components, then $\chi(G)\le l$. When the equality takes place?*

13. *Suppose that non-negative number and the direction are prescribed to every edge of a graph G. Then we say, that a system of currents is given in G. Also the rule of Kirhgoff is supposed to take place: the sum of all the currents running in any vertex is equal to the sum of all the currents running out of it.*

Prove, that if the graph G is a tree, then there could be only a trivial system of currents in G (it means all the currents should be equal to zero).

14. *Let G be a graph, G^* be its maximal tree, $r_1, r_2, \ldots, r_k$ be the cross-connections. Prove, that if we prescribe any currents to $r_1, r_2, \ldots, r_k$, we can extend them to G^* the only one way.*

§3. *Index of intersection*

Let us consider 2 examples of graphs, which cannot be embedded in the plane.

***Example* 8.** The graph "houses and wells". Six point H_1, H_2, H_3 (the houses), W_1, W_2, W_3 (the wells) are given on the plain. The task is to draw the paths from each house to each well, so that paths are not to intersect. Is it possible to do? The answer is negative. If we draw all the paths except the last one, there would be no place for this path (figure 2.7). We denote this graph as HW.

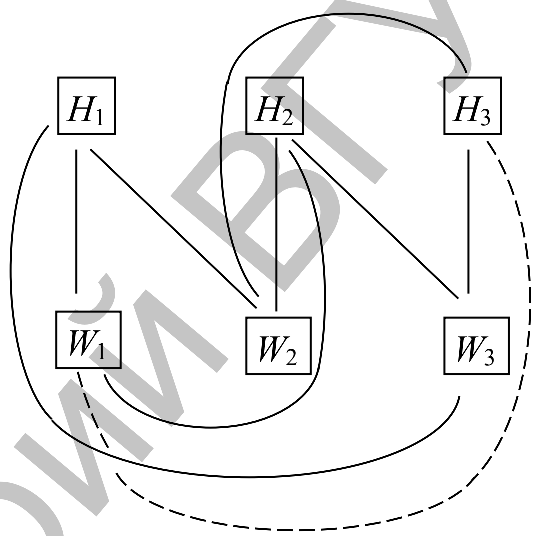

fig.2.7

***Example* 9.** Consider the full graph with 5 vertexes, were each vertex is connected with 4 other ones. We denote it as FG_5 (figure 2.8).

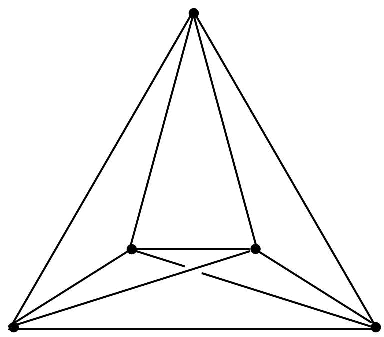

fig.2.8

It is interesting to note, that these two graphs are the standards of graphs, that couldn't be embedded in the plane. It means the following. If the graph couldn't be embedded in the plane, then it contains one of the graphs HW or FG_5.

***Exercises* 15.** *Prove, that the following graph couldn't be embedded in the plane.*

16. *Vertexes of a graph are the vertexes of the regular polygon with n angles and the edges are the sides and the shortest diagonals of the polygon. Prove, that for even n this graph could be embedded in the plane, and for odd n couldn't be embedded.*

17. *Vertexes of a graph are the vertexes of the regular polygon with* $2n$ *angles and the edges are the sides and the longest diagonals of the polygon. Prove, that for* $n \geq 3$ *this graph couldn't be embedded in the plane (figure 2.9).*

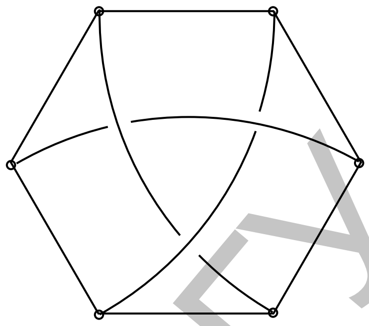

fig.2.9

We are going to submit below the full prove, that the graphs HW or FG_5 couldn't be embedded in the plane.

Definition. Let a and b are two segments on the plain and the end points of one segment do not belong to the other segment. We will write $J(a,b)=1$, if the segments intersect and $J(a,b)=0$, if the segments do not intersect. The value $J(a,b)$ is called the <u>*intersection index*</u> for the segments a and b.

Definition. A finite set of segments on the plain is called a <u>*chain*</u> or <u>*a simple chain*</u>. The segments of the chain are called the <u>*links*</u> and the end points of the links are called the <u>*vertexes*</u>. If the number of links which meet at each vertex is even, we say that the chain is a <u>*cycle*</u>. For example, any closed broken line is a cycle.

Let x, y be two chains and the vertexes of one chain do not belong to the other chain. Let chain x consist of segments $a_1, a_2, \ldots, a_m$ and the y consist of segments $b_1, b_2, \ldots, b_n$. The sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} J(a_i, b_j)$$

means the sum of all indexes of intersection of each link of x with each link of y. If this sum is even, we write $J(x,y)=0$, and if this sum is odd, we write $J(x,y)=1$. The value $J(x,y)$ is called the <u>*intersection index of two chains*</u> x and y. We are going to prove, that index of intersection of two cycles is equal to zero.

***Exercise* 18.** *Prove, that any cycle is the union of several contours, which have no common links.*

According to this result, it is sufficient to prove, the following theorem.

***Theorem* 4.** *Intersection index of two contours* x *and* y *is equal to zero.*

Proof. First we move these contours a bit in order to make each link of x be nonparallel to each link of y. This movement doesn't change the intersection index. Then we choose an arbitrary straight line l, which is not parallel to any straight line connecting a vertex of x with a vertex of y. We begin to move the contour x continuously in the direction parallel to l. In the process of movement vertexes of x can't meet the vertexes of y. So the intersection index could change only at the moment, when a vertex of one contour meet the link of the other contour. But as the result of this moment, the number of intersections can increase on 2, decrease on 2 (figure 2.10), or it can remain the same (figure 2.11). So, the number of intersections preserves its parity.

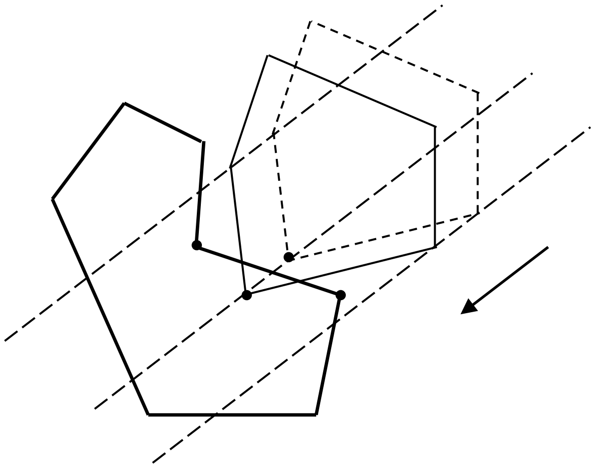

*fig.*2.10

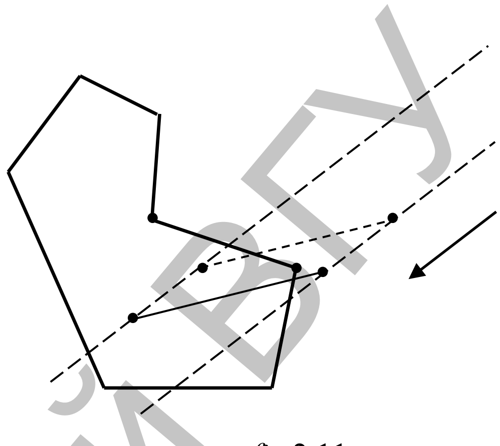

*fig.*2.11

We can move the contour x in the position, where it has no intersections with y. So $J(x,y)=0$ in this position and in any other position too. ■

Now we can prove, that HW-graph couldn't be embedded in the plain. Two paths, that leads from different houses to different wells we will call <u>*nonadjacent*</u>. Let's draw all the paths (possibly with intersection) (figure 2.12) and denote I – the number of intersections of all the pairs of nonadjacent paths. We are going to prove that for any way of drawing the paths this number is odd.

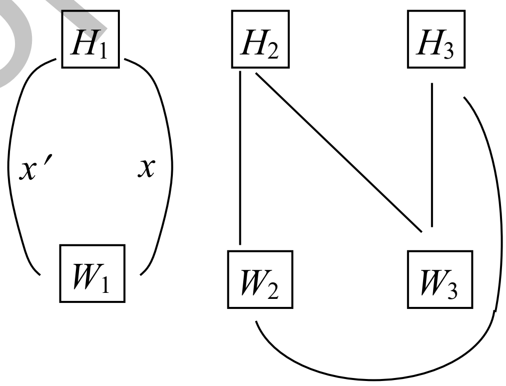

*fig.*2.12

Suppose, that we change the position of one path, for example H_1W_1. Let x be its old position, x' be a new one. Nonadjacent to x are the following paths: H_2W_2, H_2W_3, H_3W_2 and H_3W_3. They form a cycle $y=H_2W_2H_3W_3H_2$. Two paths x and x' also form a cycle. The index of intersection of these two cycles is equal to zero. It means, that the number of intersections of x with y and of x' with y has the same parity: $J(x,y)=J(x',y)$. Therefore it is obvious, that for any position of the paths on the plain the number I has the same parity. Really, we can change a position of the first path, then we change a position of the second path and so on. Finally we can obtain any prescribed position of the paths. We have the only one intersection on figure 2.7. So, for any position of the paths on the plane the number I is odd and it is not equal to zero. Therefore the graph HW couldn't be embedded in the plane

***Exercises* 19.** *Prove, that the graph FG_5 couldn't be embedded in the plane.*

20. *Prove, that the index of intersection of any two cycles on the sphere is equal to zero. Show, that there are two cycles on the torus, which has the only one intersection.*

We considered above the segments without a direction. Let now a and b be directed segments.

Definition. We move along the segment a and pay attention on the direction of b. If we note, that b intersects a from right to left we write $J(a,b)=1$. If we note that b intersects a from left to right we write $J(a,b)=-1$. If a and b have no intersection, we write $J(a,b)=0$. The number $J(a,b)$ we call *an index of intersection of the directed segments a and b*.

Definition. A finite set of directed segments on the plain we call an *integer chain* or simply *a chain*. Let the chain x consist of segments $a_1, a_2, \ldots, a_m$ and the y consist of segments $b_1, b_2, \ldots, b_n$. *Index of intersection* of two integer chains x and y is defined as the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} J(a_i, b_j),$$

We say, that an integer chain is a *cycle* if for any vertex of the chain the number of links coming into the vertex and to a number of links coming out the vertex are equal.

***Exercises* 21.** *Consider a closed broken line and let us choose a direction on of round trip along this line. We call this line a directed contour. Prove that any integer cycle is a union of finite number of directed contours, which has no common links.*

22. *Prove, that the index of intersection of integer cycles is equal to zero.*

23. Consider a cycle, which consists of two directed contours a_1 and a_2 (figure 2.13). Let the point O be in the internal part of the ring and A be an arbitrary point. Let y be a directed broken line, that connects O and A. Prove, that A belongs to the external domain of the ring if and only if $J(x, y)=2$. In what case A belongs to the shaded part of the ring?

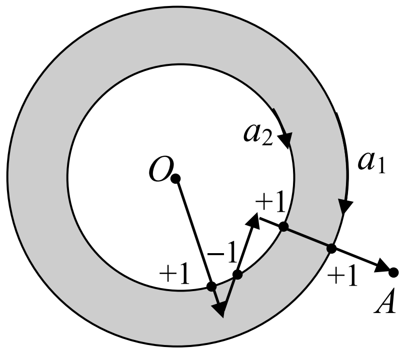

*fig.*2.13

§4. *Theorem of Jordan*

We proved above, that index of intersection of two cycles on the plain is equal to zero. It seems, that there could be a more simple prove: in each point of intersection the closed broken line comes into the internal domain of the line or comes out of it. The number of coming-in points should be equal to the number of coming-out points. Therefore the number of intersection points is even.

This prove could be admitted only if the sense of the notion "internal domain" is cleared up. But this notion is not such simple as it seems at first sight.

Definition. A closed line homeomorphic to the circle is called a <u>*simple closed line*</u>. Let l be a simple closed line. If two points A and B could be connected by a broken line, which doesn't intersect l, we say, that A and B are located in the same domain with respect to l. If any broken line with the ends at A and B intersects l, we say, that A and B are located in the different domains with respect to l.

Theorem of Jordan. *Any simple closed line divides the plain on two domains. One of them is bounded and it is called the internal domain of the line and the other one is unbounded and it is called the external domain.*

Jordan Theorem seems very simple only because we imagine to ourselves only very simple lines. If it is possible to define immediately, in what domain (internal or external) points A, B, C and D lies (figure 2.14)?

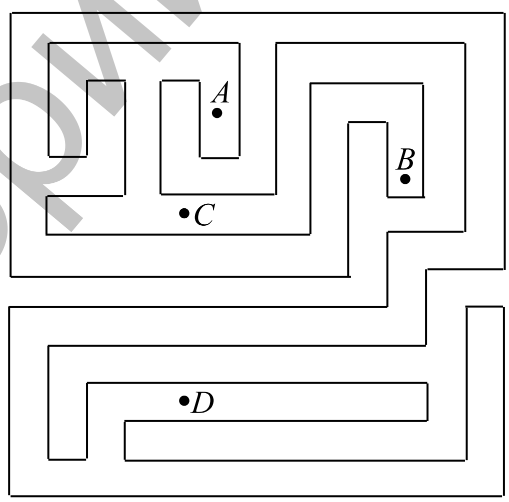

fig.2.14

We will prove Theorem of Jordan only for the case, when l is a simple closed broken line. Let a_1, a_2, …, a_n be the consecutive links. Let b_i be the bisectrix of the angle between the links a_i and a_{i+1} and b_o be the bisectrix of the angle between the links a_n and a_1. We take two points P_1 and P_2, which are symmetric to each other about the link a_1 and draw a segment c_1 parallel to a_1 from P_1 till the intersec-

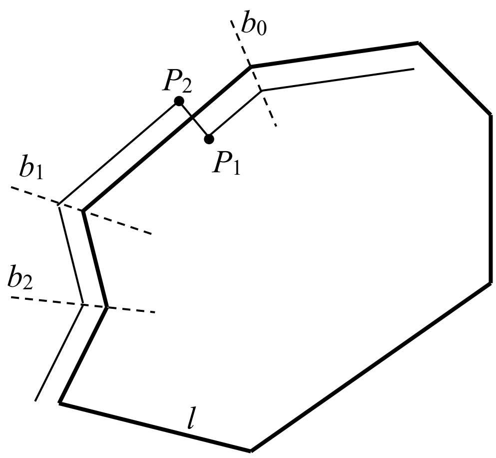

fig.2.15

tion point A_1 with b_1 (2.16). Then we draw a segment c_2 parallel to a_2 from A_1 till the intersection point A_2 with b_2 and so on. Finally we get a broken line c and all the links of c are on the same distance from l. Therefore the end point of c could be only P_1 or P_2.

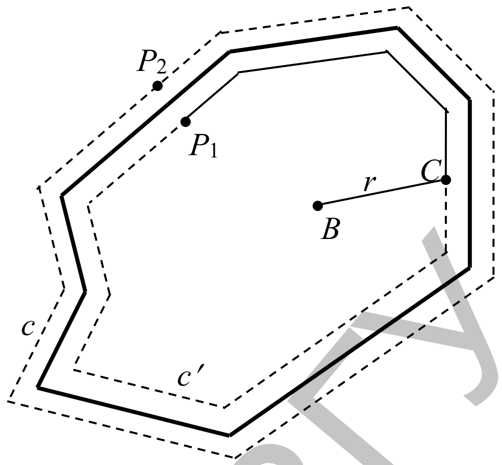

*fig.*2.16

Suppose, that the end point is P_2. We add to c the link P_1P_2 and we obtain a contour, that has an only one intersection point with the contour l (figure 2.16). We have got a contradiction. So, the end point of c could be only the point P_1.

In the similar way we obtain a broken line c' with the beginning and end point P_2. Let B be an arbitrary point, that doesn't belong to l. We draw an arbitrary ray r from B. This ray meets c or c' earlier, than it meets l. Let C be the first point of intersection of r with c or c'. We move along r and then along c or c' to the point P_1 or P_2 and we doesn't meet l. We prove, that an arbitrary point could be connected with P_1 or P_2 by a broken line, that doesn't intersect l (figure 2.16).

Suppose, that B could be connected with such broken line with both points P_1 and P_2. We unite this lines and add the segment P_1P_2. We obtain a contour, that has the only one point of intersection with l. We have got a contradiction. So, we can divide all the points of the plain on two classes. Points of the first class could be connected with P_1, points of the second class could be connected with P_2 and this classes do not intersect.

It easy to note, that all the points far from l belong to the same class. Therefore the only one class is unbounded. The other class should be bounded. ■

Exercises **24.** *Let l be a closed broken line, A be an arbitrary point, that doesn't belong to l and AB be a ray. Prove the following statement. If the ray AB has an odd number of intersection points with l, then A belongs to the internal domain of l and if AB an even number of intersection points with l, then A belongs to the external domain of l.*

25. *Prove, that any simple closed curve divides the sphere on two domains.*

26. Let k broken lines connecting two given points be drawn on the plain. Prove, that these lines divide the plain on k domains.

CHAPTER 3. TOPOLOGY OF SURFACES

§1. *Theorem of Euler*

The numbers of vertexes, edges and faces for several polyhedrons are given in the following array.

The name of the polyhedron	The number of vertexes	The number of edges	The number of faces
Tetrahedron	4	6	4
Cube	8	12	6
n-angle pyramid	6	12	8
n-angle prism	$2n$	$3n$	$n+2$

We see, that for any polyhedron the following relation takes place:

$$V - E + F = 2. \tag{4}$$

Here V is the number of vertexes, E – the number of edges and F – the number of faces. It is easy to check this relation for arbitrary pyramid or prism. Euler was the first, who noted this remarkable property of polyhedrons and proved it.

We make now the statement of the Euler theorem more precise. We note first of all, that any convex polyhedron is homeomorphic to the sphere and any face of a polyhedron is homeomorphic to the circle. We can construct a topology mapping of a polyhedron to the sphere as follows. We imbed the polyhedron into the sphere so that the center of the sphere is located inside the polyhedron (figure 3.1). Then we make a projection of the polyhedron to the sphere. Now the precise statement of the theorem looks as follows.

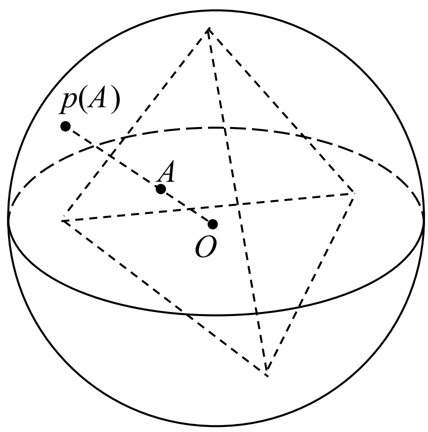

*fig.*3.1

Euler theorem. *For any polyhedron with surface homeomorphic to the sphere and each face homeomorphic to the circle relation* (4) *is true.*

We can make pure topological statement of this theorem. We note, that all the vertexes and all the edges of a polyhedron form a connected graph. This graph divides the surface of the polyhedron on the faces, which are homeomorphic to the circle.

Euler theorem. *Let* Φ *be the sphere or any surface homeomorphic to the sphere. Let a connected graph which has* V *vertexes and* E *edges be* *drawn on the surface, and this graph divides the surface* Φ *on* F *domains (faces) so that each face is homeomorphic to the circle. Then the relation* (4) *is true.*

The idea of the prove is contained in the exercise 27.

***Exercises* 27.** *Let* G *be a connected graph, drawn on the sphere,* G^* *be its maximal tree and* k *be the number of cross-pieces (i.e. the number of the edges that does not belong to* G^**). Prove, that* G^* *defines the only one domain on the sphere and so the equality* (4) *is true. If we add the cross-pieces to* G^* *one by one, we increase the number of domains on 1 on each step. Using this result prove the statement of Euler theorem.*

28. *Prove, that the equality* (4) *is true for the plain (we should add one unbounded domain to the number of domains).*

§2. *Notion of a surface*

Definition. A set $\Phi \subset E^3$ is said to be <u>*elementary surface*</u> if it is homeomorphic to some domain U on the plain. Homeomophism $\mathbf{r}\colon U \to \Phi$ is called <u>*paramerized surface*</u> or <u>*a parametrization of the elementary surface*</u> Φ (figure 3.2).

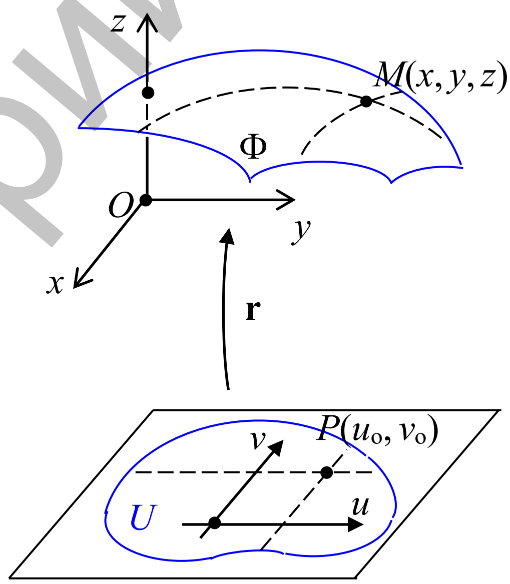

fig.3.2

We can imagine it visually as followes. We put a plain domain in the space and deform it continuously without gluing (by means of homeomophism $\mathbf{r}$).

We shell underline, that elementary surface has no boundary.

We know many surfaces, which a not elementary ones. For example, the cylinder (figure 3.3), the sphere (figure 3.4) and the torus are not elementary surfaces. Therefore we need the following definition.

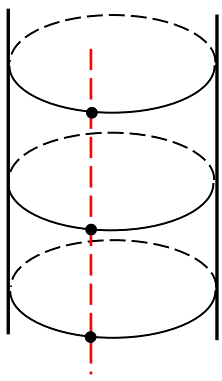

fig.3.3

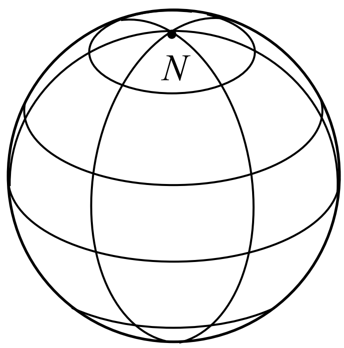

fig.3.4

Definition. A set $\Phi \subset E^3$ is said to be <u>*a simple surface*</u>, if any point $P \in \Phi$ has a neighborhood $V \subset \Phi$, which is an elementary surface.

***Examples* 1.** It is sufficient to delete one point from the sphere, and the remaining part would be homeomorphic to the plain and therefore the sphere without one point is a simple surface.

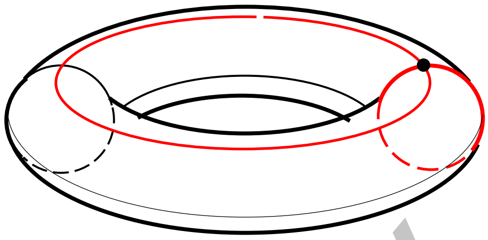

*fig.*3.5

2. It is sufficient to delete one straight line from the cylinder, and the remaining part could be developed on the plain.

3. It is sufficient to delete two circles from the torus, and the remaining part would be homeomorphic to the plain (fig. 3.5).

All these examples show us, that sphere, the cylinder and the torus are simple surfaces.

Definition. Suppose, that a set $\Phi \subset E^3$ has points of two classes. Points of the first class has a neighborhood, that is homeomorphic to a domain U on the plain, points of the second class has a neighborhood, that is homeomorphic to the open semicircle with a diameter. Then the points of the second class are said to be the boundary points, they form the boundary of the surface and the set Φ is said to be a surface with boundary (figure 3.6).

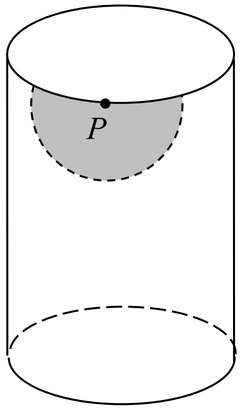

*fig.*3.6

For example, the lateral aria of the finite cylinder is a surface with boundary. If we cut out a circular hole in the sphere, or, may be several holes, we get a surface with a boundary (figure 3.7).

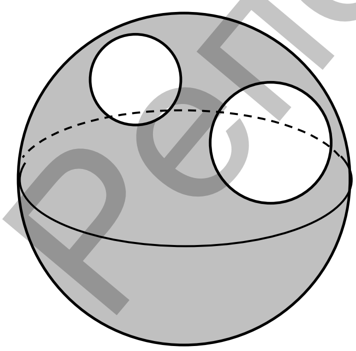

*fig.*3.7

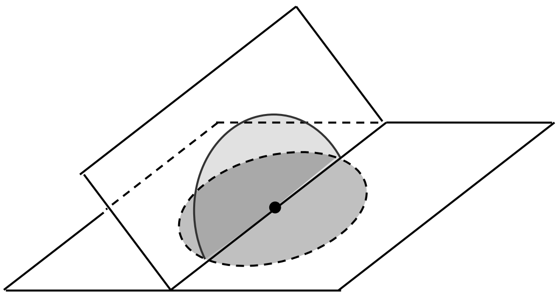

*fig.*3.8

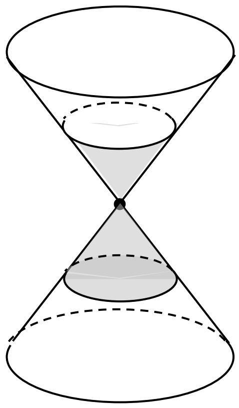

*fig.*3.9

A surface can have so called singular points, which doesn’t relate to the classes, indicated above. A neighborhood of a singular point has more complex structure. For example, it can be homeomorphic to 3 open semicircles with the common diameter (figure 3.8). The vertex of the cone is homeomorphic to 2 open circles with the common center (figure 3.9).

An interesting example of a surface with boundary was described by German mathematicians Mőbius and Listing. We can get it as follows. Let’s take a rectangular strip of paper (figure 3.10) and glue its ends so, that the indicated arrows coincide. This surface is called the Mőbius band. It has the only one side (figure 3.11). More exactly it means the following.

fig.3.10

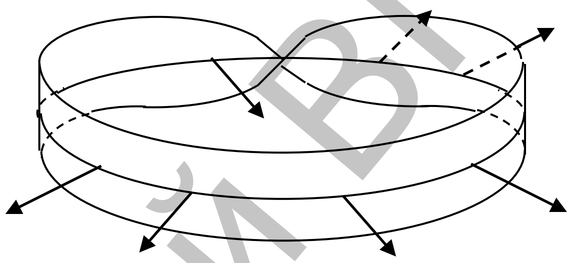

fig.3.11

Consider the middle line of the Mőbius band. Let’s choose a normal unit vector in one point on the middle line. Then we move along this line and keep an eye on the normal vector. If this vector varies continuously, we will get an opposite vector, when we return to the same point.

But we can talk about normal vectors only because the surfaces are located in the space. We will adduce now an “internal” definition of the one-side surface.

Let P an arbitrary point on a surface Φ. We choose a normal vector $\vec{\mathbf{n}}$ at point P and a little circle around P on the surface. We choose a direction on the circle, which seems counter-clockwise from the end of the vector $\vec{\mathbf{n}}$. If the point moves by the surface with the normal vector, then the circle moves as well (figure 3.12). Suppose that the point goes around by the closed curve γ and returned to its initial position. Then the circle can preserve its direction or it can change its direction. If for any closed curve on the surface the direction is always preserved, we say, that surface Φ has two sides or that the surface is orientable. Otherwise we say that it has one side or that the surface is nonorientable. For example, if we move a point with the circle by the middle line of the Mőbius band, we will get an opposite direction on the circle.

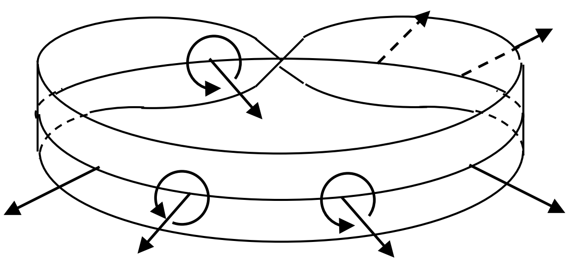

fig.3.12

Examples **4.** If we fold a surface from the following rectangle (figure 3.13), we get the cylinder (figure 3.14).

Consider now the rectangle from the figure 3.15. The sides that are to be glued together are signed by the same letter.

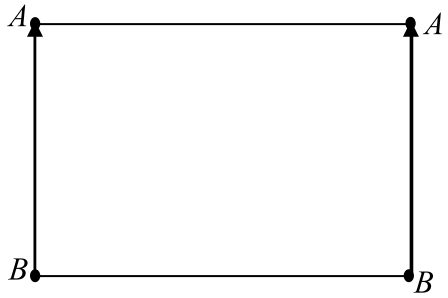

fig.3.13

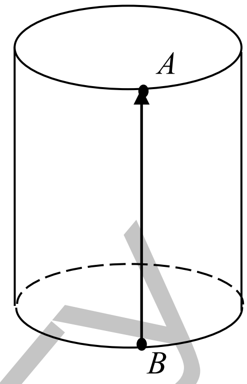

fig.3.14

We chose the direction of the round trip counter clockwise and if the direction doesn't coincide with the direction of the arrow, we sign this side as a^{-1} for example. So we can sign the whole evolvent as $aba^{-1}b^{-1}$. First we glue together sides a and we get the cylinder with the arrows on its edges (figure 3.16). Then we glue together sides b and b^{-1}. As the result we get the torus (figure 3.17).

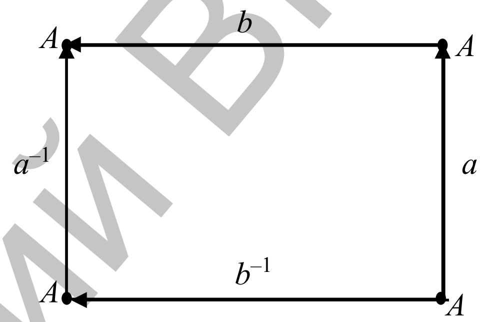

fig.3.15

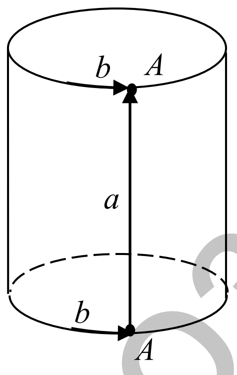

fig.3.16

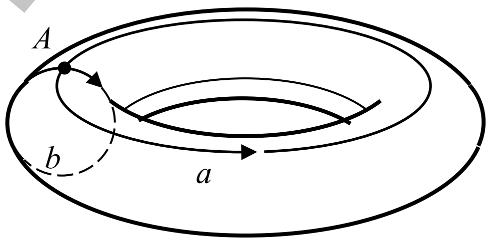

fig.3.17

5. Consider now the following unfold surface (figure 3.18). We can sign this unfold surface as $aba^{-1}b$. First we get the cylinder, but then we must glue together its sides in opposite directions (figure 3.19). It is impossible to make it in the 3-dimensional space. So, the figure we get can't be embedded into the 3-dimensional space without self-intersection. It is called 'The bottle of Klein' (figure 3.20).

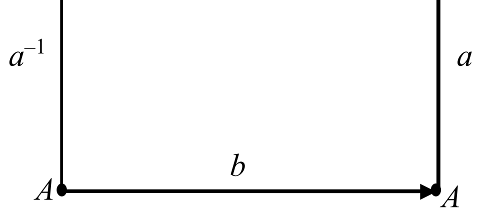

fig.3.18

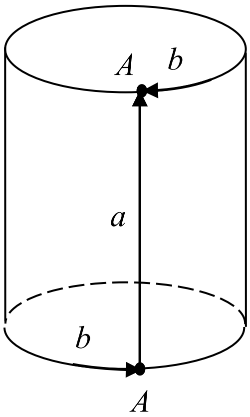

fig.3.19

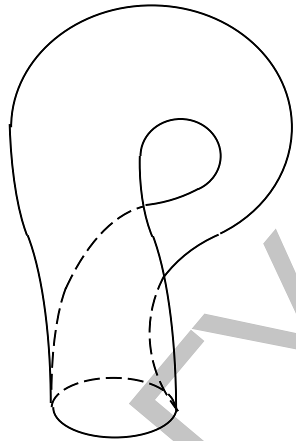

fig.3.20

6. We can make the sphere from the following unfolded surface (figure 3.21). It can be signed as aa^{-1}. Let us find out, what we can get from unfolded surface aa (figure 3.23). It would be better to consider the evolvent as semisphere and the opposite points of each diameter are to be glued together (figure 3.24).

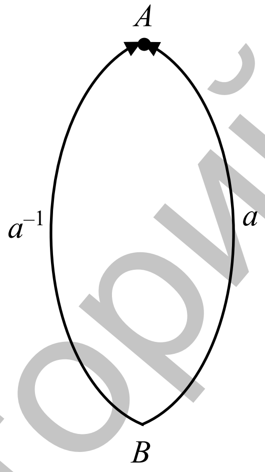

fig.3.21

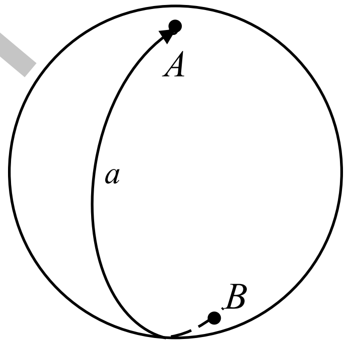

fig.3.22

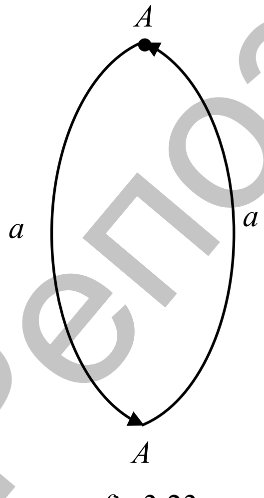

fig.3.23

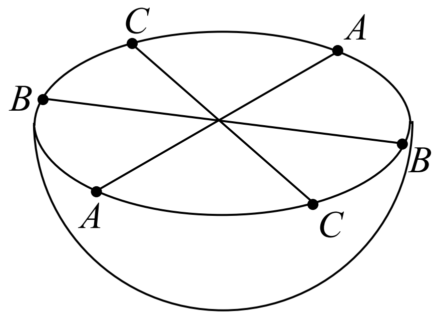

fig.3.24

The surface we get is the projective plain. It is also impossible to imbed this surface into the 3-dimensional space.

***Exercise* 29.** *Prove that the boundary of the Mőbius band is a simple closed line.*

30. *Let us construct a surface from the following evolvent (figure 3.25). We should paste 4 pairs of segments, indicated by the same letter. Prove, that the surface has one side and its boundary is a simple closed line.*

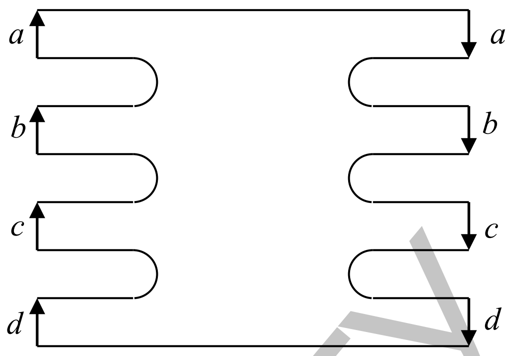

*fig.*3.25

Definition. A surface is said to be <u>*closed*</u> if it is bounded and has no boundary.

For example, the sphere and the torus are closed surfaces and elliptic paraboloid is not a closed surface. If we take its bounded part it would have a boundary.

§3. *Gluing of surfaces. A problem of topological classification.*

Let Φ_1 and Φ_2 be two surfaces with boundary and suppose, that each of the surfaces has an edge homeomorphic to a circle (figure 3.26). We join (glue, paste) the edges and we get a new surface. One can say that a hole in the first surface is pasted by the second one or vice versa. If both surfaces Φ_1 and Φ_2 are closed, the new surface is closed as well.

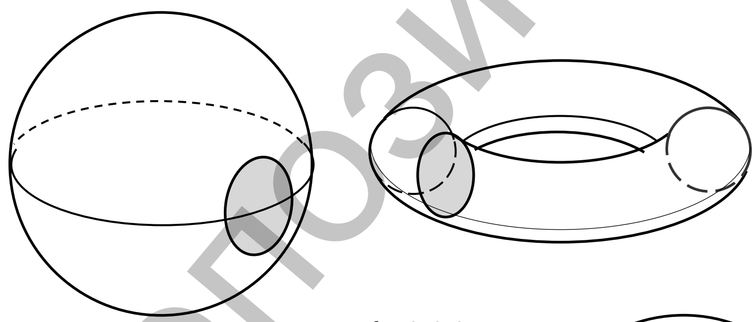
*fig.*3.26

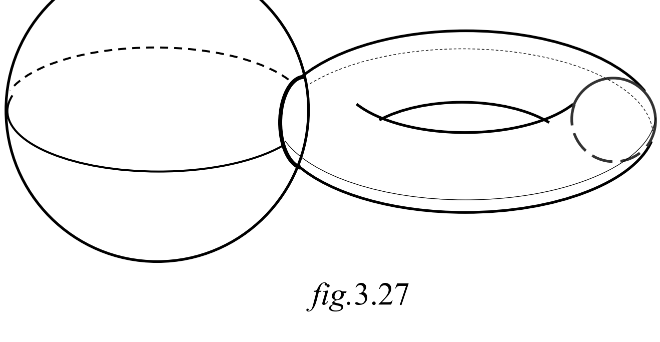
*fig.*3.27

***Example* 5.** If we paste the sphere and the torus we get the surface 'a sphere with a handle', which is homeomorphic to the torus (figure 3.27). The sphere with two handles is homeomorphic to the surface, which can be obtained by pasting of two torus.

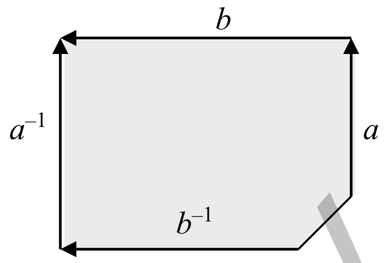

fig.3.28

***Exercise* 31.** *Consider the following unfolded surface (figure 3.28). What surface we get, if we past the sides indicated by the same letter taking into account its direction?*

A problem of topological classification of surfaces means the following. *We should submit a list of closed surfaces, which are not homeomorphic to each other, and any closed surface is homeomorphic to one of the surfaces from the list*.

Denote P_k the sphere with k hands. For example, P_o is the sphere and P_1 is the torus.

Theorem.

$$P_o, P_1, P_2, \ldots, P_k, \ldots \tag{5}$$

is a complete topological classification of the closed orientable surfaces.

We will submit the proof of the theorem later.

§4. *Eulerian characteristics of a surface.*

Let Φ be a surface with boundary or without boundary, which has 1 or 2 sides. Suppose, that Φ admits decomposition on polygonal domains, it means that we can draw a graph G on the surface, which divides the surface on a finite number of domains each of them is homeomorphic to the circle. Denote V – the number of vertexes of the graph, E – the number of edges and F – the number of polygonal domains (faces). Number

$$\chi(\Phi) = V - E + F \tag{6}$$

is called Eulerian characteristics of the surface Φ. Precisely speaking, this number is determined not by the surface itself, but by the graph G. We proved for the sphere that this number doesn't depend on the graph and it is equal to 2. We are going to show now that it is true for any surface.

Let two graphs G_1 and G_2 be drawn on the surface Φ. Each of them determines a decomposition of the surface on polygonal domains. Let V_i be the number of vertexes of the graph G_i, E_i – the number of edges and F_i – the number of faces. The graphs G_1 and G_2 could have infinite number of intersection points. But we can move the graph G_1 a bit so that two graphs would have finite number of intersection points. If the graph $G_1 \cup G_2$ is not connected, we can move G_1 and G_2 so, that they will have common points.

We add all the common points to the set of the vertexes of graph $G_1 \cup G_2$. Denote V – the number of vertexes of the graph $G_1 \cup G_2$, E – the number of its edges and F – the number of faces. We are going to prove two equalities

$$\begin{cases} V_1 - E_1 + F_1 = V - E + F \\ V_2 - E_2 + F_2 = V - E + F \end{cases} \tag{7}$$

and they imply $V_1 - E_1 + F_1 = V_2 - E_2 + F_2$. Both equalities (7) have the same proof. So we will prove only the first one.

Let M be an arbitrary face determined by graph G_1 (figure 3.29). Denote V', E' the number of vertexes and the number of edges of the graph $G_1 \cup G_2$ located inside M (not on the border) and let this graph decompose M on F' faces. Denote q the number of vertexes and the number of edges at the same time located on the border of M. Let M_o be the same polygon, but without any decomposition. Let us cut the polygon M out of the surface Φ and paste it by the border with M_o. We get the surface homeomorphic to the sphere (figure 3.30). Each of the polygons M and M_o is homeomorphic to the hemisphere. We have a connected graph drawn on the sphere, which has $V'+q$ vertexes, $E'+q$ edges and $F'+1$ faces, because we added to F' faces of M one more face M_o.

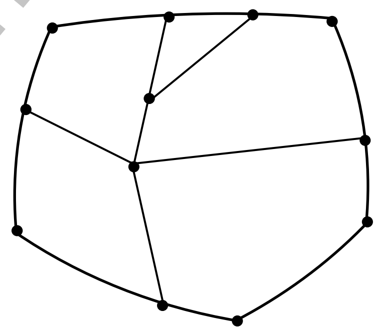

*fig.*3.29

According to the Euler theorem we have

$$(V'+q)-(E'+q)+(F'+1)=2.$$

Therefore

$$V'-E'+F'=1. \tag{8}$$

Let us return to the surface Φ, where the graph $G_1 \cup G_2$ is drawn. We delete from the graph its part located inside M. Instead of V' vertexes, E' edges and

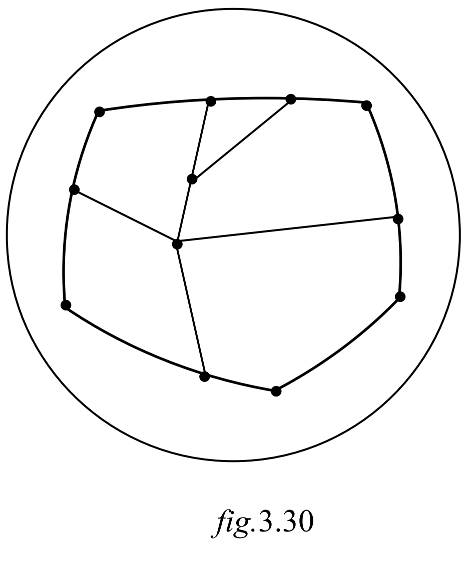

*fig.*3.30

F' faces we get 0 vertexes, 0 edges and 1 face (M itself). I.e. the number $V'-E'+F'$ will be replaced by the number $0-0+1=1$. According to (8) nothing will change!

We delete now all the parts of graph $G_1 \cup G_2$ located inside all the faces of graph G_1. We get a new graph G^* and the number $V^*-E^*+F^*$ remains the same, i.e. $V^*-E^*+F^*=V-E+F$.

What is the difference between G_1 and G^*? G^* has some additional vertexes on the edges of G_1. If one adds a vertex on the edge of a graph, he adds 1 more edge at the same time. So this action does not change the number $V-E+F$. Therefore $V^*-E^*+F^*=V_1-E_1+F_1$ and the first of the equalities (7) is true.

We proved, that Eulerian characteristics does not depend on a decomposition of the surface on polygonal domains and it is determined by the surface itself. Moreover, it is topological invariant.

Really, let a finite graph G be drawn on the closed surface Φ_1 and let $f:\Phi_1 \to \Phi_2$ be a homeomorphism. Then $G'=f(G)$ is the graph on the surface Φ_2 which has the same number of edges and the same number of vertexes as G. And G' decomposes Φ_2 on the same number of domains as G decomposes Φ_1. Therefore $\chi(\Phi_1)=\chi(\Phi_2)$.

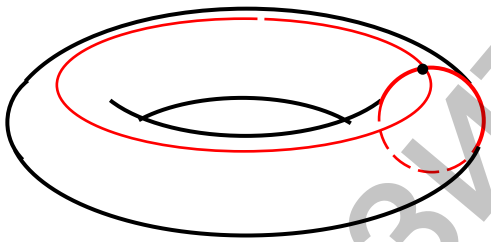

*fig.*3.31

Now it is easy to prove, that the sphere is not homeomorphic to the torus. Consider the following graph on the torus (figure 3.31). It has 1 vertex, 1 edge and there is only one face. Therefore $\chi(T^2)=1-2+1=0$. And we know, that $\chi(S^2)=2$.

***Exercises* 32.** *Prove, that the sphere with q holes has Eulerian characteristics $2-q$.*

33. *Let Φ_1 and Φ_2 be two surfaces, that have boundary homeomorphic to the circle. Prove, that if we paste them by there's boundaries, we will get the surface with the Eulerian characteristics $\chi(\Phi_1)+\chi(\Phi_2)$.*

34. *Find the Eulerian characteristics of circle, and of a hand and of the Mőbius band.*

35. *Prove, that the Eulerian characteristics of the surface P_k is equal to $2-2k$. Therefore all the surfaces from the list (5) are not topologically equivalent.*

34. *A graph is drawn on a surface and it has* V *vertexes,* E *edges. Suppose, that the graph decomposes the surface on* F *domains, but some of the domains are not homeomorphic to a circle. Prove, that* $V-E+F\geq\chi(\Phi)$.

§5. *Topological classification of nonorientable closed surfaces*

Consider the edge of the Mőbius band. As the matter of fact it is a circle twisted like the figure-of-eight (figure 3.32). Let Φ be an arbitrary surface. We can make the round hole in Φ and glue the edge of the Mőbius band with the edge of the hole. So we can seal (stick) the hole by the Mőbius band.

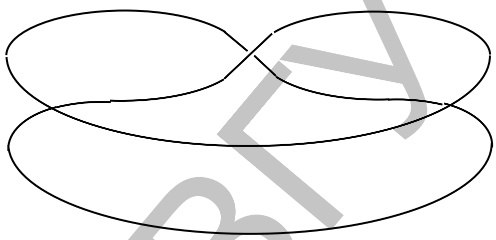

fig.3.32

We can accomplish this process in a several steps. First we make some preparations with the evolvent of the Mőbius band. Let d be the medium line. We cut the evolvent through this line (figure 3.33) and glue it again as it is demonstrated on the figure 3.34. The edge of the Mőbius band is not signed.

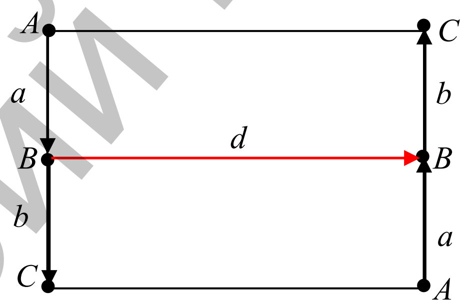

fig.3.33

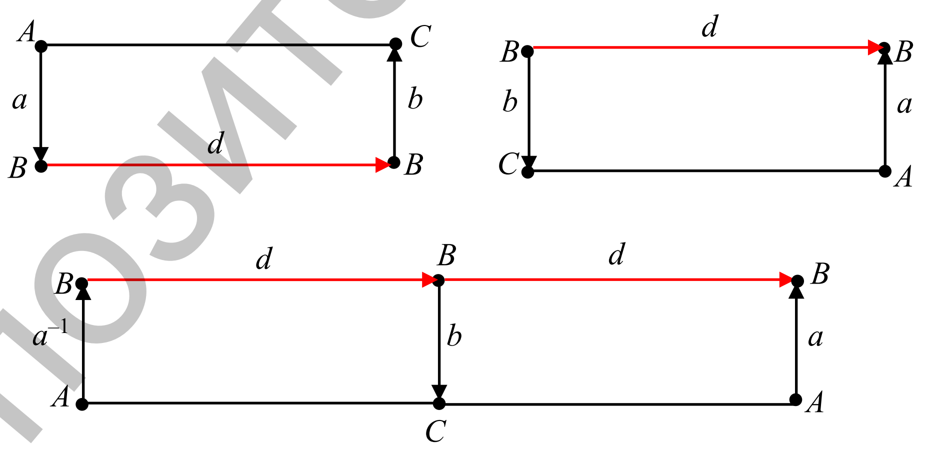

fig.3.34

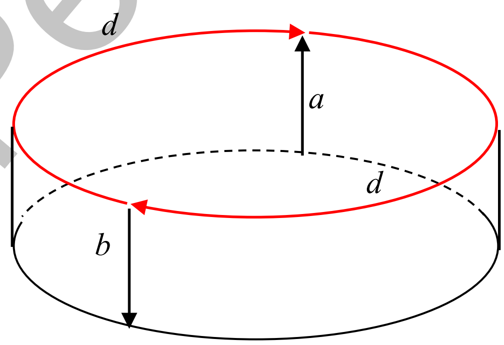

fig.3.35

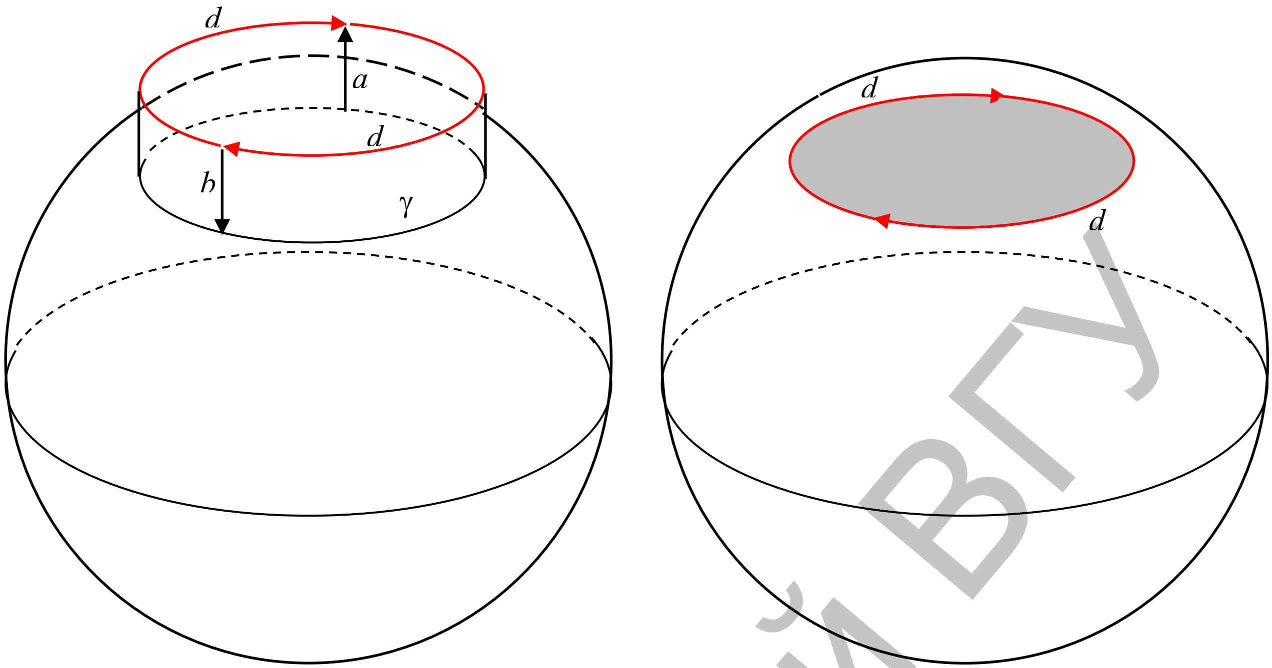

*fig.*3.36

We get the band where upper side marked by letter d should be pasted with itself.

We glue the edge of the Mőbius band of the to the edge γ of the round hole (figure 3.36). We see, that it is the same, as if we identify the opposite points of γ. If our surface Φ is the sphere, we get the projective plane.

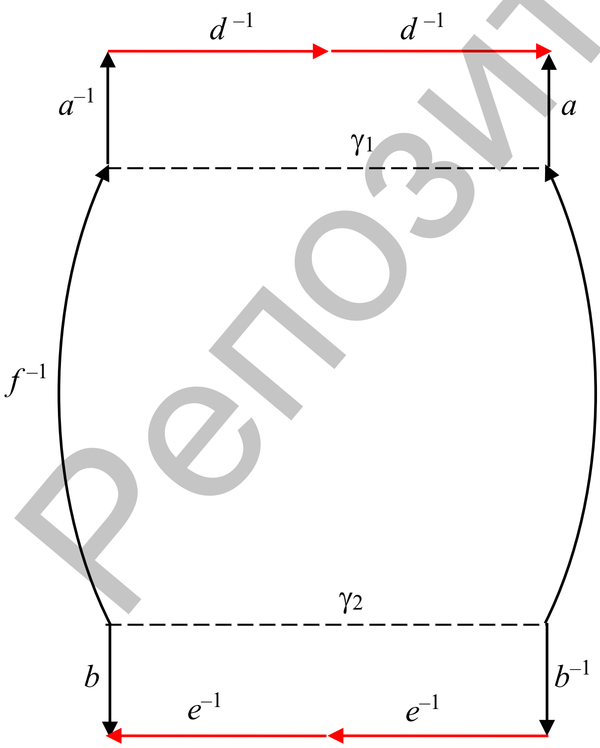

*fig.*3.37

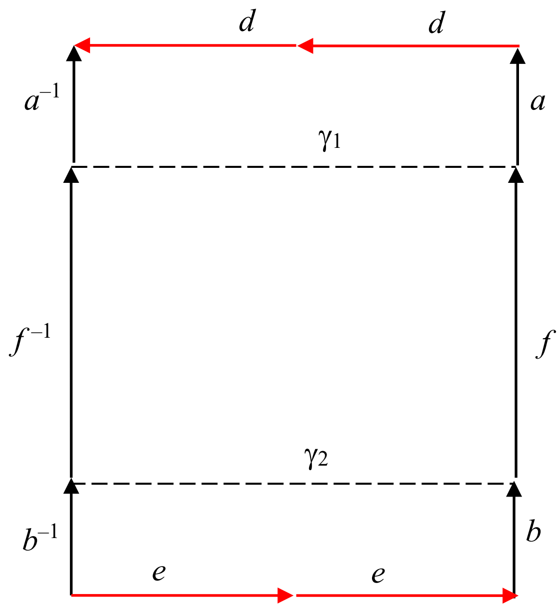

*fig.*3.38

What will we get, if we seal two round holes on the sphere by the Mőbius band? Let's see on the evolvent. Edges of the round holes are signed as γ_1 and γ_2 (figure 3.37). We can change the directions of two arrows signed by the same letter simultaneously. So we get the evolvent 3.38.

Then we unite the edges b, f and a in the one edge g, and in the same way we unite a^{-1}, f^{-1} and b^{-1} in the one edge g^{-1}: $(bfa)^{-1} = a^{-1}f^{-1}b^{-1}$.

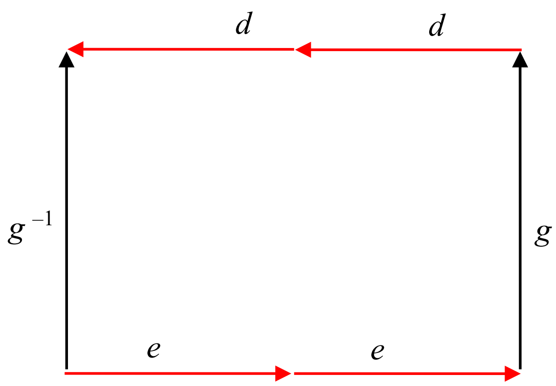

*fig.*3.39

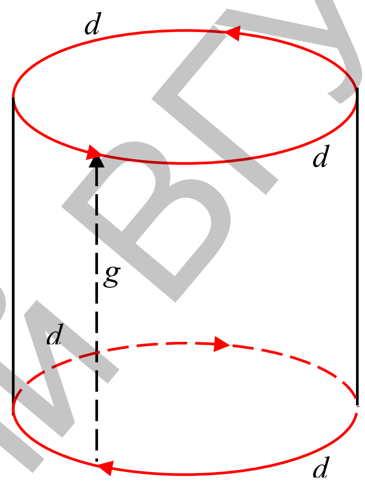

*fig.*3.40

So we get the evolvent 3.39. It is the same situation, as if we identify the opposite points on two bases of the cylinder (figure 3.40). Then we cut the evolvent 3.39 through the medium line and paste the two parts (figures 3.41, 3.42). Later on we join $h^{-1}g^{-1}$ in one edge k (figure 3.43). Finally we get the evolvent of the Klein bottle.

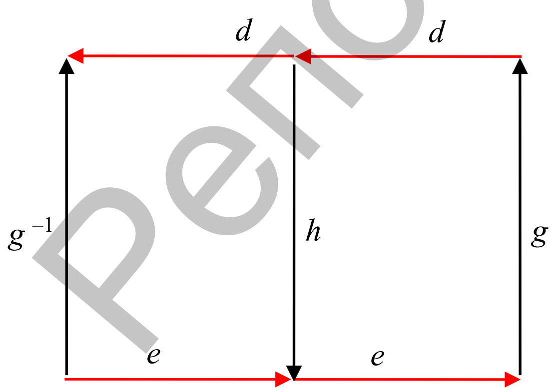

*fig.*3.41

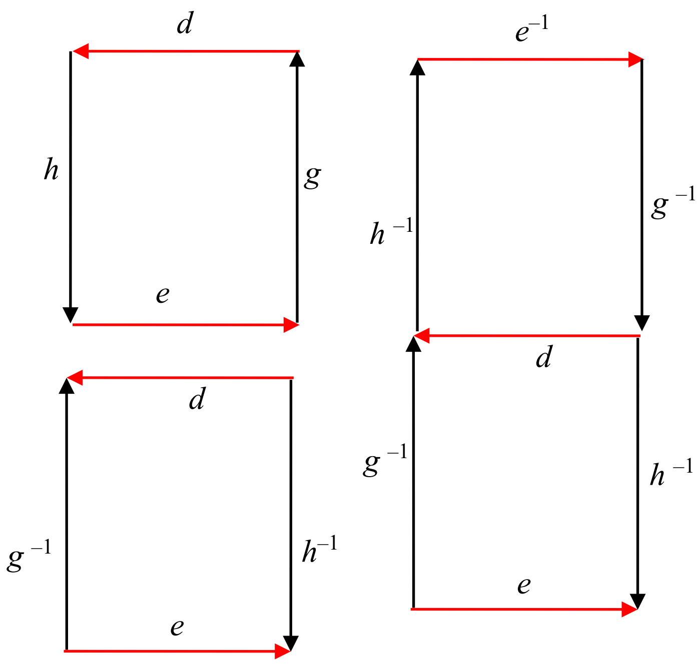

*fig.*3.42

It means, that the sphere with two round holes sealed by the Mőbius bands is topologically equivalent to the Klein bottle. We accept without proof, that sealing of three round holes by the Mőbius bands gives the same surface as pasting one hand and sealing of one round hole by the Mőbius band.

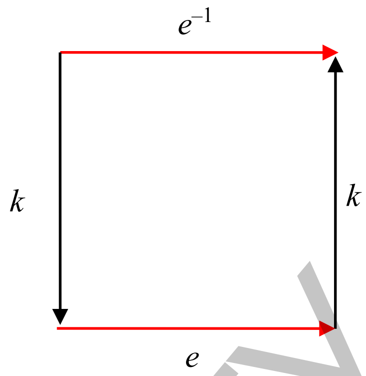

fig.3.43

Theorem of Mőbius-Jordan. *Denote* N_q *the surface which is obtained from the sphere as a result of sealing* q *round holes by the Mőbius bands. Then*

$$N_1, N_2, \ldots, N_q, \ldots \qquad (9)$$

is the complete topological classification of closed nonorientable surfaces. It means, that an arbitrary closed nonorientable surface is homeomorphic to one of the surfaces $N_1, N_2, \ldots, N_q, \ldots$.

***Exercise* 37.** *Prove, that the Eulerian characteristics of the surface* N_q *is equal to* $2-q$.

The latter result means, that all the surfaces (9) are not topologically equivalent.

§6. *Vector fields on closed surfaces*

Let Φ be an arbitrary surface. Denote $T_M\Phi$ the tangent plain to the surface Φ at the point $M\in\Phi$. Suppose, that each point $M\in\Phi$ is assigned a vector $\vec{v}_M$. Then we say, that *a vector field* $\vec{v}$ is given on the surface Φ. If for each point $M\in\Phi$ $\vec{v}_M$ is a vector from tangent plane $T_M\Phi$, then we say that the *tangent vector field* $\vec{v}$ is given on the surface Φ. We say that a vector field $\vec{v}$ is *continuous*, if $\vec{v}_M$ varies continuously from point to point.

***Example* 6.** Let's try to define continuous vector field on the sphere. The first attempt is to define it along the meridians, and the second attempt is to define it along the parallels (fig. 3.44).

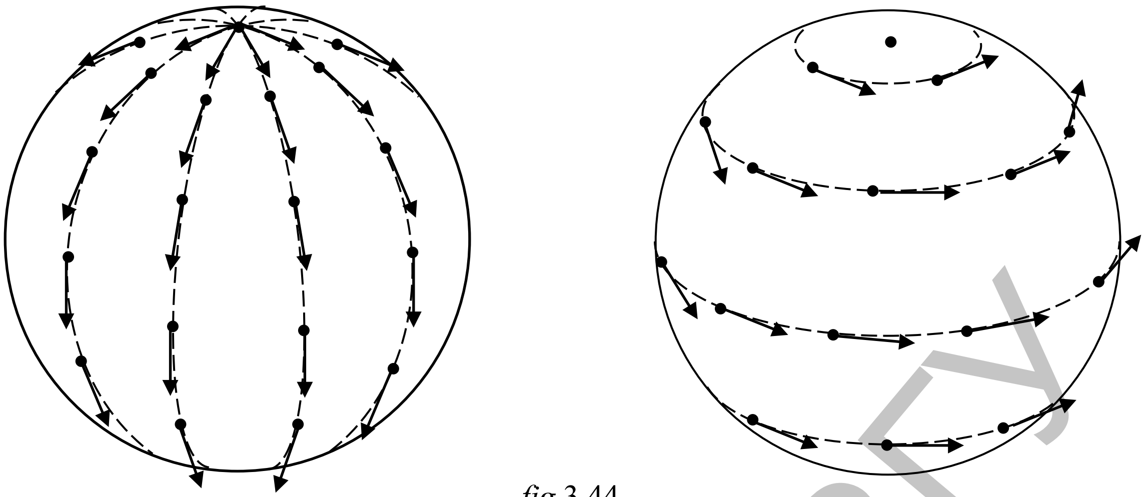

fig.3.44

Let's have a look at the poles. We see, that these vector fields can be continuous at the poles N and S only if $|\vec{v}_N| = |\vec{v}_S| = \vec{o}$ (figure 3.45). This points are called <u>*the singular points*</u> of vector field $\vec{v}$.

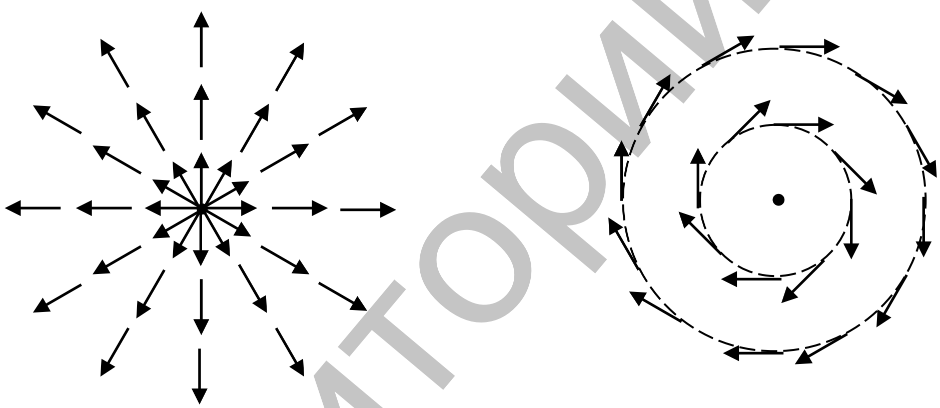

fig.3.45

We are going to prove later, that there is no continuous field of directions on the sphere without singular points. This result can be formulated as follows.

Hedgehog Theorem. *Suppose, that a continuous nonzero vector field is given on the sphere* S^2. *Then there is a point* $x_o \in S^2$, *were* $\vec{v}(x_o)$ *is a normal vector* (*i.e. it is perpendicular to the tangent plane*).

Really, we can decompose this vector field as a sum of two vector fields: $\vec{v} = \vec{n} + \vec{t}$, where $\vec{n}$ is the normal one, and $\vec{t}$ is the tangent one (figure 3.46). The vector field $\vec{t}$ must have the singular point, where $\vec{t} = \vec{o}$ and at this point we have $\vec{v} = \vec{n}$.

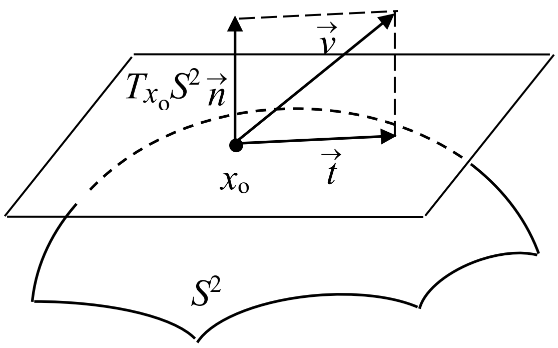

fig.3.46

On the figure 68 one can see more complex singular point, then on the figure 3.47. It is so called 'saddle point'.

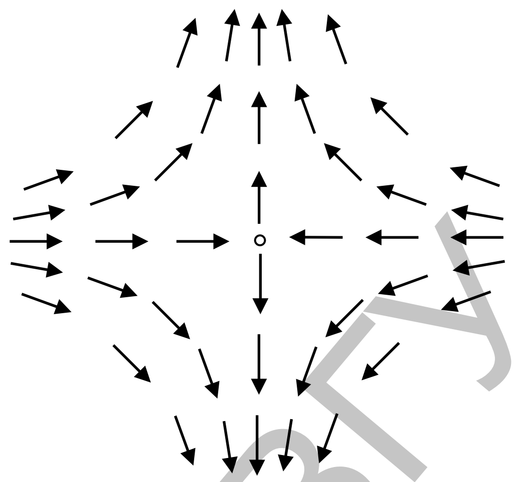

*fig.*3.47

Let's choose a little circle around a singular point and chose direction of the circuit around the point (for instance, counterclockwise). We go along the circle and observe the direction of the vector field. If the vector field turns n times in the same direction, we say, that the singular point has index n. If the vector field turns n times in the opposite direction, we say, that the singular point has the index $-n$.

Let's have a look, what we have for the singular points drawn in the figures 65 and 67. For the first two points we have the index +1, and for the third point we have the index –1 (fig. 3.48).

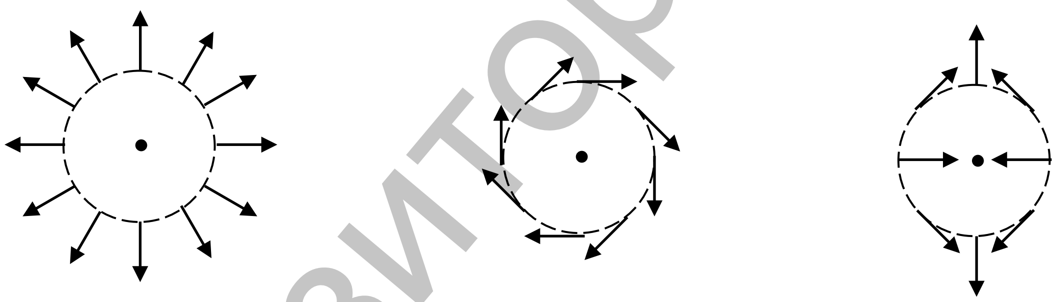

*fig.*3.48

We are going to prove the following theorem.

Theorem of Poincare. *Suppose that a tangent vector field is defined on a closed surface* Φ*, and this vector field is continuous everywhere except the finite number of singular points. Then the sum of indexes of all the singular points is equal to the* $\chi(\Phi)$.

***Example* 7.** We know, that $\chi(P_k) = 2 - k$ for the orientable closed surface P_k. So the only orientable closed surface, which can have a tangent vector field without singular points is P_1, i.e. the torus. And, of course it is impossible to define such vector field on the sphere.

We can define the desirable vector field on the torus, for example, along the parallels or along the meridians. Any constant vector field on the evolvent of the torus generates the desirable vector field on the torus (figure 3.49).

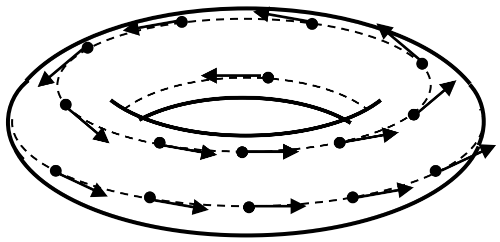

fig.3.49

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